

DE BROGLIE'S HYPOTHESIS AND THE WAVE EQUATION

TRISTRAM DE PIRO

ABSTRACT.

Lemma 0.1. *We have that;*

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^n(y)dy = \frac{n!\pi}{2^n[(\frac{n}{2})!]^2}, \text{ if } n \text{ is even}$$

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^n(y)dy = \frac{[\frac{n-1}{2}]!2^n}{n!}, \text{ if } n \text{ is odd}$$

Proof. Let $I_n = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^n(y)dy$, then for $n \geq 2$, we have that, using integration by parts;

$$\begin{aligned} I_n &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^n(y)dy \\ &= [\cos^{n-1}(y)\sin(y)]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} + \int -\frac{\pi}{2}^{\frac{\pi}{2}}(n-1)\cos^{n-2}(y)\sin^2(y)dy \\ &= \int -\frac{\pi}{2}^{\frac{\pi}{2}}(n-1)\cos^{n-2}(y)(1-\cos^2(y))dy \\ &= (n-1)I_{n-2} - (n-1)I_n \end{aligned}$$

so that, rearranging;

$$I_n = \frac{n-1}{n}I_{n-2}$$

and, using the fact $I_0 = \pi$, $I_1 = 2$, we have that, for n even;

$$I_n = \frac{n!}{2^n[(\frac{n}{2})!]^2}\pi$$

and, for n odd;

$$I_n = \frac{[\frac{n-1}{2}]!2^n}{n!}$$

□

Lemma 0.2. *Let $n \in \mathcal{N}$, $\epsilon > 0$, and let $\gamma_{n,\epsilon}$ be defined by;*

$$\gamma_{n,\epsilon}(x) = \frac{1}{\epsilon} \cos^n\left(\frac{\pi x}{2\epsilon}\right), \text{ for } x \in [-\epsilon, \epsilon]$$

$$\gamma_{n,\epsilon}(x) = 0, \text{ otherwise}$$

Then $\gamma_{n,\epsilon}$ has the following properties;

$$(i). \gamma_{n,\epsilon} \in C^{m-1}(\mathcal{R}).$$

$$(ii). \gamma_{n,\epsilon} \geq 0.$$

$$(iii). \int_{\mathcal{R}} \gamma_{n,\epsilon}(x) dx = \frac{n!}{2^{n-1}[(\frac{n}{2})!]^2}, \text{ } n \text{ even}$$

$$\int_{\mathcal{R}} \gamma_{n,\epsilon}(x) dx = \frac{[\frac{n-1}{2}]!^2 2^{n+1}}{\pi n!}, \text{ } n \text{ odd}$$

$$(iv) \gamma_{n,\epsilon} \text{ is supported on } [-\epsilon, \epsilon].$$

Proof. (ii) is clear as $\cos(y) \geq 0$ for $y \in [-\frac{\pi}{2}, \frac{\pi}{2}]$, (iv) is clear by the definition of $\gamma_{n,\epsilon}$. To prove (i), it is sufficient to show that;

$$\cos^n\left(\frac{\pi x}{2\epsilon}\right)^{(m)}(\epsilon) = \cos^n\left(\frac{\pi x}{2\epsilon}\right)^{(m)}(-\epsilon) = 0$$

for $0 \leq m \leq n-1$. We can prove this by induction on n , as for $n=1$, we have that;

$$\cos\left(\frac{\pi x}{2\epsilon}\right)(\epsilon) = \cos\left(\frac{\pi}{2}\right) = \cos\left(\frac{\pi x}{2\epsilon}\right)(-\epsilon) = \cos\left(-\frac{\pi}{2}\right) = 0$$

and, if the inductive hypothesis holds for $n \in \mathcal{N}$, then, for $1 \leq m \leq n$;

$$\begin{aligned} & \cos^{n+1}\left(\frac{\pi x}{2\epsilon}\right)^{(m)}(\epsilon) \\ &= -\left[\frac{\pi(n+1)}{2\epsilon} \cos^n\left(\frac{\pi x}{2\epsilon}\right) \sin\left(\frac{\pi x}{2\epsilon}\right)\right]^{(m-1)}(\epsilon) \\ &= -\frac{\pi(n+1)}{2\epsilon} \left[\sum_{k=0}^{m-1} C_k^{m-1} \cos^n\left(\frac{\pi x}{2\epsilon}\right)^{(m-1-k)} \sin\left(\frac{\pi x}{2\epsilon}\right)^{(k)}\right](\epsilon) \\ &= 0 \end{aligned}$$

and similarly;

$$\cos^{n+1}\left(\frac{\pi x}{2\epsilon}\right)^{(m)}(-\epsilon) = 0$$

while, clearly;

$$\cos^{n+1}\left(\frac{\pi x}{2\epsilon}\right)(\epsilon) = \cos^{n+1}\left(\frac{\pi x}{2\epsilon}\right)(-\epsilon) = 0$$

To prove (iii), we have that, for $n \in \mathcal{N}$;

$$\begin{aligned} & \int_{\mathcal{R}} \gamma_{n,\epsilon}(x) dx \\ &= \frac{1}{\epsilon} \int_{-\epsilon}^{\epsilon} \cos^n\left(\frac{\pi x}{2\epsilon}\right) \\ &= \frac{1}{\epsilon} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^n(y) \frac{2\epsilon}{\pi} dy, \quad (y = \frac{\pi x}{2\epsilon}) \\ &= \frac{2}{\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^n(y) dy \end{aligned}$$

so that, using Lemma 0.1, for n even;

$$\begin{aligned} \int_{\mathcal{R}} \gamma_{n,\epsilon}(x) dx &= \frac{2}{\pi} \frac{n!}{2^{n-1}[(\frac{n}{2})!]^2} \pi \\ &= \frac{n!}{2^{n-1}[(\frac{n}{2})!]^2} \end{aligned}$$

and, for n odd;

$$\begin{aligned} \int_{\mathcal{R}} \gamma_{n,\epsilon}(x) dx &= \frac{2}{\pi} \frac{[\frac{n-1}{2}]!^2 2^n}{n!} \\ &= \frac{[\frac{n-1}{2}]!^2 2^{n+1}}{\pi n!} \end{aligned}$$

□

Lemma 0.3. *Let $\delta_{n,\epsilon}(x)$ be defined by;*

$$\delta_{n,\epsilon}(x) = \frac{2^{n-1}[\frac{n}{2}]!^2}{n!} \gamma_{n,\epsilon}, \text{ for } n \text{ even}$$

and by;

$$\delta_{n,\epsilon}(x) = \frac{\pi n!}{[(\frac{n-1}{2})!]^2 2^{n+1}} \gamma_{n,\epsilon}, \text{ for } n \text{ odd}$$

Then the properties (i), (ii), (iv) of Lemma 0.2 hold, with (iii) changed to;

$$(iii)'. \int_{\mathcal{R}} \gamma_{n,\epsilon}(x) dx = 1, n \in \mathcal{N}$$

and, for $n \in \mathcal{N}$;

$$\lim_{\epsilon \rightarrow 0} \delta_{n,\epsilon} = \delta$$

in the sense of distributions, where δ is the Dirac delta function on \mathcal{R} .

Proof. The first claim is clear as we have just normalised $\gamma_{n,\epsilon}$. For the remaining claim, let $f \in C_c^\infty(\mathcal{R})$, and write;

$$f = f^+ + f^-$$

where;

$$f^+(x) = f(x), \text{ if } f(x) \geq 0$$

$$f^+(x) = 0 \text{ otherwise}$$

$$f^-(x) = f(x), \text{ if } f(x) \leq 0$$

$$f^-(x) = 0 \text{ otherwise}$$

Then, using properties (ii), (iii)', (iv) of $\delta_{n,\epsilon}$ and continuity of f ;

$$\begin{aligned} \min_{[-\epsilon,\epsilon]} f^+ + \min_{[-\epsilon,\epsilon]} f^- &\leq \delta_{n,\epsilon}(f) = \int_{-\epsilon}^{\epsilon} \delta_{n,\epsilon}(x) f^+(x) dx + \int_{-\epsilon}^{\epsilon} \delta_{n,\epsilon}(x) f^-(x) dx \\ &\leq \max_{[-\epsilon,\epsilon]} f^+ + \max_{[-\epsilon,\epsilon]} f^- \end{aligned}$$

with;

$$\begin{aligned} \lim_{\epsilon \rightarrow 0} \min_{[-\epsilon,\epsilon]} f^+ + \min_{[-\epsilon,\epsilon]} f^- &= \lim_{\epsilon \rightarrow 0} \max_{[-\epsilon,\epsilon]} f^+ + \max_{[-\epsilon,\epsilon]} f^- \\ &= f(0) \end{aligned}$$

so that $\lim_{\epsilon \rightarrow 0} \delta_{n,\epsilon}(f) = f(0) = \delta(f)$, as required.

□

Lemma 0.4. *We define the time derivative δ'_t of the delta function δ to be;*

$$\frac{d}{dt}\delta(x - vt)$$

where v is the velocity, so that;

$$\delta'_t = -v\delta'$$

in the sense of distributions. Similarly, we define the time derivative $\delta'_{n,\epsilon,t}$ of the approximations by;

$$\frac{d}{dt}\delta_{n,\epsilon}(x - vt)$$

so that, by the chain rule;

$$\delta'_{n,\epsilon,t}(x) = -v\delta'_{n,\epsilon}(x)$$

Then;

$$\lim_{\epsilon \rightarrow 0}\delta'_{n,\epsilon,t} = \delta'_t$$

in the sense of distributions.

Moreover, for n even, $n \geq 2$;

$$\delta'_{n,\epsilon,t}(x) = \frac{v2^{n-1}\pi[\frac{n}{2}]!^2}{2\epsilon(n-1)!}\gamma_{n-1,\epsilon}\sin(\frac{\pi x}{2\epsilon})$$

and, for n odd, $n \geq 3$;

$$\delta'_{n,\epsilon,t}(x) = \frac{v\pi^2 n!n}{2\epsilon[(\frac{n-1}{2})!]^2 2^{n+1}}\gamma_{n-1,\epsilon}\sin(\frac{\pi x}{2\epsilon})$$

In particular, for $n \geq 2$;

$$\delta'_{n,\epsilon,t} \in C_c^{n-2}(\mathcal{R})$$

Proof. For the first claim, let $f \in C_c^\infty(\mathcal{R})$, then, using integration by parts, (iv) of Lemma 0.3;

$$\delta'_{n,\epsilon,t}(f) = -v \int_{-\epsilon}^{\epsilon} \delta'_{n,\epsilon}(x) f(x) dx$$

$$\begin{aligned}
&= -v([\delta'_{n,\epsilon}(x)f(x)]_{-\epsilon}^{\epsilon} - \int_{-\epsilon}^{\epsilon} \delta_{n,\epsilon}(x)f'(x)dx) \\
&= v \int_{-\epsilon}^{\epsilon} \delta_{n,\epsilon}(x)f'(x)dx
\end{aligned}$$

so that, using the main result of Lemma 0.3;

$$\begin{aligned}
&\lim_{\epsilon \rightarrow 0} \delta'_{n,\epsilon,t}(f) \\
&= v \lim_{\epsilon \rightarrow 0} \int_{-\epsilon}^{\epsilon} \delta_{n,\epsilon}(x)f'(x)dx \\
&= v f'(0) \\
&= -v \delta'(f) \\
&= \delta'_t(f)
\end{aligned}$$

as required. For the second claim, we have that, for n even, $n \geq 2$;

$$\begin{aligned}
\delta'_{n,\epsilon,t}(x) &= -v \delta'_{n,\epsilon}(x) \\
&= -v \frac{2^{n-1} \lfloor \frac{n}{2} \rfloor!^2}{n!} \gamma'_{n,\epsilon}(x) \\
&= -v \frac{2^{n-1} \lfloor \frac{n}{2} \rfloor!^2}{n!} (-n \cos^{n-1}(\frac{\pi x}{2\epsilon}) \sin(\frac{\pi x}{2\epsilon}) \frac{\pi}{2\epsilon^2}) \\
&= v \frac{2^{n-1} \pi \lfloor \frac{n}{2} \rfloor!^2}{2\epsilon(n-1)!} (\frac{1}{\epsilon} \cos^{n-1}(\frac{\pi x}{2\epsilon}) \sin(\frac{\pi x}{2\epsilon})) \\
&= \frac{v 2^{n-1} \pi \lfloor \frac{n}{2} \rfloor!^2}{2\epsilon(n-1)!} \gamma_{n-1,\epsilon}(x) \sin(\frac{\pi x}{2\epsilon})
\end{aligned}$$

and, for n odd, $n \geq 3$;

$$\begin{aligned}
\delta'_{n,\epsilon,t}(x) &= -v \delta'_{n,\epsilon}(x) \\
&= -v \frac{\pi n!}{[\lfloor \frac{n-1}{2} \rfloor!]^2 2^{n+1}} \gamma'_{n,\epsilon}(x) \\
&= -v \frac{\pi n!}{[\lfloor \frac{n-1}{2} \rfloor!]^2 2^{n+1}} (-n \cos^{n-1}(\frac{\pi x}{2\epsilon}) \sin(\frac{\pi x}{2\epsilon}) \frac{\pi}{2\epsilon^2}) \\
&= v \frac{\pi^2 n! n}{2\epsilon [\lfloor \frac{n-1}{2} \rfloor!]^2 2^{n+1}} (\frac{1}{\epsilon} \cos^{n-1}(\frac{\pi x}{2\epsilon}) \sin(\frac{\pi x}{2\epsilon})) \\
&= \frac{v \pi^2 n! n}{2\epsilon [\lfloor \frac{n-1}{2} \rfloor!]^2 2^{n+1}} \gamma_{n-1,\epsilon}(x) \sin(\frac{\pi x}{2\epsilon})
\end{aligned}$$

For the final claim, we have that, by the product rule, for $0 \leq m \leq n - 2$;

$$\begin{aligned} & (\gamma_{n-1,\epsilon}(x) \sin(\frac{\pi x}{2\epsilon}))^{(m)} \\ &= \sum_{k=0}^m \gamma_{n-1,\epsilon}^{(m-k)} (\sin(\frac{\pi x}{2\epsilon}))^k \end{aligned}$$

and use the fact that $\gamma_{n-1,\epsilon} \in C_c^{n-2}(\mathcal{R})$ □

Lemma 0.5. *Let D denote the 3-dimensional Dirac delta function. For $n \in \mathcal{N}$, $\epsilon > 0$, let $D_{n,\epsilon}$ be defined by;*

$$D_{n,\epsilon}(x, y, z) = \delta_{n,\epsilon}(x) \delta_{n,\epsilon}(y) \delta_{n,\epsilon}(z)$$

Then $D_{n,\epsilon}$ has the following properties;

- (i). $D_{n,\epsilon} \in C^{n-1}(\mathcal{R}^3)$.
- (ii). $D_{n,\epsilon} \geq 0$.
- (iii). $\int_{\mathcal{R}^3} D_{n,\epsilon}(x, y, z) dx dy dz = 1$
- (iv) $D_{n,\epsilon}$ is supported on $[-\epsilon, \epsilon]^3$.

Moreover, for $n \in \mathcal{N}$, $\lim_{\epsilon \rightarrow 0} D_{n,\epsilon} = D$ in the sense of distributions, (*).

We define the time derivative D'_t of the delta function D to be;

$$\frac{d}{dt} D(\bar{x} - \bar{v}t)$$

where \bar{v} is the velocity vector, so that;

$$D'_t = -v_1 D_x - v_2 D_y - v_3 D_z$$

in the sense of distributions. Similarly, we define the time derivative $D'_{n,\epsilon,t}$ of the approximations by;

$$\frac{d}{dt} D_{n,\epsilon}(\bar{x} - \bar{v}t)$$

so that, by the chain rule;

$$\begin{aligned}
D'_{n,\epsilon,t}(\bar{x}) &= -v_1 D_{n,\epsilon,x}(\bar{x}) - v_2 D_{n,\epsilon,y}(\bar{x}) - v_3 D_{n,\epsilon,z}(\bar{x}) \\
&= -v_1 \delta'_{n,\epsilon}(x) \delta_{n,\epsilon}(y) \delta_{n,\epsilon}(z) - v_2 \delta_{n,\epsilon}(x) \delta'_{n,\epsilon}(y) \delta_{n,\epsilon}(z) - v_3 \delta_{n,\epsilon}(x) \delta_{n,\epsilon}(y) \delta'_{n,\epsilon}(z)
\end{aligned}$$

Then;

$$\lim_{\epsilon \rightarrow 0} D'_{n,\epsilon,t} = D'_t, (**)$$

in the sense of distributions.

Finally, we have that;

$$D_{n,\epsilon}(x, y, z) = \left(\frac{2^{n-1} [\frac{n}{2}]!^2}{n!} \right)^3 \gamma_{n,\epsilon}(x) \gamma_{n,\epsilon}(y) \gamma_{n,\epsilon}(z), \text{ for } n \text{ even}$$

and;

$$D_{n,\epsilon}(x, y, z) = \left(\frac{\pi n!}{[(\frac{n-1}{2})!]^2 2^{2n+1}} \right)^3 \gamma_{n,\epsilon}(x) \gamma_{n,\epsilon}(y) \gamma_{n,\epsilon}(z), \text{ for } n \text{ odd}$$

We have that;

$$\begin{aligned}
D'_{n,\epsilon,t} &= \frac{2^{n-1} \pi [\frac{n}{2}]!^2}{2\epsilon(n-1)!} \left(\frac{2^{n-1} [\frac{n}{2}]!^2}{n!} \right)^2 [v_1 \gamma_{n-1,\epsilon}(x) \gamma_{n,\epsilon}(y) \gamma_{n,\epsilon}(z) \sin(\frac{\pi x}{2\epsilon}) \\
&+ v_2 \gamma_{n,\epsilon}(x) \gamma_{n-1,\epsilon}(y) \gamma_{n,\epsilon}(z) \sin(\frac{\pi y}{2\epsilon}) + v_3 \gamma_{n,\epsilon}(x) \gamma_{n,\epsilon}(y) \gamma_{n-1,\epsilon}(z) \sin(\frac{\pi z}{2\epsilon})]
\end{aligned}$$

for n even.

$$\begin{aligned}
D'_{n,\epsilon,t} &= \frac{\pi^2 n! n}{2\epsilon [(\frac{n-1}{2})!]^2 2^{2n+1}} \left(\frac{\pi n!}{[(\frac{n-1}{2})!]^2 2^{2n+1}} \right)^2 [v_1 \gamma_{n-1,\epsilon}(x) \gamma_{n,\epsilon}(y) \gamma_{n,\epsilon}(z) \sin(\frac{\pi x}{2\epsilon}) \\
&+ v_2 \gamma_{n,\epsilon}(x) \gamma_{n-1,\epsilon}(y) \gamma_{n,\epsilon}(z) \sin(\frac{\pi y}{2\epsilon}) + v_3 \gamma_{n,\epsilon}(x) \gamma_{n,\epsilon}(y) \gamma_{n-1,\epsilon}(z) \sin(\frac{\pi z}{2\epsilon})]
\end{aligned}$$

for n odd.

In particular, $D'_{n,\epsilon,t} \in C_c^{n-2}(\mathcal{R}^3)$.

Proof. For the first claim, (i) follows from the fact that $\delta_{n,\epsilon} \in C^{n-1}(\mathcal{R})$ and the fact that if $i + j + k = n - 1$;

$$\frac{\partial^{i+j+k} D_{n,\epsilon}}{\partial x^i \partial y^j \partial z^k}(x, y, z) = \delta_{n,\epsilon}^{(i)}(x) \delta_{n,\epsilon}^{(j)}(y) \delta_{n,\epsilon}^{(k)}(z)$$

(ii) is trivial from the corresponding property of the $\delta_{n,\epsilon}$. (iii) follows from Fubini's theorem;

$$\int_{[-\epsilon,\epsilon]} \int_{[-\epsilon,\epsilon]} \int_{[-\epsilon,\epsilon]} D_{n,\epsilon}(x, y, z) dx dy dz = \int_{[-\epsilon,\epsilon]} \delta_{n,\epsilon}(x) dx \int_{[-\epsilon,\epsilon]} \delta_{n,\epsilon}(y) dy \int_{[-\epsilon,\epsilon]} \delta_{n,\epsilon}(z) dz$$

and the corresponding property (iii) of $\delta_{n,\epsilon}$. (iv) is again trivial from the the corresponding property of $\delta_{n,\epsilon}$. The claim (*) is a consequence of properties (ii), (iii), (iv), see the proof above for the 1-dimensional case. For the claim (**), we have that, for $f \in C_c^\infty(\mathcal{R}^3)$, using integration by parts and Tonelli's theorem;

$$\begin{aligned} D'_{n,\epsilon,t}(f) &= \int_{[-\epsilon,\epsilon]} \int_{[-\epsilon,\epsilon]} \int_{[-\epsilon,\epsilon]} [-v_1 \delta'_{n,\epsilon}(x) \delta_{n,\epsilon}(y) \delta_{n,\epsilon}(z) - v_2 \delta_{n,\epsilon}(x) \delta'_{n,\epsilon}(y) \delta_{n,\epsilon}(z) \\ &\quad - v_3 \delta_{n,\epsilon}(x) \delta_{n,\epsilon}(y) \delta'_{n,\epsilon}(z)] f dx dy dz \\ &= \int_{[-\epsilon,\epsilon]} \int_{[-\epsilon,\epsilon]} \int_{[-\epsilon,\epsilon]} [v_1 \delta_{n,\epsilon}(x) \delta_{n,\epsilon}(y) \delta_{n,\epsilon}(z) f_x + v_2 \delta_{n,\epsilon}(x) \delta_{n,\epsilon}(y) \delta_{n,\epsilon}(z) f_y \\ &\quad + v_3 \delta_{n,\epsilon}(x) \delta_{n,\epsilon}(y) \delta_{n,\epsilon}(z) f_z] dx dy dz \end{aligned}$$

so that, by the claim (*);

$$\begin{aligned} \lim_{\epsilon \rightarrow 0} D'_{n,\epsilon,t}(f) &= v_1 f_x(0) + v_2 f_y(0) + v_3 f_z(0) \\ &= (-v_1 D_x - v_2 D_y - v_3 D_z)(f) \\ &= D'_t(f) \end{aligned}$$

The first computational claim follows from Lemma 0.3 and the definition of $D_{n,\epsilon}$. For the second computational claim;

$$D'_{n,\epsilon,t}(\bar{x}) = -v_1 \delta'_{n,\epsilon}(x) \delta_{n,\epsilon}(y) \delta_{n,\epsilon}(z) - v_2 \delta_{n,\epsilon}(x) \delta'_{n,\epsilon}(y) \delta_{n,\epsilon}(z) - v_3 \delta_{n,\epsilon}(x) \delta_{n,\epsilon}(y) \delta'_{n,\epsilon}(z)$$

so that, for n even;

$$\begin{aligned} D'_{n,\epsilon,t}(\bar{x}) &= -v_1 \left[-\frac{2^{n-1} \pi \left[\frac{n}{2}\right]^2}{2\epsilon(n-1)!} \gamma_{n-1,\epsilon}(x) \sin\left(\frac{\pi x}{2\epsilon}\right) \right] \left[\left(\frac{2^{n-1} \left[\frac{n}{2}\right]^2}{n!}\right) \gamma_{n,\epsilon}(y) \right] \left[\left(\frac{2^{n-1} \left[\frac{n}{2}\right]^2}{n!}\right) \gamma_{n,\epsilon}(z) \right] \\ &\quad - v_2 \left[\left(\frac{2^{n-1} \left[\frac{n}{2}\right]^2}{n!}\right) \gamma_{n,\epsilon}(x) \right] \left[-\frac{2^{n-1} \pi \left[\frac{n}{2}\right]^2}{2\epsilon(n-1)!} \gamma_{n-1,\epsilon}(y) \sin\left(\frac{\pi y}{2\epsilon}\right) \right] \left[\left(\frac{2^{n-1} \left[\frac{n}{2}\right]^2}{n!}\right) \gamma_{n,\epsilon}(z) \right] \\ &\quad - v_3 \left[\left(\frac{2^{n-1} \left[\frac{n}{2}\right]^2}{n!}\right) \gamma_{n,\epsilon}(x) \right] \left[\left(\frac{2^{n-1} \left[\frac{n}{2}\right]^2}{n!}\right) \gamma_{n,\epsilon}(y) \right] \left[-\frac{2^{n-1} \pi \left[\frac{n}{2}\right]^2}{2\epsilon(n-1)!} \gamma_{n-1,\epsilon}(z) \sin\left(\frac{\pi z}{2\epsilon}\right) \right] \end{aligned}$$

$$\begin{aligned}
&= v_1 \frac{2^{n-1} \pi [\frac{n}{2}]!^2}{2\epsilon(n-1)!} \left(\frac{2^{n-1} [\frac{n}{2}]!^2}{n!} \right)^2 \gamma_{n-1,\epsilon}(x) \gamma_{n,\epsilon}(y) \gamma_{n,\epsilon}(z) \sin\left(\frac{\pi x}{2\epsilon}\right) \\
&+ v_2 \frac{2^{n-1} \pi [\frac{n}{2}]!^2}{2\epsilon(n-1)!} \left(\frac{2^{n-1} [\frac{n}{2}]!^2}{n!} \right)^2 \gamma_{n,\epsilon}(x) \gamma_{n-1,\epsilon}(y) \gamma_{n,\epsilon}(z) \sin\left(\frac{\pi y}{2\epsilon}\right) \\
&+ v_3 \frac{2^{n-1} \pi [\frac{n}{2}]!^2}{2\epsilon(n-1)!} \left(\frac{2^{n-1} [\frac{n}{2}]!^2}{n!} \right)^2 \gamma_{n,\epsilon}(x) \gamma_{n,\epsilon}(y) \gamma_{n-1,\epsilon}(z) \sin\left(\frac{\pi z}{2\epsilon}\right)
\end{aligned}$$

and, for n odd;

$$\begin{aligned}
D'_{n,\epsilon,t}(\bar{x}) &= -v_1 \left[-\frac{\pi^2 n! n}{2\epsilon [(\frac{n-1}{2})!]^2 2^{n+1}} \gamma_{n-1,\epsilon}(x) \sin\left(\frac{\pi x}{2\epsilon}\right) \right] \left[\left(\frac{\pi n!}{[(\frac{n-1}{2})!]^2 2^{n+1}} \right) \gamma_{n,\epsilon}(y) \right] \left[\left(\frac{\pi n!}{[(\frac{n-1}{2})!]^2 2^{n+1}} \right) \gamma_{n,\epsilon}(z) \right] \\
&- v_2 \left[\left(\frac{\pi n!}{[(\frac{n-1}{2})!]^2 2^{n+1}} \right) \gamma_{n,\epsilon}(x) \right] \left[-\frac{\pi^2 n! n}{2\epsilon [(\frac{n-1}{2})!]^2 2^{n+1}} \gamma_{n-1,\epsilon}(y) \sin\left(\frac{\pi y}{2\epsilon}\right) \right] \left[\left(\frac{\pi n!}{[(\frac{n-1}{2})!]^2 2^{n+1}} \right) \gamma_{n,\epsilon}(z) \right] \\
&- v_3 \left[\left(\frac{\pi n!}{[(\frac{n-1}{2})!]^2 2^{n+1}} \right) \gamma_{n,\epsilon}(x) \right] \left[\left(\frac{\pi n!}{[(\frac{n-1}{2})!]^2 2^{n+1}} \right) \gamma_{n,\epsilon}(y) \right] \left[-\frac{\pi^2 n! n}{2\epsilon [(\frac{n-1}{2})!]^2 2^{n+1}} \gamma_{n-1,\epsilon}(z) \sin\left(\frac{\pi z}{2\epsilon}\right) \right] \\
&= v_1 \frac{\pi^2 n! n}{2\epsilon [(\frac{n-1}{2})!]^2 2^{n+1}} \left(\frac{\pi n!}{[(\frac{n-1}{2})!]^2 2^{n+1}} \right)^2 \gamma_{n-1,\epsilon}(x) \gamma_{n,\epsilon}(y) \gamma_{n,\epsilon}(z) \sin\left(\frac{\pi x}{2\epsilon}\right) \\
&+ v_2 \frac{\pi^2 n! n}{2\epsilon [(\frac{n-1}{2})!]^2 2^{n+1}} \left(\frac{\pi n!}{[(\frac{n-1}{2})!]^2 2^{n+1}} \right)^2 \gamma_{n,\epsilon}(x) \gamma_{n-1,\epsilon}(y) \gamma_{n,\epsilon}(z) \sin\left(\frac{\pi y}{2\epsilon}\right) \\
&+ v_3 \frac{\pi^2 n! n}{2\epsilon [(\frac{n-1}{2})!]^2 2^{n+1}} \left(\frac{\pi n!}{[(\frac{n-1}{2})!]^2 2^{n+1}} \right)^2 \gamma_{n,\epsilon}(x) \gamma_{n,\epsilon}(y) \gamma_{n-1,\epsilon}(z) \sin\left(\frac{\pi z}{2\epsilon}\right)
\end{aligned}$$

The final claim follows as above, by repeated application of the product rule, using the fact that $\gamma_{n-1,\epsilon} \in C_c^{n-2}(\mathcal{R})$ and $\gamma_{n,\epsilon} \in C_c^{n-1}(\mathcal{R})$ \square

Definition 0.6. For $n \geq 4$, we let $\rho_{n,\epsilon}$ be the unique charge distribution on $\mathcal{R}^3 \times \mathcal{R}_{\geq 0}$ defined by the initial conditions $\{qD_{n,\epsilon}, qD'_{n,\epsilon,t}\}$, satisfying the wave equation with velocity c , $\square^2(\rho_{n,\epsilon}) = 0$, on $\mathcal{R}^3 \times \mathcal{R}_{\geq 0}$, where q is the total charge.

Lemma 0.7. For $n \geq 6$, we have that;

$$\rho_{n,\epsilon}(\bar{x}, t) = \frac{1}{(2\pi)^3} \int_{\mathcal{R}^3} (A(\bar{k})e^{ikct} + B(\bar{k})e^{-ikct}) e^{i\bar{k} \cdot \bar{x}} d\bar{k}$$

where;

$$A(\bar{k}) = \frac{\mathcal{F}(qD_{n,\epsilon})}{2} + \frac{\mathcal{F}(qD'_{n,\epsilon,t})}{2ikc}$$

$$B(\bar{k}) = \frac{\mathcal{F}(qD_{n,\epsilon})}{2} - \frac{\mathcal{F}(qD'_{n,\epsilon,t})}{2ikc}$$

and \mathcal{F} denotes the usual 3 dimensional Fourier transform;

$$\mathcal{F}(g) = \int_{\mathcal{R}^3} g(\bar{x}) e^{-i\bar{k} \cdot \bar{x}} d\bar{x}$$

for $g \in L^1(\mathcal{R}^3)$.

Proof. We have that the existence of $\rho_{n,\epsilon}$ is guaranteed by Kirchoff's formula, for $n \geq 4$, as the initial conditions $D_{n,\epsilon} \in C_c^3(\mathcal{R}^3)$ and $D'_{n,\epsilon,t} \in C_c^2(\mathcal{R}^4)$ for $n \geq 4$, see [2]. Using the fact that the initial conditions $\{qD_{n,\epsilon}, qD'_{n,\epsilon,t}\}$ have compact support, $\rho_{n,\epsilon,t}$ has compact support as a process, in particular $\rho_{n,\epsilon,t} \in L^1(\mathcal{R}^3)$, for $t \geq 0$. Using Kirchoff's formula, see [2], we have that, for $t > 0$;

$$\rho_{n,\epsilon}(\bar{x}, t) = \frac{1}{4\pi c^2 t^2} \int_{\delta B(\bar{x}, ct)} [ctqD'_{n,\epsilon,t}(\bar{y}) + qD_{n,\epsilon}(\bar{y}) + DqD_{n,\epsilon}(\bar{y}) \cdot (\bar{y} - \bar{x})] dS(\bar{y})$$

Then, we have that, using the substitution $\bar{z} = \bar{y} - (h, 0, 0)$, $d\bar{z} = d\bar{y}$ and interchanging limits;

$$\begin{aligned} \frac{\partial \rho_{n,\epsilon}}{\partial x} &= \lim_{h \rightarrow 0} \frac{1}{h} \left(\frac{1}{4\pi c^2 t^2} \int_{\delta B(\bar{x} + (h, 0, 0), ct)} [ctqD'_{n,\epsilon,t}(\bar{y}) + qD_{n,\epsilon}(\bar{y}) + DqD_{n,\epsilon}(\bar{y}) \cdot (\bar{y} - (\bar{x} + (h, 0, 0)))] dS(\bar{y}) - \frac{1}{4\pi c^2 t^2} \int_{\delta B(\bar{x}, ct)} [ctqD'_{n,\epsilon,t}(\bar{y}) + qD_{n,\epsilon}(\bar{y}) + DqD_{n,\epsilon}(\bar{y}) \cdot (\bar{y} - \bar{x})] dS(\bar{y}) \right) \\ &= \lim_{h \rightarrow 0} \frac{1}{h} \left(\frac{1}{4\pi c^2 t^2} \int_{\delta B(\bar{x}, ct)} [ctqD'_{n,\epsilon,t}(\bar{y} + (h, 0, 0)) + qD_{n,\epsilon}(\bar{y} + (h, 0, 0)) + DqD_{n,\epsilon}(\bar{y} + (h, 0, 0)) \cdot (\bar{y} + (h, 0, 0) - (\bar{x} + (h, 0, 0)))] dS(\bar{y}) - \frac{1}{4\pi c^2 t^2} \int_{\delta B(\bar{x}, ct)} [ctqD'_{n,\epsilon,t}(\bar{y}) + qD_{n,\epsilon}(\bar{y}) + DqD_{n,\epsilon}(\bar{y}) \cdot (\bar{y} - \bar{x})] dS(\bar{y}) \right) \\ &= \frac{1}{4\pi c^2 t^2} \int_{\delta B(\bar{x}, ct)} [\lim_{h \rightarrow 0} \frac{1}{h} (ctqD'_{n,\epsilon,t}(\bar{y} + (h, 0, 0)) - ctqD'_{n,\epsilon,t}(\bar{y})) + \lim_{h \rightarrow 0} \frac{1}{h} (qD_{n,\epsilon}(\bar{y} + (h, 0, 0)) - qD_{n,\epsilon}(\bar{y})) + \lim_{h \rightarrow 0} \frac{1}{h} (DqD_{n,\epsilon}(\bar{y} + (h, 0, 0)) - DqD_{n,\epsilon}(\bar{y})) \cdot (\bar{y} - \bar{x})] dS(\bar{y}) \\ &= \frac{1}{4\pi c^2 t^2} \int_{\delta B(\bar{x}, ct)} [ctqD'_{n,\epsilon,t,x}(\bar{y}) + qD_{n,\epsilon,x}(\bar{y}) + DqD_{n,\epsilon,x}(\bar{y}) \cdot (\bar{y} - \bar{x})] dS(\bar{y}) \end{aligned}$$

so that, as the initial conditions $\{qD_{n,\epsilon}, qD'_{n,\epsilon,t}\}$ are in $C_c^{n-1}(\mathcal{R}^3)$ and $C_c^{n-2}(\mathcal{R}^3)$ respectively, it follows that for $t > 0$, $\rho_{n,\epsilon,t} \in C_c^{n-3}(\mathcal{R}^3)$. In particular, for $n \geq 6$, $\rho_{n,\epsilon,t} \in C_c^3(\mathcal{R}^3)$, (*). We have that;

$$\square^2(\rho_{n,\epsilon,t}) = \nabla^2(\rho_{n,\epsilon,t}) - \frac{1}{c^2} \frac{\partial^2 \rho_{n,\epsilon,t}}{\partial t^2} = 0$$

so we can apply \mathcal{F} to both sides and obtain, and differentiating under the integral sign;

$$\mathcal{F}(\nabla^2(\rho_{n,\epsilon,t}))(\bar{k}, t) - \frac{1}{c^2} \frac{\partial^2 \mathcal{F}(\rho_{n,\epsilon,t})(\bar{k}, t)}{\partial t^2} = 0$$

As $\rho_{n,\epsilon,t} \in C_c^2(\mathcal{R}^3)$, by (*), we have, using integration by parts, that;

$$\mathcal{F}(\nabla^2(\rho_{n,\epsilon,t}))(\bar{k}, t) = -k^2 \mathcal{F}(\rho_{n,\epsilon,t})(\bar{k}, t)$$

so that;

$$-k^2 \mathcal{F}(\rho_{n,\epsilon,t})(\bar{k}, t) - \frac{1}{c^2} \frac{\partial^2 \mathcal{F}(\rho_{n,\epsilon,t})(\bar{k}, t)}{\partial t^2} = 0$$

and we can use Peano's theorem to solve the ODE in time, to obtain;

$$\mathcal{F}(\rho_{n,\epsilon,t})(\bar{k}, t) = A(\bar{k})e^{ikct} + B(\bar{k})e^{-ikct}$$

where, at $t = 0$;

$$A(\bar{k}) + B(\bar{k}) = \mathcal{F}(qD_{n,\epsilon})$$

and, taking the time derivative at $t = 0$;

$$ikcA(\bar{k}) - ikcB(\bar{k}) = \mathcal{F}(qD'_{n,\epsilon,t})$$

Now we can solve the simultaneous equations to get the expressions for $\{A(\bar{k}), B(\bar{k})\}$, We have that $\mathcal{F}(qD_{n,\epsilon}) \in C^\infty(\mathcal{R}^3)$ as $qD_{n,\epsilon}$ has compact support and similarly for $\mathcal{F}(qD'_{n,\epsilon,t})$. Moreover, we have that, using integration by parts, for $k_1 \neq 0$;

$$\mathcal{F}(qD_{n,\epsilon,x}) = ik_1 \mathcal{F}(qD_{n,\epsilon})$$

so that, for $k \neq 0$, using the fact that for $n \geq 6$, $qD_{n,\epsilon} \in C_c^5(\mathcal{R}^3)$, $qD'_{n,\epsilon,t} \in C^4(\mathcal{R}^3)$;

$$\mathcal{F}((\nabla^2)^2 qD_{n,\epsilon}) = k^4 \mathcal{F}(qD_{n,\epsilon})$$

$$|\mathcal{F}(qD_{n,\epsilon})| \leq \frac{|\mathcal{F}((\nabla^2)^2 qD_{n,\epsilon})|}{k^4}$$

$$\leq \frac{C}{k^4}$$

$$\mathcal{F}((\nabla^2)^2 qD'_{n,\epsilon,t}) = k^4 \mathcal{F}(qD'_{n,\epsilon,t})$$

$$|\mathcal{F}(qD'_{n,\epsilon,t})| \leq \frac{|\mathcal{F}((\nabla^2)^2 qD'_{n,\epsilon,t})|}{k^4}$$

$$\leq \frac{D}{k^4} \quad (\dagger)$$

Converting to polar coordinates, this proves that $\{\mathcal{F}(qD_{n,\epsilon}), \mathcal{F}(qD'_{n,\epsilon,t})\} \subset L^1(\mathcal{R}^3)$. Now we can use the fact that, for $k \neq 0$;

$$|A(\bar{k})| \leq \frac{|\mathcal{F}(qD_{n,\epsilon})|}{2} + \frac{|\mathcal{F}(qD'_{n,\epsilon,t})|}{2kc}$$

$$|B(\bar{k})| \leq \frac{|\mathcal{F}(qD_{n,\epsilon})|}{2} + \frac{|\mathcal{F}(qD'_{n,\epsilon,t})|}{2kc}$$

the fact that $\{\mathcal{F}(qD_{n,\epsilon}), \mathcal{F}(qD'_{n,\epsilon,t})\} \subset C^\infty(\mathcal{R}^3)$, and (\dagger) , using polar coordinates again, to prove that $\{A(\bar{k}), B(\bar{k})\} \subset L^1(\mathcal{R}^3)$. Finally, for $t > 0$;

$$|\mathcal{F}(\rho_{n,\epsilon})| = |A(\bar{k})e^{ikct} + B(\bar{k})e^{-ikct}|$$

$$\leq |A(\bar{k})| + |B(\bar{k})|$$

so that $\mathcal{F}(\rho_{n,\epsilon}) \in L^1(\mathcal{R}^3)$ for $t > 0$. It follows that we can apply the inversion theorem in the last step.

□

Lemma 0.8. *Let \mathcal{F} be the 1-dimensional Fourier transform, then, for n even, $k \in \mathcal{R}$, $k \neq \frac{\pi}{\epsilon}(\frac{n}{2} - j)$, $0 \leq j \leq n$;*

$$\mathcal{F}(\gamma_{n,\epsilon})(k) = -\frac{1}{2^{n-1}} \sum_{j=0}^n C_j^n \frac{1}{\pi(\frac{n}{2}-j)-\epsilon k} [(-1)^{\frac{n}{2}-j} \sin(\epsilon k)]$$

and, for $0 \leq j_0 \leq n$;

$$\mathcal{F}(\gamma_{n,\epsilon})(\frac{\pi}{\epsilon}(\frac{n}{2} - j_0)) = \frac{1}{2^{n-1}} C_{j_0}^n$$

In particular;

$$\lim_{\epsilon \rightarrow 0} \mathcal{F}(\gamma_{n,\epsilon}) = \frac{n!}{2^{n-1}(\frac{n}{2}!)^2}$$

uniformly on compact subsets of \mathcal{R} . For n even;

$$\mathcal{F}(\delta_{n,\epsilon}) = \frac{2^{n-1}[\frac{n}{2}]!}{n!} \mathcal{F}(\gamma_{n,\epsilon})$$

in particular;

$$\lim_{\epsilon \rightarrow 0} \mathcal{F}(\delta_{n,\epsilon}) = 1$$

uniformly on compact subsets of \mathcal{R} . For n even;

$$\mathcal{F}(\delta'_{n,\epsilon,t})(k) = ivk \mathcal{F}(\delta_{n,\epsilon})$$

in particular;

$$\lim_{\epsilon \rightarrow 0} \mathcal{F}(\delta'_{n,\epsilon,t})(k) = ivk$$

uniformly on compact subsets of \mathcal{R} .

Proof. We have that, for $k \in \mathcal{R}$, n even, $k \neq \frac{\pi}{\epsilon}(\frac{n}{2} - j)$, $0 \leq j \leq n$, by the definition of $\gamma_{n,\epsilon}$ and \mathcal{F} ;

$$\begin{aligned} \mathcal{F}(\gamma_{n,\epsilon}) &= \frac{1}{\epsilon} \int_{-\epsilon}^{\epsilon} \cos^n\left(\frac{\pi x}{2\epsilon}\right) e^{-ixk} dx \\ &= \frac{1}{\epsilon} \int_{-\epsilon}^{\epsilon} \left(e^{\frac{i\pi x}{2\epsilon}} + e^{-\frac{i\pi x}{2\epsilon}} \right)^n e^{-ixk} dx \\ &= \frac{1}{2^n \epsilon} \int_{-\epsilon}^{\epsilon} \sum_{j=0}^n C_j^n e^{\frac{i\pi x(n-j)}{2\epsilon}} e^{-\frac{i\pi x j}{2\epsilon}} e^{-ixk} dx \\ &= \frac{1}{2^n \epsilon} \sum_{j=0}^n C_j^n \int_{-\epsilon}^{\epsilon} e^{\frac{i\pi x(n-2j-2\epsilon k)}{2\epsilon}} dx \\ &= \frac{1}{2^n \epsilon} \sum_{j=0}^n C_j^n \left[e^{\frac{i\pi x(n-2j-2\epsilon k)}{2\epsilon}} \right]_{-\epsilon}^{\epsilon} \\ &= \frac{1}{2^n \epsilon} \sum_{j=0}^n C_j^n \frac{1}{\frac{i\pi}{2\epsilon}(n-2j-\frac{2\epsilon k}{\pi})} \left[e^{\frac{i\pi x(n-2j-2\epsilon k)}{2\epsilon}} \right]_{-\epsilon}^{\epsilon} \\ &= \frac{1}{2^n \epsilon} \sum_{j=0}^n C_j^n \frac{1}{\frac{i\pi}{2\epsilon}(n-2j)-ik} \left[i^{n-2j-\frac{2\epsilon k}{\pi}} - (-i)^{n-2j-\frac{2\epsilon k}{\pi}} \right] \\ &= \frac{1}{2^n \epsilon} \sum_{j=0}^n C_j^n \frac{1}{\frac{i\pi}{2\epsilon}(n-2j)-ik} \left[i^{n-2j} e^{-i\epsilon k} - (-i)^{n-2j} e^{i\epsilon k} \right] \\ &= \frac{1}{2^n \epsilon} \sum_{j=0}^n C_j^n \frac{1}{\frac{i\pi}{2\epsilon}(n-2j)-ik} \left[i^{n-2j} e^{-i\epsilon k} - i^{n-2j} e^{i\epsilon k} \right], \quad (n-2j) \text{ even} \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{2^n \epsilon} \sum_{j=0}^n C_j^n \frac{1}{\frac{i\pi}{2\epsilon}(n-2j)-ik} [-2i^{n-2j+1} \sin(\epsilon k)] \\
&= \frac{1}{2^n \epsilon} \sum_{j=0}^n C_j^n \frac{1}{\frac{\pi}{2\epsilon}(n-2j)-k} [-2i^{n-2j} \sin(\epsilon k)] \\
&= -\frac{1}{2^{n-1} \epsilon} \sum_{j=0}^n C_j^n \frac{1}{\frac{\pi}{2\epsilon}(n-2j)-k} [(-1)^{\frac{n}{2}-j} \sin(\epsilon k)] \\
&= -\frac{1}{2^{n-1}} \sum_{j=0}^n C_j^n \frac{1}{\pi(\frac{n}{2}-j)-\epsilon k} [(-1)^{\frac{n}{2}-j} \sin(\epsilon k)]
\end{aligned}$$

We know that $\mathcal{F}(\gamma_{n,\epsilon})$ is continuous, so that, for fixed ϵ , $0 \leq j_0 \leq n$, using L'Hopital's rule;

$$\begin{aligned}
\mathcal{F}(\gamma_{n,\epsilon})\left(\frac{\pi}{\epsilon}\left(\frac{n}{2}-j_0\right)\right) &= \lim_{k \rightarrow \frac{\pi}{\epsilon}\left(\frac{n}{2}-j_0\right)} \mathcal{F}(\gamma_{n,\epsilon})(k) \\
&= -\frac{1}{2^{n-1}} \left(\sum_{j=0, j \neq j_0}^n C_j^n \frac{1}{\pi(j_0-j)} [(-1)^{\frac{n}{2}-j} \sin(\pi(\frac{n}{2}-j_0))] \right. \\
&\quad \left. - \frac{1}{2^{n-1}} C_{j_0}^n \frac{1}{-\epsilon} [(-1)^{\frac{n}{2}-j_0} \epsilon \cos(\pi(\frac{n}{2}-j_0))] \right) \\
&= \frac{1}{2^{n-1}} C_{j_0}^n [(-1)^{\frac{n}{2}-j_0} (-1)^{\frac{n}{2}-j_0}] \\
&= \frac{1}{2^{n-1}} C_{j_0}^n
\end{aligned}$$

In particular;

$$\begin{aligned}
\mathcal{F}(\gamma_{n,\epsilon})(0) &= \frac{1}{2^{n-1}} C_{\frac{n}{2}}^n \\
&= \frac{n!}{2^{n-1}(\frac{n}{2}!)^2}, (*)
\end{aligned}$$

independently of ϵ .

It follows that, for $k \neq 0$, using L'Hopital's rule;

$$\begin{aligned}
\lim_{\epsilon \rightarrow 0} \mathcal{F}(\gamma_{n,\epsilon})(k) &= \lim_{\epsilon \rightarrow 0} \frac{1}{2^{n-1}} C_{\frac{n}{2}}^n \frac{1}{\epsilon k} \sin(\epsilon k) \\
&= \frac{1}{2^{n-1}} \frac{n!}{(\frac{n}{2}!)^2} \lim_{\epsilon \rightarrow 0} \frac{1}{k} \cos(\epsilon k) k \\
&= \frac{n!}{2^{n-1}(\frac{n}{2}!)^2} \lim_{\epsilon \rightarrow 0} \cos(\epsilon k) \\
&= \frac{n!}{2^{n-1}(\frac{n}{2}!)^2}
\end{aligned}$$

and clearly;

$$\lim_{\epsilon \rightarrow 0} \mathcal{F}(\gamma_{n,\epsilon})(0) = \frac{n!}{2^{n-1}(\frac{n!}{2})^2}$$

by (*).

It follows that;

$$\lim_{\epsilon \rightarrow 0} \mathcal{F}(\gamma_{n,\epsilon}) = \frac{n!}{2^{n-1}(\frac{n!}{2})^2} \text{ (pointwise convergence)}$$

To obtain uniform convergence on compact subsets, note that;

$$\frac{1}{\epsilon k} \sin(\epsilon k)$$

converges uniformly to 1 on compact subsets as, for $|k| \leq K$;

$$\begin{aligned} \left| \frac{1}{\epsilon k} \sin(\epsilon k) - 1 \right| &= \frac{|\sin(\epsilon k) - \epsilon k|}{|\epsilon k|} \\ &= \frac{|\epsilon k + O((\epsilon k)^3) - \epsilon k|}{|\epsilon k|} \end{aligned}$$

$$\leq C|\epsilon k|^2$$

$$\leq C\epsilon^2 K^2$$

and $\sin(\epsilon k)$

converges uniformly to 0 on compact subsets as, for $|k| \leq K$;

$$|\sin(\epsilon k)| \leq |\epsilon k|$$

$$\leq \epsilon K$$

By Lemma 0.3, we have that, for n even;

$$\mathcal{F}(\delta_{n,\epsilon}) = \frac{2^{n-1}[\frac{n!}{2}]^2}{n!} \mathcal{F}(\gamma_{n,\epsilon})$$

so that, by the previous claim in this lemma;

$$\lim_{\epsilon \rightarrow 0} \mathcal{F}(\delta_{n,\epsilon}) = \frac{2^{n-1}[\frac{n!}{2}]^2}{n!} \frac{n!}{2^{n-1}(\frac{n!}{2})^2} = 1$$

on compact subsets of \mathcal{R} . We have that, using integration by parts and the definition of $\delta'_{n,\epsilon,t}$;

$$\begin{aligned}
\mathcal{F}(\delta'_{n,\epsilon,t})(k) &= \mathcal{F}(-v\delta'_{n,\epsilon}) \\
&= -v(-ik)\mathcal{F}(\delta_{n,\epsilon}) \\
&= ivk\mathcal{F}(\delta_{n,\epsilon})
\end{aligned}$$

so that, by the previous claim;

$$\begin{aligned}
\lim_{\epsilon \rightarrow 0} \mathcal{F}(\delta_{n,\epsilon}) &= ivk \lim_{\epsilon \rightarrow 0} \mathcal{F}(\delta_{n,\epsilon}) \\
&= ivk
\end{aligned}$$

uniformly on compact subsets of \mathcal{R} .

□

Lemma 0.9. *Let \mathcal{F} be the three dimensional Fourier transform, then;*

$$\mathcal{F}(D_{n,\epsilon}) = \mathcal{F}_1(\delta_{n,\epsilon})(k_1)\mathcal{F}_1(\delta_{n,\epsilon})(k_2)\mathcal{F}_1(\delta_{n,\epsilon})(k_3)$$

where \mathcal{F}_1 is the 1-dimensional Fourier transform.

In particular;

$$\lim_{\epsilon \rightarrow 0} \mathcal{F}(D_{n,\epsilon}) = 1$$

uniformly on compact subsets of \mathcal{R}^3 .

$$\begin{aligned}
\mathcal{F}(D'_{n,\epsilon,t})(k_1, k_2, k_3) &= -v_1\mathcal{F}_1(\delta'_{n,\epsilon})(k_1)\mathcal{F}_1(\delta_{n,\epsilon})(k_2)\mathcal{F}_1(\delta_{n,\epsilon})(k_3) \\
&\quad -v_2\mathcal{F}_1(\delta_{n,\epsilon})(k_1)\mathcal{F}_1(\delta'_{n,\epsilon})(k_2)\mathcal{F}_1(\delta_{n,\epsilon})(k_3) -v_3\mathcal{F}_1(\delta_{n,\epsilon})(k_1)\mathcal{F}_1(\delta_{n,\epsilon})(k_2)\mathcal{F}_1(\delta'_{n,\epsilon})(k_3)
\end{aligned}$$

In particular;

$$\lim_{\epsilon \rightarrow 0} \mathcal{F}(D'_{n,\epsilon,t})(\bar{k}) = i\bar{v} \cdot \bar{k}$$

uniformly on compact subsets of \mathcal{R}^3 .

Proof. By Definition 0.5 and the theorem of Fubini;

$$\mathcal{F}(D_{n,\epsilon})(k_1, k_2, k_3) = \mathcal{F}_1(\delta_{n,\epsilon})(k_1)\mathcal{F}_1(\delta_{n,\epsilon})(k_2)\mathcal{F}_1(\delta_{n,\epsilon})(k_3)$$

where \mathcal{F}_1 is the 1-dimensional Fourier transform. In particular, as the projections of a compact subset are compact, and using the result of Lemma 0.8;

$$\begin{aligned} \lim_{\epsilon \rightarrow 0} \mathcal{F}(D_{n\epsilon})(\bar{k}) &= 1.1.1 \\ &= 1 \end{aligned}$$

uniformly on compact subsets of \mathcal{R}^3 .

By definition 0.5 again and Fubini's theorem;

$$\begin{aligned} \mathcal{F}(D'_{n,\epsilon,t})(k_1, k_2, k_3) &= -v_1 \mathcal{F}_1(\delta'_{n,\epsilon})(k_1) \mathcal{F}_1(\delta_{n,\epsilon})(k_2) \mathcal{F}_1(\delta_{n,\epsilon})(k_3) \\ &\quad -v_2 \mathcal{F}_1(\delta_{n,\epsilon})(k_1) \mathcal{F}_1(\delta'_{n,\epsilon})(k_2) \mathcal{F}_1(\delta_{n,\epsilon})(k_3) -v_3 \mathcal{F}_1(\delta_{n,\epsilon})(k_1) \mathcal{F}_1(\delta_{n,\epsilon})(k_2) \mathcal{F}_1(\delta'_{n,\epsilon})(k_3) \end{aligned}$$

so that, using the result of Lemma 0.8, integration by parts and the previous observation;

$$\begin{aligned} \lim_{\epsilon \rightarrow 0} \mathcal{F}(D'_{n,\epsilon,t})(k_1, k_2, k_3) &= -v_1(-ik_1).1.1 - v_2(-ik_2).1.1 - v_3(-ik_3).1.1 \\ &= i\bar{v} \cdot \bar{k} \end{aligned}$$

uniformly on compact subsets of \mathcal{R}^3 .

□

Lemma 0.10. *For the charge distribution $\rho_{n,\epsilon}$ of Lemma 0.7, we have that;*

$$\begin{aligned} \lim_{\epsilon \rightarrow 0} A_{n,\epsilon}(\bar{k}) &= \frac{q}{2} + \frac{q\bar{v} \cdot \bar{k}}{2kc} \\ \lim_{\epsilon \rightarrow 0} B_{n,\epsilon}(\bar{k}) &= \frac{q}{2} - \frac{q\bar{v} \cdot \bar{k}}{2kc} \end{aligned}$$

uniformly on compact subsets of $\mathcal{R}^3 \setminus \{0\}$.

Proof. We have that, by Lemma 0.7 and the results of Lemma 0.9;

$$\begin{aligned} \lim_{\epsilon \rightarrow 0} A_{n,\epsilon}(\bar{k}) &= \lim_{\epsilon \rightarrow 0} \left(\frac{\mathcal{F}(qD_{n,\epsilon})}{2} + \frac{\mathcal{F}(qD'_{n,\epsilon,t})}{2ikc} \right) \\ &= \frac{q}{2} + \frac{qi\bar{v} \cdot \bar{k}}{2ikc} \end{aligned}$$

$$\begin{aligned}
&= \frac{q}{2} + \frac{q\bar{v}\cdot\bar{k}}{2kc} \\
\lim_{\epsilon \rightarrow 0} B_{n,\epsilon}(\bar{k}) &= \lim_{\epsilon \rightarrow 0} \left(\frac{\mathcal{F}(qD_{n,\epsilon})}{2} - \frac{\mathcal{F}(qD'_{n,\epsilon,t})}{2ikc} \right) \\
&= \frac{q}{2} - \frac{q\bar{v}\cdot\bar{k}}{2ikc} \\
&= \frac{q}{2} - \frac{q\bar{v}\cdot\bar{k}}{2kc}
\end{aligned}$$

uniformly on compact subsets of $\mathcal{R}^3 \setminus \{0\}$.

□

Lemma 0.11. For $n \geq 6$, for $k \in \mathcal{R}_{>0}$, we define the intensity $I_{n,\epsilon}(\bar{x}, t, k)$ at k to be;

$$I_{n,\epsilon}(\bar{x}, t, k) = \frac{1}{(2\pi)^3} \int_{S_k} (A(\bar{k})e^{ikct} + B(\bar{k})e^{-ikct}) e^{i\bar{k}\cdot\bar{x}} dS(\bar{k})$$

so that, by Lemma 0.7;

$$\rho_{n,\epsilon}(\bar{x}, t) = \int_{\mathcal{R}} I(\bar{x}, t, k) dk$$

Then;

$$\begin{aligned}
\lim_{\epsilon \rightarrow 0} I_{n,\epsilon}(s\bar{v}, t, k) &= \frac{q}{(2\pi)^3} \left(\frac{4\pi k}{sv} \sin(sv k) \cos(ctk) - \frac{4\pi}{s^2vc} \sin(sv k) \sin(ctk) \right. \\
&\quad \left. + \frac{4\pi k}{cs} \cos(sv k) \sin(ctk) \right)
\end{aligned}$$

For the electron at time t moving a distance d , we obtain local maxima in the wave number when;

$$2kd = -D \tan(2kd)$$

$$\text{where } D = \frac{c-v}{c+v}$$

Proof. By Lemma 0.10 and the definition of $I_{n,\epsilon}$, we have that;

$$\begin{aligned}
\lim_{\epsilon \rightarrow 0} I_{n,\epsilon}(s\bar{v}, t, k) &= \frac{1}{(2\pi)^3} \int_{S_k} (A_{n,\epsilon}(\bar{k})e^{ikct} + B_{n,\epsilon}(\bar{k})e^{-ikct}) e^{i\bar{k}\cdot s\bar{v}} dS(\bar{k}) \\
&= \frac{q}{(2\pi)^3} e^{ikct} \int_{S_k} \left(\frac{1}{2} + \frac{\bar{v}\cdot\bar{k}}{2kc} \right) e^{is\bar{v}\cdot\bar{k}} dS(\bar{k}) + \frac{q}{(2\pi)^3} e^{-ikct} \int_{S_k} \left(\frac{1}{2} - \frac{\bar{v}\cdot\bar{k}}{2kc} \right) e^{is\bar{v}\cdot\bar{k}} dS(\bar{k}), \\
(*) &
\end{aligned}$$

Switching to polar coordinates, letting θ be the angle between \bar{v} and $\bar{k} \in S_k$, we have that;

$$\begin{aligned}
\int_{S_k} \frac{1}{2} e^{i\bar{v}\cdot\bar{k}} dS(\bar{k}) &= \frac{1}{2} \int_0^\pi \int_{-\pi}^\pi e^{isvk\cos(\theta)} k^2 \sin(\theta) d\theta d\phi \\
&= \pi k^2 \int_0^\pi e^{isvk\cos(\theta)} \sin(\theta) d\theta \\
&= \pi k^2 \int_{-1}^1 e^{isvku} du, \quad (u = \cos(\theta), du = -\sin(\theta) d\theta) \\
&= \pi k^2 \left[\frac{e^{isvku}}{isvk} \right]_{-1}^1 \\
&= \frac{\pi k^2}{isvk} (e^{isvk} - e^{-isvk}) \\
&= \frac{\pi k^2}{isvk} 2i \sin(svk) \\
&= \frac{2\pi k \sin(svk)}{sv}
\end{aligned}$$

and;

$$\begin{aligned}
\int_{S_k} \frac{\bar{v}\cdot\bar{k}}{2kc} e^{i\bar{v}\cdot\bar{k}} dS(\bar{k}) &= \frac{1}{2} \int_0^\pi \int_{-\pi}^\pi \frac{vk\cos(\theta)}{2kc} e^{isvk\cos(\theta)} k^2 \sin(\theta) d\theta d\phi \\
&= \frac{\pi vk^2}{c} \int_0^\pi e^{isvk\cos(\theta)} \cos(\theta) \sin(\theta) d\theta \\
&= \frac{\pi vk^2}{c} \int_{-1}^1 u e^{isvku} du, \quad (u = \cos(\theta), du = -\sin(\theta) d\theta) \\
&= \frac{\pi vk^2}{c} \left(\left[\frac{u e^{isvku}}{isvk} \right]_{-1}^1 - \int_{-1}^1 \frac{e^{isvku}}{isvk} du \right) \\
&= \frac{\pi vk^2}{c} \left(\frac{e^{isvk} + e^{-isvk}}{isvk} - \left[\frac{e^{isvku}}{(isvk)^2} \right]_{-1}^1 \right) \\
&= \frac{\pi vk^2}{cisvk} (2\cos(svk)) - \frac{\pi vk^2}{c(isvk)^2} 2i \sin(svk) \\
&= \frac{2\pi i \sin(svk)}{s^2vc} - \frac{2\pi ik \cos(svk)}{cs}
\end{aligned}$$

It follows from (*), that;

$$\begin{aligned}
\lim_{\epsilon \rightarrow 0} I_{n,\epsilon}(s\bar{v}, t, k) &= \frac{q}{(2\pi)^3} e^{ikct} \left(\frac{2\pi k \sin(svk)}{sv} + \frac{2\pi i \sin(svk)}{s^2vc} - \frac{2\pi ik \cos(svk)}{cs} \right) \\
&+ \frac{q}{(2\pi)^3} e^{-ikct} \left(\frac{2\pi k \sin(svk)}{sv} - \frac{2\pi i \sin(svk)}{s^2vc} + \frac{2\pi ik \cos(svk)}{cs} \right) \\
&= \frac{q}{(2\pi)^3} \left(\frac{2\pi k}{sv} \sin(svk) 2\cos(kct) + \frac{2\pi i^2}{s^2vc} \sin(svk) 2\sin(kct) - \frac{2\pi ik}{cs} \cos(svk) 2i \sin(kct) \right)
\end{aligned}$$

$$= \frac{q}{(2\pi)^3} \left(\frac{4\pi k}{sv} \sin(sv k) \cos(kct) - \frac{4\pi}{s^2 v c} \sin(sv k) \sin(kct) + \frac{4\pi k}{cs} \cos(sv k) \sin(kct) \right)$$

(†)

The electron wave propagates at speed c , so we are interested in the case when $|s\bar{v}| = ct$, so that $s = \frac{ct}{v}$. Making this substitution in (†), we obtain that;

$$\begin{aligned} \lim_{\epsilon \rightarrow 0} I_{n,\epsilon} \left(\frac{ct}{v} \bar{v}, t, k \right) &= \frac{q}{(2\pi)^3} \left(\frac{4\pi k}{ct} \sin(kct) \cos(kct) - \frac{4\pi v}{c^3 t^2} \sin^2(kct) + \frac{4\pi kv}{c^2 t} \cos(kct) \sin(kct) \right) \\ &= \frac{q}{(2\pi)^3} \left(\left[\frac{4\pi k}{ct} + \frac{4\pi kv}{c^2 t} \right] \sin(kct) \cos(kct) - \frac{4\pi v}{c^3 t^2} \sin^2(kct) \right) \\ &= \frac{q}{(2\pi)^3} \left(\left[\frac{2\pi k}{ct} + \frac{2\pi kv}{c^2 t} \right] \sin(2kct) - \frac{4\pi v}{c^3 t^2} \left(\frac{1 - \cos(2kct)}{2} \right) \right) \\ &= \frac{q}{(2\pi)^3} \left(\left[\frac{2\pi k}{ct} + \frac{2\pi kv}{c^2 t} \right] \sin(2kct) + \frac{2\pi v}{c^3 t^2} \cos(2kct) - \frac{2\pi v}{c^3 t^2} \right) \end{aligned}$$

We look for a local maximum in k , so that;

$$\begin{aligned} \frac{d}{dk} \lim_{\epsilon \rightarrow 0} I_{n,\epsilon} \left(\frac{ct}{v} \bar{v}, t, k \right) &= \frac{d}{dk} \left(\frac{q}{(2\pi)^3} \left(\left[\frac{2\pi k}{ct} + \frac{2\pi kv}{c^2 t} \right] \sin(2kct) + \frac{2\pi v}{c^3 t^2} \cos(2kct) \right. \right. \\ &\quad \left. \left. - \frac{2\pi v}{c^3 t^2} \right) \right) \\ &= \frac{q}{(2\pi)^3} \left(\left[\frac{2\pi}{ct} + \frac{2\pi v}{c^2 t} \right] \sin(2kct) + 2ct \left[\frac{2\pi k}{ct} + \frac{2\pi kv}{c^2 t} \right] \cos(2kct) - 2ct \frac{2\pi v}{c^3 t^2} \sin(2kct) \right) \\ &= \frac{q}{(2\pi)^3} \left(\left[\frac{2\pi}{ct} + \frac{2\pi v}{c^2 t} \right] \sin(2kct) + \left[4\pi k + \frac{4\pi kv}{c} \right] \cos(2kct) - \frac{4\pi v}{c^2 t} \sin(2kct) \right) \\ &= \frac{q}{(2\pi)^3} \left(\left[\frac{2\pi}{ct} - \frac{2\pi v}{c^2 t} \right] \sin(2kct) + \left[4\pi k + \frac{4\pi kv}{c} \right] \cos(2kct) \right) \\ &= 0 \end{aligned}$$

so that, rearranging;

$$k = -D \tan(2kct)$$

$$\text{where } D = \frac{\frac{2\pi}{ct} - \frac{2\pi v}{c^2 t}}{4\pi \left(1 + \frac{v}{c}\right)} = \frac{c-v}{2ct(c+v)}$$

Now use the fact that the distance at time t is $d = ct$.

□

Lemma 0.12. *For an electron moving at velocity $(v, 0, 0)$ in the laboratory frame S , the amplitudes in the spectrum of the stationary electron*

are shifted by the Doppler factor $\frac{k'}{k}$ in the frame S' of the electron, moving at velocity $(v, 0, 0)$ relative to S .

Proof. We consider a frame S' moving with velocity $(v, 0, 0)$ relative to the base frame S . For $n \geq 6$, by results of [3], [4] and [5], given $\rho_{n,\epsilon}$ in the base frame S , there exists a unique current $\bar{J}_{n,\epsilon}$ with compact support, such that the standard relations are satisfied for $(\rho_{n,\epsilon}, \bar{J}_{n,\epsilon})$. By results of [3], we have that $\square^2(\bar{J}_{n,\epsilon}) = \bar{0}$ and moreover, $\bar{J}_{n,\epsilon}$ has the wave representaion;

$$\bar{J}_{n,\epsilon} = \frac{1}{(2\pi)^3} \int_{\mathcal{R}^3} (\bar{A}_{n,\epsilon}(\bar{k})e^{ikct} + \bar{B}_{n,\epsilon}(\bar{k})e^{-ikct})e^{i\bar{k}\cdot\bar{x}}d\bar{k} \quad (\dagger)$$

where, for $k \neq 0$;

$$\bar{A}_{n,\epsilon}(\bar{k}) = -\frac{cA_{n,\epsilon}(\bar{k})}{k}\bar{k}$$

$$\bar{B}_{n,\epsilon}(\bar{k}) = \frac{cB_{n,\epsilon}(\bar{k})}{k}\bar{k} \quad (*)$$

If the electron travels with velocity vector $(v, 0, 0)$ in the base frame S , then by Lemma 0.10, we have that;

$$\lim_{\epsilon \rightarrow 0} A_{n,\epsilon}(\bar{k}) = \frac{q}{2} + \frac{qv k_1}{2kc}$$

$$\lim_{\epsilon \rightarrow 0} B_{n,\epsilon}(\bar{k}) = \frac{q}{2} - \frac{qv k_1}{2kc}$$

so that, by $(*)$, uniformly on compact subsets of $\mathcal{R}^3 \setminus \{\bar{0}\}$;

$$\lim_{\epsilon \rightarrow 0} \bar{A}_{n,\epsilon}(\bar{k}) = \left[-\frac{cq}{2k} - \frac{qv k_1}{2k^2}\right]\bar{k}$$

$$\lim_{\epsilon \rightarrow 0} \bar{B}_{n,\epsilon}(\bar{k}) = \left[\frac{cq}{2k} - \frac{qv k_1}{2k^2}\right]\bar{k} \quad (D)$$

We have that $\rho_{n,\epsilon}$ transforms to S' as;

$$\rho'_{n,\epsilon}(\bar{x}', t') = \gamma_v(\rho_{n,\epsilon} - \frac{vj_{1,n,\epsilon}}{c^2})(\bar{x}, t)$$

see [1], so that, using (\dagger) and the wave representation for $\rho_{n,\epsilon}$;

$$\begin{aligned} \rho'_{n,\epsilon}(\bar{x}', t') &= \frac{1}{(2\pi)^3} \int_{\mathcal{R}^3} (\gamma_v A_{n,\epsilon}(\bar{k})e^{ikct} + \gamma_v B_{n,\epsilon}(\bar{k})e^{-ikct})e^{i\bar{k}\cdot\bar{x}}d\bar{k} \\ &\quad - \frac{1}{(2\pi)^3} \int_{\mathcal{R}^3} \left(\frac{\gamma_v v}{c^2} \bar{A}_{1,n,\epsilon}(\bar{k})e^{ikct} + \frac{\gamma_v v}{c^2} \bar{B}_{1,n,\epsilon}(\bar{k})e^{-ikct}\right)e^{i\bar{k}\cdot\bar{x}}d\bar{k} \end{aligned}$$

$$= \frac{1}{(2\pi)^3} \int_{\mathcal{R}^3} ([\gamma_v A_{n,\epsilon}(\bar{k}) - \frac{\gamma_v v}{c^2} \bar{A}_{1,n,\epsilon}(\bar{k})] e^{ikct} + [\gamma_v B_{n,\epsilon}(\bar{k}) - \frac{\gamma_v v}{c^2} \bar{B}_{1,n,\epsilon}(\bar{k})] e^{-ikct}) e^{i\bar{k}\cdot\bar{x}} d\bar{k}$$

(††)

We have that, using the inverse Lorentz transformation;

$$t = \gamma_v t' + \frac{\gamma_v v x'_1}{c^2}$$

$$x_1 = \gamma_v x'_1 + \gamma_v v t'$$

$$x_2 = x'_2$$

$$x_3 = x'_3$$

so that, substituting into (††);

$$\begin{aligned} \rho'_{n,\epsilon}(\bar{x}', t') &= \frac{1}{(2\pi)^3} \int_{\mathcal{R}^3} ([\gamma_v A_{n,\epsilon}(\bar{k}) - \frac{\gamma_v v}{c^2} \bar{A}_{1,n,\epsilon}(\bar{k})] e^{ikc(\gamma_v t' + \frac{\gamma_v v x'_1}{c^2})} \\ &+ [\gamma_v B_{n,\epsilon}(\bar{k}) - \frac{\gamma_v v}{c^2} \bar{B}_{1,n,\epsilon}(\bar{k})] e^{-ikc(\gamma_v t' + \frac{\gamma_v v x'_1}{c^2})}) e^{ik_1(\gamma_v x'_1 + \gamma_v v t')} e^{ik_2 x'_2} e^{ik_3 x'_3} d\bar{k} \\ &= \frac{1}{(2\pi)^3} \int_{\mathcal{R}^3} [\gamma_v A_{n,\epsilon}(\bar{k}) - \frac{\gamma_v v}{c^2} \bar{A}_{1,n,\epsilon}(\bar{k})] e^{i(kc\gamma_v + k_1\gamma_v v)t'} e^{i(k_1\gamma_v + \frac{k_1\gamma_v v}{c})x'_1} e^{ik_2 x'_2} e^{ik_3 x'_3} d\bar{k} \\ &+ \frac{1}{(2\pi)^3} \int_{\mathcal{R}^3} [\gamma_v B_{n,\epsilon}(\bar{k}) - \frac{\gamma_v v}{c^2} \bar{B}_{1,n,\epsilon}(\bar{k})] e^{-i(kc\gamma_v - k_1\gamma_v v)t'} e^{i(k_1\gamma_v - \frac{k_1\gamma_v v}{c})x'_1} e^{ik_2 x'_2} e^{ik_3 x'_3} d\bar{k} \end{aligned}$$

(A)

We make the change of variables, in the first line of (A);

$$k'c = kc\gamma_v + k_1\gamma_v v$$

$$k' = k\gamma_v + \frac{k_1\gamma_v v}{c}$$

$$k'_1 = k_1\gamma_v + \frac{k_1\gamma_v v}{c}$$

$$k'_2 = k_2$$

$$k'_3 = k_3$$

and, in the second line of (A);

$$k'c = kc\gamma_v - k_1\gamma_v v$$

$$k' = k\gamma_v - \frac{k_1\gamma_v v}{c}$$

$$k'_1 = k_1\gamma_v - \frac{k\gamma_v v}{c}$$

$$k'_2 = k_2$$

$$k'_3 = k_3$$

which are the Doppler shift transformations given in [7], for the velocity vector $(v, 0, 0)$. Calculating the Jacobian, and using the chain rule, we have that, for the first and second lines of (A) respectively;

$$dk_1 dk_2 dk_3 = (\gamma_v - \frac{\gamma_v k'_1 v}{k'c}) dk'_1 dk'_2 dk'_3$$

$$dk_1 dk_2 dk_3 = (\gamma_v + \frac{\gamma_v k'_1 v}{k'c}) dk'_1 dk'_2 dk'_3$$

so that, substituting in (A);

$$\begin{aligned} \rho'_{n,\epsilon}(\bar{x}', t') &= \frac{1}{(2\pi)^3} \int_{\mathcal{R}^3} [\gamma_v A_{n,\epsilon}(\bar{k}') - \frac{\gamma_v v}{c^2} \bar{A}_{1,n,\epsilon}(\bar{k}')] e^{ik'ct'} e^{ik'_1 x'_1} e^{ik'_2 x'_2} e^{ik'_3 x'_3} \\ & (\gamma_v - \frac{\gamma_v k'_1 v}{k'c}) d\bar{k}' \\ & + \frac{1}{(2\pi)^3} \int_{\mathcal{R}^3} [\gamma_v B_{n,\epsilon}(\bar{k}') - \frac{\gamma_v v}{c^2} \bar{B}_{1,n,\epsilon}(\bar{k}')] e^{-ik'ct'} e^{ik'_1 x'_1} e^{ik'_2 x'_2} e^{ik'_3 x'_3} (\gamma_v + \frac{\gamma_v k'_1 v}{k'c}) d\bar{k}' \\ & = \frac{1}{(2\pi)^3} \int_{\mathcal{R}^3} [\gamma_v A_{n,\epsilon}(\bar{k}') - \frac{\gamma_v v}{c^2} \bar{A}_{1,n,\epsilon}(\bar{k}')] e^{ik'ct'} e^{i\bar{k}' \cdot \bar{x}'} (\gamma_v - \frac{\gamma_v k'_1 v}{k'c}) d\bar{k}' \\ & + \frac{1}{(2\pi)^3} \int_{\mathcal{R}^3} [\gamma_v B_{n,\epsilon}(\bar{k}') - \frac{\gamma_v v}{c^2} \bar{B}_{1,n,\epsilon}(\bar{k}')] e^{-ik'ct'} e^{i\bar{k}' \cdot \bar{x}'} (\gamma_v + \frac{\gamma_v k'_1 v}{k'c}) d\bar{k}' \end{aligned}$$

so that equating coefficients, and using the wave representation for $\rho'_{n,\epsilon}(\bar{x}', t')$, we have that;

$$A'_{n,\epsilon}(\bar{k}') = (\gamma_v - \frac{\gamma_v k'_1 v}{k'c}) [\gamma_v A_{n,\epsilon}(\bar{k}') - \frac{\gamma_v v}{c^2} \bar{A}_{1,n,\epsilon}(\bar{k}')]$$

$$B'_{n,\epsilon}(\bar{k}') = (\gamma_v + \frac{\gamma_v k'_1 v}{k'c}) [\gamma_v B_{n,\epsilon}(\bar{k}') - \frac{\gamma_v v}{c^2} \bar{B}_{1,n,\epsilon}(\bar{k}')] \quad (E)$$

and, using (D), with the inverse relations for the first line;

$$k = k'\gamma_v - \frac{k'_1\gamma_v v}{c}$$

$$k_1 = k'_1\gamma_v - \frac{k'\gamma_v v}{c}$$

$$k'_2 = k_2$$

$$k'_3 = k_3$$

$$\lim_{\epsilon \rightarrow 0} A_{n,\epsilon}(\bar{k}') = \frac{q}{2} + \frac{qv k_1}{2k c}$$

$$= \frac{q}{2} + \frac{qv(k'_1 \gamma_v - \frac{k'_1 \gamma_v v}{c})}{2(k' \gamma_v - \frac{k'_1 \gamma_v v}{c})c}$$

$$\lim_{\epsilon \rightarrow 0} \bar{A}_{1,n,\epsilon}(\bar{k}') = [-\frac{cq}{2k} - \frac{qv k_1}{2k^2}] k_1$$

$$= -\frac{cq(k'_1 \gamma_v - \frac{k'_1 \gamma_v v}{c})}{2(k' \gamma_v - \frac{k'_1 \gamma_v v}{c})} - \frac{qv(k'_1 \gamma_v - \frac{k'_1 \gamma_v v}{c})^2}{2(k' \gamma_v - \frac{k'_1 \gamma_v v}{c})^2}$$

and, substituting into the first line of (E);

$$\begin{aligned} \lim_{\epsilon \rightarrow 0} A'_{n,\epsilon}(\bar{k}') &= (\gamma_v - \frac{\gamma_v k'_1 v}{k' c}) [\gamma_v (\frac{q}{2} + \frac{qv(k'_1 \gamma_v - \frac{k'_1 \gamma_v v}{c})}{2(k' \gamma_v - \frac{k'_1 \gamma_v v}{c})c}) - \frac{\gamma_v v}{c^2} (-\frac{cq(k'_1 \gamma_v - \frac{k'_1 \gamma_v v}{c})}{2(k' \gamma_v - \frac{k'_1 \gamma_v v}{c})} \\ &\quad - \frac{qv(k'_1 \gamma_v - \frac{k'_1 \gamma_v v}{c})^2}{2(k' \gamma_v - \frac{k'_1 \gamma_v v}{c})^2})] \end{aligned}$$

Again, using (D), with the inverse relations for the second line;

$$k = k' \gamma_v + \frac{k'_1 \gamma_v v}{c}$$

$$k_1 = k'_1 \gamma_v + \frac{k'_1 \gamma_v v}{c}$$

$$k'_2 = k_2$$

$$k'_3 = k_3$$

$$\lim_{\epsilon \rightarrow 0} B_{n,\epsilon}(\bar{k}') = \frac{q}{2} - \frac{qv k_1}{2k c}$$

$$= \frac{q}{2} - \frac{qv(k'_1 \gamma_v + \frac{k'_1 \gamma_v v}{c})}{2(k' \gamma_v + \frac{k'_1 \gamma_v v}{c})c}$$

$$\lim_{\epsilon \rightarrow 0} \bar{B}_{1,n,\epsilon}(\bar{k}') = [\frac{cq}{2k} - \frac{qv k_1}{2k^2}] k_1$$

$$= \frac{cq(k'_1 \gamma_v + \frac{k'_1 \gamma_v v}{c})}{2(k' \gamma_v + \frac{k'_1 \gamma_v v}{c})} - \frac{qv(k'_1 \gamma_v + \frac{k'_1 \gamma_v v}{c})^2}{2(k' \gamma_v + \frac{k'_1 \gamma_v v}{c})^2}$$

and, substituting into the second line of (E);

$$\lim_{\epsilon \rightarrow 0} B'_{n,\epsilon}(\bar{k}') = (\gamma_v + \frac{\gamma_v k'_1 v}{k' c}) [\gamma_v (\frac{q}{2} - \frac{qv(k'_1 \gamma_v + \frac{k'_1 \gamma_v v}{c})}{2(k' \gamma_v + \frac{k'_1 \gamma_v v}{c})c}) - \frac{\gamma_v v}{c^2} (\frac{cq(k'_1 \gamma_v + \frac{k'_1 \gamma_v v}{c})}{2(k' \gamma_v + \frac{k'_1 \gamma_v v}{c})} - \frac{qv(k'_1 \gamma_v + \frac{k'_1 \gamma_v v}{c})^2}{2(k' \gamma_v + \frac{k'_1 \gamma_v v}{c})^2})]$$

$$-\frac{qv(k'_1\gamma_v + \frac{k'_1\gamma_v v}{c})^2}{2(k'\gamma_v + \frac{k'_1\gamma_v v}{c})^2}])$$

Rearranging, we have that;

$$\begin{aligned} \lim_{\epsilon \rightarrow 0} A'_{n,\epsilon}(\bar{k}') &= q\gamma_v^2 \left(1 - \frac{k'_1 v}{k'c}\right) \left[\left(\frac{1}{2} + \frac{vk'_1(1 - \frac{k'_1 v}{k'c})}{2k'(1 - \frac{k'_1 v}{k'c})c}\right) - \frac{v}{c^2} \left(-\frac{ck'_1(1 - \frac{k'_1 v}{k'c})}{2k'(1 - \frac{k'_1 v}{k'c})}\right) \right. \\ &\quad \left. - \frac{vk_1'^2(1 - \frac{k'_1 v}{k'c})^2}{2k'^2(1 - \frac{k'_1 v}{k'c})^2} \right] \\ &= q\gamma_v^2 \left(1 - \frac{k'_1 v}{k'c}\right) \left[\frac{1}{2} + \frac{vk'_1(1 - \frac{k'_1 v}{k'c})}{k'(1 - \frac{k'_1 v}{k'c})c} + \frac{v^2 k_1'^2(1 - \frac{k'_1 v}{k'c})^2}{2k'^2 c^2 (1 - \frac{k'_1 v}{k'c})^2} \right] \\ &= \frac{q\gamma_v^2}{2} \left(1 - \frac{k'_1 v}{k'c}\right) \left[1 + \frac{vk'_1(1 - \frac{k'_1 v}{k'c})}{k'(1 - \frac{k'_1 v}{k'c})c} \right]^2 \\ &= \frac{q\gamma_v^2}{2} \left(1 - \frac{k'_1 v}{k'c}\right) \left[\frac{k'(1 - \frac{k'_1 v}{k'c})c + vk'_1(1 - \frac{k'_1 v}{k'c})}{k'(1 - \frac{k'_1 v}{k'c})c} \right]^2 \\ &= \frac{q\gamma_v^2}{2} \left(1 - \frac{k'_1 v}{k'c}\right) \left[\frac{k'c - k'_1 v + k'_1 v - \frac{v^2 k'_1}{c}}{k'(1 - \frac{k'_1 v}{k'c})c} \right]^2 \\ &= \frac{q\gamma_v^2}{2} \left(1 - \frac{k'_1 v}{k'c}\right) \left[\frac{k'c(1 - \frac{v^2}{c^2})}{k'c(1 - \frac{k'_1 v}{k'c})} \right]^2 \\ &= \frac{q\gamma_v^2}{2} \left(1 - \frac{k'_1 v}{k'c}\right) \left[\frac{(1 - \frac{v^2}{c^2})}{(1 - \frac{k'_1 v}{k'c})} \right]^2 \\ &= \frac{q}{2} \left[\frac{(1 - \frac{v^2}{c^2})}{(1 - \frac{k'_1 v}{k'c})} \right] \\ &= \frac{q}{2\gamma_v} \left[\frac{1}{\gamma_v(1 - \frac{k'_1 v}{k'c})} \right] \\ &= \frac{q'}{2} \frac{k'}{k} \end{aligned}$$

and, using the symmetry;

$$\lim_{\epsilon \rightarrow 0} A'_{n,\epsilon}(\bar{k}')(v) = \lim_{\epsilon \rightarrow 0} B'_{n,\epsilon}(\bar{k}')(-v)$$

we have;

$$\begin{aligned} \lim_{\epsilon \rightarrow 0} B'_{n,\epsilon}(\bar{k}') &= \frac{q\gamma_v^2}{2} \left(1 + \frac{k'_1 v}{k'c}\right) \left[1 - \frac{vk'_1(1 + \frac{k'_1 v}{k'c})}{k'(1 + \frac{k'_1 v}{k'c})c} \right]^2 \\ &= \frac{q'}{2} \frac{k'}{k} \end{aligned}$$

where $q' = \frac{q}{\gamma_v}$ is the new charge due to the change in mass of the electron, $m' = \frac{m}{\gamma_v}$ and $\frac{k'}{k}$ is the Doppler shift in amplitude of the spectrum of the stationary electron.

□

Lemma 0.13. *Let an electron move at velocity $(w, 0, 0)$ in the base frame S , and let S' move at velocity $(v, 0, 0)$ relative to S , then the amplitudes in the spectrum of the electron in S' are shifted by the background Doppler factor $\frac{k'}{k}$.*

Proof. The proof is almost the same as Lemma 0.12. We consider a frame S' moving with velocity $(v, 0, 0)$ relative to the base frame S . For $n \geq 6$, by results of [3], [4] and [5], given $\rho_{n,\epsilon}$ in the base frame S , there exists a unique current $\bar{J}_{n,\epsilon}$ with compact support, such that the standard relations are satisfied for $(\rho_{n,\epsilon}, \bar{J}_{n,\epsilon})$. By results of [3], we have that $\square^2(\bar{J}_{n,\epsilon}) = \bar{0}$ and moreover, $\bar{J}_{n,\epsilon}$ has the wave representaion;

$$\bar{J}_{n,\epsilon} = \frac{1}{(2\pi)^3} \int_{\mathcal{R}^3} (\bar{A}_{n,\epsilon}(\bar{k})e^{ikct} + \bar{B}_{n,\epsilon}(\bar{k})e^{-ikct})e^{i\bar{k}\cdot\bar{x}} d\bar{k} \quad (\dagger)$$

where, for $k \neq 0$;

$$\bar{A}_{n,\epsilon}(\bar{k}) = -\frac{cA_{n,\epsilon}(\bar{k})}{k}\bar{k}$$

$$\bar{B}_{n,\epsilon}(\bar{k}) = \frac{cB_{n,\epsilon}(\bar{k})}{k}\bar{k} \quad (*)$$

If the electron travels with velocity vector $(w, 0, 0)$ in the base frame S , then by Lemma 0.10, we have that;

$$\lim_{\epsilon \rightarrow 0} A_{n,\epsilon}(\bar{k}) = \frac{q}{2} + \frac{qw k_1}{2kc}$$

$$\lim_{\epsilon \rightarrow 0} B_{n,\epsilon}(\bar{k}) = \frac{q}{2} - \frac{qw k_1}{2kc}$$

so that, by (*), uniformly on compact subsets of $\mathcal{R}^3 \setminus \{\bar{0}\}$;

$$\lim_{\epsilon \rightarrow 0} \bar{A}_{n,\epsilon}(\bar{k}) = \left[-\frac{cq}{2k} - \frac{qw k_1}{2k^2}\right]\bar{k}$$

$$\lim_{\epsilon \rightarrow 0} \bar{B}_{n,\epsilon}(\bar{k}) = \left[\frac{cq}{2k} - \frac{qw k_1}{2k^2}\right]\bar{k} \quad (D)$$

We have that $\rho_{n,\epsilon}$ transforms to S' as;

$$\rho'_{n,\epsilon}(\bar{x}', t') = \gamma_v(\rho_{n,\epsilon} - \frac{vj_{1,n,\epsilon}}{c^2})(\bar{x}, t)$$

see [1], so that, using (†) and the wave representation for $\rho_{n,\epsilon}$;

$$\begin{aligned} \rho'_{n,\epsilon}(\bar{x}', t') &= \frac{1}{(2\pi)^3} \int_{\mathcal{R}^3} (\gamma_v A_{n,\epsilon}(\bar{k}) e^{ikct} + \gamma_v B_{n,\epsilon}(\bar{k}) e^{-ikct}) e^{i\bar{k}\cdot\bar{x}} d\bar{k} \\ &\quad - \frac{1}{(2\pi)^3} \int_{\mathcal{R}^3} (\frac{\gamma_v v}{c^2} \bar{A}_{1,n,\epsilon}(\bar{k}) e^{ikct} + \frac{\gamma_v v}{c^2} \bar{B}_{1,n,\epsilon}(\bar{k}) e^{-ikct}) e^{i\bar{k}\cdot\bar{x}} d\bar{k} \\ &= \frac{1}{(2\pi)^3} \int_{\mathcal{R}^3} ([\gamma_v A_{n,\epsilon}(\bar{k}) - \frac{\gamma_v v}{c^2} \bar{A}_{1,n,\epsilon}(\bar{k})] e^{ikct} + [\gamma_v B_{n,\epsilon}(\bar{k}) - \frac{\gamma_v v}{c^2} \bar{B}_{1,n,\epsilon}(\bar{k})] e^{-ikct}) e^{i\bar{k}\cdot\bar{x}} d\bar{k} \end{aligned}$$

(††)

We have that, using the inverse Lorentz transformation;

$$t = \gamma_v t' + \frac{\gamma_v v x'_1}{c^2}$$

$$x_1 = \gamma_v x'_1 + \gamma_v v t'$$

$$x_2 = x'_2$$

$$x_3 = x'_3$$

so that, substituting into (††);

$$\begin{aligned} \rho'_{n,\epsilon}(\bar{x}', t') &= \frac{1}{(2\pi)^3} \int_{\mathcal{R}^3} ([\gamma_v A_{n,\epsilon}(\bar{k}) - \frac{\gamma_v v}{c^2} \bar{A}_{1,n,\epsilon}(\bar{k})] e^{ikc(\gamma_v t' + \frac{\gamma_v v x'_1}{c^2})} \\ &\quad + [\gamma_v B_{n,\epsilon}(\bar{k}) - \frac{\gamma_v v}{c^2} \bar{B}_{1,n,\epsilon}(\bar{k})] e^{-ikc(\gamma_v t' + \frac{\gamma_v v x'_1}{c^2})}) e^{ik_1(\gamma_v x'_1 + \gamma_v v t')} e^{ik_2 x'_2} e^{ik_3 x'_3} d\bar{k} \\ &= \frac{1}{(2\pi)^3} \int_{\mathcal{R}^3} [\gamma_v A_{n,\epsilon}(\bar{k}) - \frac{\gamma_v v}{c^2} \bar{A}_{1,n,\epsilon}(\bar{k})] e^{i(kc\gamma_v + k_1\gamma_v v)t'} e^{i(k_1\gamma_v + \frac{k\gamma_v v}{c})x'_1} e^{ik_2 x'_2} e^{ik_3 x'_3} d\bar{k} \\ &\quad + \frac{1}{(2\pi)^3} \int_{\mathcal{R}^3} [\gamma_v B_{n,\epsilon}(\bar{k}) - \frac{\gamma_v v}{c^2} \bar{B}_{1,n,\epsilon}(\bar{k})] e^{-i(kc\gamma_v - k_1\gamma_v v)t'} e^{i(k_1\gamma_v - \frac{k\gamma_v v}{c})x'_1} e^{ik_2 x'_2} e^{ik_3 x'_3} d\bar{k} \end{aligned}$$

(A)

We make the change of variables, in the first line of (A);

$$k'c = kc\gamma_v + k_1\gamma_v v$$

$$k' = k\gamma_v + \frac{k_1\gamma_v v}{c}$$

$$k'_1 = k_1\gamma_v + \frac{k\gamma_v v}{c}$$

$$k'_2 = k_2$$

$$k'_3 = k_3$$

and, in the second line of (A);

$$k'c = kc\gamma_v - k_1\gamma_v v$$

$$k' = k\gamma_v - \frac{k_1\gamma_v v}{c}$$

$$k'_1 = k_1\gamma_v - \frac{k\gamma_v v}{c}$$

$$k'_2 = k_2$$

$$k'_3 = k_3$$

which are the Doppler shift transformations given in [7], for the velocity vector $(v, 0, 0)$. Calculating the Jacobian, and using the chain rule, we have that, for the first and second lines of (A) respectively;

$$dk_1 dk_2 dk_3 = (\gamma_v - \frac{\gamma_v k'_1 v}{k'c}) dk'_1 dk'_2 dk'_3$$

$$dk_1 dk_2 dk_3 = (\gamma_v + \frac{\gamma_v k'_1 v}{k'c}) dk'_1 dk'_2 dk'_3$$

so that, substituting in (A);

$$\begin{aligned} \rho'_{n,\epsilon}(\bar{x}', t') &= \frac{1}{(2\pi)^3} \int_{\mathcal{R}^3} [\gamma_v A_{n,\epsilon}(\bar{k}') - \frac{\gamma_v v}{c^2} \bar{A}_{1,n,\epsilon}(\bar{k}')] e^{ik'ct'} e^{ik'_1 x'_1} e^{ik'_2 x'_2} e^{ik'_3 x'_3} \\ & (\gamma_v - \frac{\gamma_v k'_1 v}{k'c}) d\bar{k}' \\ & + \frac{1}{(2\pi)^3} \int_{\mathcal{R}^3} [\gamma_v B_{n,\epsilon}(\bar{k}') - \frac{\gamma_v v}{c^2} \bar{B}_{1,n,\epsilon}(\bar{k}')] e^{-ik'ct'} e^{ik'_1 x'_1} e^{ik'_2 x'_2} e^{ik'_3 x'_3} (\gamma_v + \frac{\gamma_v k'_1 v}{k'c}) d\bar{k}' \\ & = \frac{1}{(2\pi)^3} \int_{\mathcal{R}^3} [\gamma_v A_{n,\epsilon}(\bar{k}') - \frac{\gamma_v v}{c^2} \bar{A}_{1,n,\epsilon}(\bar{k}')] e^{ik'ct'} e^{i\bar{k}' \cdot \bar{x}'} (\gamma_v - \frac{\gamma_v k'_1 v}{k'c}) d\bar{k}' \\ & + \frac{1}{(2\pi)^3} \int_{\mathcal{R}^3} [\gamma_v B_{n,\epsilon}(\bar{k}') - \frac{\gamma_v v}{c^2} \bar{B}_{1,n,\epsilon}(\bar{k}')] e^{-ik'ct'} e^{i\bar{k}' \cdot \bar{x}'} (\gamma_v + \frac{\gamma_v k'_1 v}{k'c}) d\bar{k}' \end{aligned}$$

so that equating coefficients, and using the wave representation for $\rho'_{n,\epsilon}(\bar{x}', t')$, we have that;

$$A'_{n,\epsilon}(\bar{k}') = (\gamma_v - \frac{\gamma_v k'_1 v}{k'c}) [\gamma_v A_{n,\epsilon}(\bar{k}') - \frac{\gamma_v v}{c^2} \bar{A}_{1,n,\epsilon}(\bar{k}')]$$

$$B'_{n,\epsilon}(\bar{k}') = (\gamma_v + \frac{\gamma_v k'_1 v}{k'c}) [\gamma_v B_{n,\epsilon}(\bar{k}') - \frac{\gamma_v v}{c^2} \bar{B}_{1,n,\epsilon}(\bar{k}')] \quad (E)$$

and, using (D), with the inverse relations for the first line;

$$k = k' \gamma_v - \frac{k'_1 \gamma_v v}{c}$$

$$k_1 = k'_1 \gamma_v - \frac{k' \gamma_v v}{c}$$

$$k'_2 = k_2$$

$$k'_3 = k_3$$

$$\lim_{\epsilon \rightarrow 0} A_{n,\epsilon}(\bar{k}') = \frac{q}{2} + \frac{qw k_1}{2kc}$$

$$= \frac{q}{2} + \frac{qw(k'_1 \gamma_v - \frac{k' \gamma_v v}{c})}{2(k' \gamma_v - \frac{k'_1 \gamma_v v}{c})c}$$

$$\lim_{\epsilon \rightarrow 0} \bar{A}_{1,n,\epsilon}(\bar{k}') = [-\frac{cq}{2k} - \frac{qw k_1}{2k^2}] k_1$$

$$= -\frac{cq(k'_1 \gamma_v - \frac{k' \gamma_v v}{c})}{2(k' \gamma_v - \frac{k'_1 \gamma_v v}{c})} - \frac{qw(k'_1 \gamma_v - \frac{k' \gamma_v v}{c})^2}{2(k' \gamma_v - \frac{k'_1 \gamma_v v}{c})^2}$$

and, substituting into the first line of (E);

$$\begin{aligned} \lim_{\epsilon \rightarrow 0} A'_{n,\epsilon}(\bar{k}') &= (\gamma_v - \frac{\gamma_v k'_1 v}{k'c}) [\gamma_v (\frac{q}{2} + \frac{qw(k'_1 \gamma_v - \frac{k' \gamma_v v}{c})}{2(k' \gamma_v - \frac{k'_1 \gamma_v v}{c})c}) - \frac{\gamma_v v}{c^2} (-\frac{cq(k'_1 \gamma_v - \frac{k' \gamma_v v}{c})}{2(k' \gamma_v - \frac{k'_1 \gamma_v v}{c})} \\ &\quad - \frac{qw(k'_1 \gamma_v - \frac{k' \gamma_v v}{c})^2}{2(k' \gamma_v - \frac{k'_1 \gamma_v v}{c})^2})] \end{aligned}$$

Again, using (D), with the inverse relations for the second line;

$$k = k' \gamma_v + \frac{k'_1 \gamma_v v}{c}$$

$$k_1 = k'_1 \gamma_v + \frac{k' \gamma_v v}{c}$$

$$k'_2 = k_2$$

$$k'_3 = k_3$$

$$\lim_{\epsilon \rightarrow 0} B_{n,\epsilon}(\bar{k}') = \frac{q}{2} - \frac{qw k_1}{2kc}$$

$$= \frac{q}{2} - \frac{qw(k'_1 \gamma_v + \frac{k' \gamma_v v}{c})}{2(k' \gamma_v + \frac{k'_1 \gamma_v v}{c})c}$$

$$\begin{aligned} \lim_{\epsilon \rightarrow 0} \bar{B}_{1,n,\epsilon}(\bar{k}') &= \left[\frac{cq}{2k} - \frac{qw k_1}{2k^2} \right] k_1 \\ &= \frac{cq(k_1' \gamma_v + \frac{k_1' \gamma_{vv}}{c})}{2(k_1' \gamma_v + \frac{k_1' \gamma_{vv}}{c})} - \frac{qw(k_1' \gamma_v + \frac{k_1' \gamma_{vv}}{c})^2}{2(k_1' \gamma_v + \frac{k_1' \gamma_{vv}}{c})^2} \end{aligned}$$

and, substituting into the second line of (E);

$$\begin{aligned} \lim_{\epsilon \rightarrow 0} B'_{n,\epsilon}(\bar{k}') &= (\gamma_v + \frac{\gamma_v k_1' v}{k_1' c}) \left[\gamma_v \left(\frac{q}{2} - \frac{qw(k_1' \gamma_v + \frac{k_1' \gamma_{vv}}{c})}{2(k_1' \gamma_v + \frac{k_1' \gamma_{vv}}{c})c} \right) - \frac{\gamma_{vv}}{c^2} \left(\frac{cq(k_1' \gamma_v + \frac{k_1' \gamma_{vv}}{c})}{2(k_1' \gamma_v + \frac{k_1' \gamma_{vv}}{c})} \right. \right. \\ &\quad \left. \left. - \frac{qw(k_1' \gamma_v + \frac{k_1' \gamma_{vv}}{c})^2}{2(k_1' \gamma_v + \frac{k_1' \gamma_{vv}}{c})^2} \right) \right] \end{aligned}$$

Rearranging, we have that;

$$\begin{aligned} \lim_{\epsilon \rightarrow 0} A'_{n,\epsilon}(\bar{k}') &= q \gamma_v^2 \left(1 - \frac{k_1' v}{k_1' c} \right) \left[\left(\frac{1}{2} + \frac{w k_1' (1 - \frac{k_1' v}{k_1' c})}{2k_1' (1 - \frac{k_1' v}{k_1' c})c} \right) - \frac{v}{c^2} \left(- \frac{c k_1' (1 - \frac{k_1' v}{k_1' c})}{2k_1' (1 - \frac{k_1' v}{k_1' c})} \right. \right. \\ &\quad \left. \left. - \frac{w k_1'^2 (1 - \frac{k_1' v}{k_1' c})^2}{2k_1'^2 (1 - \frac{k_1' v}{k_1' c})^2} \right) \right] \\ &= q \gamma_v^2 \left(1 - \frac{k_1' v}{k_1' c} \right) \left[\frac{1}{2} + \frac{w k_1' (1 - \frac{k_1' v}{k_1' c})}{2k_1' (1 - \frac{k_1' v}{k_1' c})c} + \frac{v k_1' (1 - \frac{k_1' v}{k_1' c})}{2k_1' (1 - \frac{k_1' v}{k_1' c})c} + \frac{v w k_1'^2 (1 - \frac{k_1' v}{k_1' c})^2}{2k_1'^2 c^2 (1 - \frac{k_1' v}{k_1' c})^2} \right] \\ &= \frac{q \gamma_v^2}{2} \left(1 - \frac{k_1' v}{k_1' c} \right) \left[1 + \frac{v k_1' (1 - \frac{k_1' v}{k_1' c})}{k_1' (1 - \frac{k_1' v}{k_1' c})c} \right] \left[1 + \frac{w k_1' (1 - \frac{k_1' v}{k_1' c})}{k_1' (1 - \frac{k_1' v}{k_1' c})c} \right] \\ &= \frac{q \gamma_v^2}{2} \left(1 - \frac{k_1' v}{k_1' c} \right) \left[\frac{(1 - \frac{v^2}{c^2})}{(1 - \frac{k_1' v}{k_1' c})} \right] \left[1 + \frac{w k_1' (1 - \frac{k_1' v}{k_1' c})}{k_1' (1 - \frac{k_1' v}{k_1' c})c} \right] \\ &= \frac{q}{2} \left[1 + \frac{w k_1' (1 - \frac{k_1' v}{k_1' c})}{k_1' (1 - \frac{k_1' v}{k_1' c})c} \right] \\ &= \frac{q}{2} \left[\frac{k_1' c - k_1' v + k_1' w - \frac{k_1' v w}{c}}{k_1' (1 - \frac{k_1' v}{k_1' c})c} \right] \\ &= \frac{q}{2} \left[\frac{k_1' c (1 - \frac{v w}{c^2}) + k_1' (w - v)}{k_1' (1 - \frac{k_1' v}{k_1' c})c} \right] \\ &= \frac{q}{2} \frac{[k_1' c (1 - \frac{v w}{c^2})] [1 + \frac{k_1'}{k_1' c} (\frac{w - v}{1 - \frac{v w}{c^2}})]}{k_1' (1 - \frac{k_1' v}{k_1' c})c} \\ &= \frac{q}{2} \frac{(1 - \frac{v w}{c^2}) [1 + \frac{k_1'}{k_1' c} w']}{(1 - \frac{k_1' v}{k_1' c})} \\ &= \frac{q}{2} \frac{(1 - \frac{v w}{c^2}) [1 + \frac{k_1'}{k_1' c} w']}{(1 - \frac{k_1' v}{k_1' c})} \\ &= \frac{q}{2} \frac{\gamma_{w \gamma_v} \lim_{\epsilon \rightarrow 0} A'_{n,\epsilon}}{(1 - \frac{k_1' v}{k_1' c})} \end{aligned}$$

$$\begin{aligned}
&= \frac{q'}{2} \frac{\lim_{\epsilon \rightarrow 0} A''_{n,\epsilon}(\bar{k}')}{\gamma_v (1 - \frac{k'v}{k'c})} \\
&= \frac{q'}{2} \lim_{\epsilon \rightarrow 0} A''_{n,\epsilon}(\bar{k}') \frac{k'}{k}
\end{aligned}$$

and, using the symmetries;

$$\lim_{\epsilon \rightarrow 0} A''_{n,\epsilon}(\bar{k}')(\omega) = \lim_{\epsilon \rightarrow 0} B''_{n,\epsilon}(\bar{k}')(-\omega)$$

$$\lim_{\epsilon \rightarrow 0} A'_{n,\epsilon}(\bar{k}')(\nu) = \lim_{\epsilon \rightarrow 0} B'_{n,\epsilon}(\bar{k}')(-\nu)$$

we have;

$$\lim_{\epsilon \rightarrow 0} B'_{n,\epsilon}(\bar{k}') = \frac{q'}{2} \lim_{\epsilon \rightarrow 0} B''_{n,\epsilon}(\bar{k}') \frac{k'}{k}$$

where $q' = q \frac{\gamma_{\bar{w}^* - \bar{v}}}{\gamma_w}$ is the new charge in S' due to the change in mass of the electron, $m' = \frac{m \gamma_{\bar{w}^* - \bar{v}}}{\gamma_v}$, $\{A''_{n,\epsilon}(\bar{k}'), B''_{n,\epsilon}(\bar{k}')\}$ are the factors in the wave equation for an electron moving with the transformed velocity $w' = \frac{w-v}{1 - \frac{vw}{c^2}}$ in S' and $\frac{k'}{k}$ is the background Doppler shift in amplitude, transferring from S to S' .

□

Lemma 0.14. *Let an electron move at velocity \bar{w} in the base frame S , and let S' be a rotation of S by $g \in O(3)$, then there is no Doppler shift in the amplitudes of the spectrum of the electron in S' , and the electron moves with velocity \bar{w}^g in S' .*

Proof. We consider a rotated frame S' relative to the base frame S . For $n \geq 6$, if the electron travels with velocity vector \bar{w} in the base frame S , then by Lemma 0.10, we have that;

$$\lim_{\epsilon \rightarrow 0} A_{n,\epsilon}(\bar{k}) = \frac{q}{2} + \frac{q\bar{w} \cdot \bar{k}}{2kc}$$

$$\lim_{\epsilon \rightarrow 0} B_{n,\epsilon}(\bar{k}) = \frac{q}{2} - \frac{q\bar{w} \cdot \bar{k}}{2kc} \quad (D)$$

We have that $\rho_{n,\epsilon}$ transforms to S' as;

$$\rho'_{n,\epsilon}(\bar{x}', t') = \rho_{n,\epsilon}(\bar{x}, t), \text{ where } \bar{x}' = \bar{x}^g$$

see [6], so that, using (†) and the wave representation for $\rho_{n,\epsilon}$;

$$\rho'_{n,\epsilon}(\bar{x}', t') = \frac{1}{(2\pi)^3} \int_{\mathcal{R}^3} (A_{n,\epsilon}(\bar{k})e^{ikct} + B_{n,\epsilon}(\bar{k})e^{-ikct}) e^{i\bar{k}\cdot\bar{x}} d\bar{k}$$

(††)

We have that, using the inverse rotation, $\bar{x} = \bar{x}'g^{-1}$ and substituting into (††), using the fact that $g \in O(3)$;

$$\begin{aligned} \rho'_{n,\epsilon}(\bar{x}', t') &= \frac{1}{(2\pi)^3} \int_{\mathcal{R}^3} \int_{\mathcal{R}^3} (A_{n,\epsilon}(\bar{k})e^{ikct} + B_{n,\epsilon}(\bar{k})e^{-ikct}) e^{i\bar{k}\cdot\bar{x}'g^{-1}} d\bar{k} \\ &= \frac{1}{(2\pi)^3} \int_{\mathcal{R}^3} \int_{\mathcal{R}^3} (A_{n,\epsilon}(\bar{k})e^{ikct} + B_{n,\epsilon}(\bar{k})e^{-ikct}) e^{i\bar{k}^g\cdot\bar{x}'} d\bar{k}, \quad (A) \end{aligned}$$

We make the change of variables $\bar{k}' = \bar{k}^g$. Calculating the Jacobian, and using the fact that $g \in O(3)$, we have that;

$$d\bar{k} = d\bar{k}'$$

so that, substituting in (A);

$$\rho'_{n,\epsilon}(\bar{x}', t') = \frac{1}{(2\pi)^3} \int_{\mathcal{R}^3} \int_{\mathcal{R}^3} (A_{n,\epsilon}(\bar{k}'g^{-1})e^{ik'ct} + B_{n,\epsilon}(\bar{k}'g^{-1})e^{-ik'ct}) e^{i\bar{k}'\cdot\bar{x}'} d\bar{k}'$$

and equating coefficients, using the wave representation for $\rho'_{n,\epsilon}(\bar{x}', t')$, we have that;

$$\begin{aligned} A'_{n,\epsilon}(\bar{k}') &= A_{n,\epsilon}(\bar{k}'g^{-1}) \\ B'_{n,\epsilon}(\bar{k}') &= B_{n,\epsilon}(\bar{k}'g^{-1}) \quad (E) \end{aligned}$$

so that, using (D) and the fact that $g \in O(3)$, $k = k'$;

$$\begin{aligned} \lim_{\epsilon \rightarrow 0} A'_{n,\epsilon}(\bar{k}') &= \frac{q}{2} + \frac{q\bar{w}\cdot\bar{k}'g^{-1}}{2k'c} \\ &= \frac{q}{2} + \frac{q\bar{w}^g\cdot\bar{k}'}{2k'c} \\ \lim_{\epsilon \rightarrow 0} B'_{n,\epsilon}(\bar{k}') &= \frac{q}{2} - \frac{q\bar{w}\cdot\bar{k}'g^{-1}}{2k'c} \\ &= \frac{q}{2} - \frac{q\bar{w}^g\cdot\bar{k}'}{2k'c} \end{aligned}$$

which represents an electron with charge q and velocity vector \bar{w}^g moving in S' . There is no mass change and no Doppler effect as $k' = k$. \square

Lemma 0.15. *Let an electron move at velocity \bar{w} in the base frame S , and let S' move at velocity \bar{v} relative to S , then the amplitudes in the spectrum of the electron in S' are shifted by the background Doppler factor $\frac{k'}{k}$.*

Proof. The proof generalises Lemma 0.13. We consider a frame S' moving with velocity \bar{v} relative to the base frame S and let \bar{w} be the velocity of the electron in S . Choose $g \in SO(3)$, with $g(\bar{v}) = (v, 0, 0)$ and $g(\bar{w}) = \bar{p} = (p_1, p_2, 0)$, with $w = \sqrt{p_1^2 + p_2^2}$. Then, by a result in [6], we have that;

$$R_g B_{\bar{v}} = B_{g(\bar{v})} R_g = B_{(v,0,0)} R_g$$

and the electron travels at velocity $(p_1, p_2, 0)$ in the rotated by g frame S_1 with no Doppler shift. Let S_2 be the frame connected to S_1 by the boost $B_{(v,0,0)}$. For $n \geq 6$, by results of [3], [4] and [5] again, given $\rho_{n,\epsilon}$ in the base frame S_1 , there exists a unique current $\bar{J}_{n,\epsilon}$ with compact support, such that the standard relations are satisfied for $(\rho_{n,\epsilon}, \bar{J}_{n,\epsilon})$. By results of [3], we have that $\square^2(\bar{J}_{n,\epsilon}) = \bar{0}$ and moreover, $\bar{J}_{n,\epsilon}$ has the wave representation;

$$\bar{J}_{n,\epsilon} = \frac{1}{(2\pi)^3} \int_{\mathcal{R}^3} (\bar{A}_{n,\epsilon}(\bar{k}) e^{ikct} + \bar{B}_{n,\epsilon}(\bar{k}) e^{-ikct}) e^{i\bar{k}\cdot\bar{x}} d\bar{k} \quad (\dagger)$$

where, for $k \neq 0$;

$$\bar{A}_{n,\epsilon}(\bar{k}) = -\frac{cA_{n,\epsilon}(\bar{k})}{k} \bar{k}$$

$$\bar{B}_{n,\epsilon}(\bar{k}) = \frac{cB_{n,\epsilon}(\bar{k})}{k} \bar{k} \quad (*)$$

If the electron travels with velocity vector \bar{p} in the frame S_1 , then by Lemma 0.10, we have that;

$$\lim_{\epsilon \rightarrow 0} A_{n,\epsilon}(\bar{k}) = \frac{q}{2} + \frac{q\bar{p}\cdot\bar{k}}{2kc}$$

$$\lim_{\epsilon \rightarrow 0} B_{n,\epsilon}(\bar{k}) = \frac{q}{2} - \frac{q\bar{p}\cdot\bar{k}}{2kc}$$

so that, by (*), uniformly on compact subsets of $\mathcal{R}^3 \setminus \{\bar{0}\}$;

$$\lim_{\epsilon \rightarrow 0} \bar{A}_{n,\epsilon}(\bar{k}) = \left[-\frac{cq}{2k} - \frac{q\bar{p}\cdot\bar{k}}{2k^2} \right] \bar{k}$$

$$\lim_{\epsilon \rightarrow 0} \bar{B}_{n,\epsilon}(\bar{k}) = \left[\frac{cq}{2k} - \frac{q\bar{p}\cdot\bar{k}}{2k^2} \right] \bar{k} \quad (D)$$

We have that $\rho_{n,\epsilon}$ transforms to S_2 as;

$$\rho'_{n,\epsilon}(\bar{x}', t') = \gamma_v(\rho_{n,\epsilon} - \frac{vj_{1,n,\epsilon}}{c^2})(\bar{x}, t)$$

see [1], so that, using (†) and the wave representation for $\rho_{n,\epsilon}$;

$$\begin{aligned} \rho'_{n,\epsilon}(\bar{x}', t') &= \frac{1}{(2\pi)^3} \int_{\mathcal{R}^3} (\gamma_v A_{n,\epsilon}(\bar{k}) e^{ikct} + \gamma_v B_{n,\epsilon}(\bar{k}) e^{-ikct}) e^{i\bar{k}\cdot\bar{x}} d\bar{k} \\ &\quad - \frac{1}{(2\pi)^3} \int_{\mathcal{R}^3} (\frac{\gamma_v v}{c^2} \bar{A}_{1,n,\epsilon}(\bar{k}) e^{ikct} + \frac{\gamma_v v}{c^2} \bar{B}_{1,n,\epsilon}(\bar{k}) e^{-ikct}) e^{i\bar{k}\cdot\bar{x}} d\bar{k} \\ &= \frac{1}{(2\pi)^3} \int_{\mathcal{R}^3} ([\gamma_v A_{n,\epsilon}(\bar{k}) - \frac{\gamma_v v}{c^2} \bar{A}_{1,n,\epsilon}(\bar{k})] e^{ikct} + [\gamma_v B_{n,\epsilon}(\bar{k}) - \frac{\gamma_v v}{c^2} \bar{B}_{1,n,\epsilon}(\bar{k})] e^{-ikct}) e^{i\bar{k}\cdot\bar{x}} d\bar{k} \end{aligned}$$

(††)

We have that, using the inverse Lorentz transformation;

$$t = \gamma_v t' + \frac{\gamma_v v x'_1}{c^2}$$

$$x_1 = \gamma_v x'_1 + \gamma_v v t'$$

$$x_2 = x'_2$$

$$x_3 = x'_3$$

so that, substituting into (††);

$$\begin{aligned} \rho'_{n,\epsilon}(\bar{x}', t') &= \frac{1}{(2\pi)^3} \int_{\mathcal{R}^3} ([\gamma_v A_{n,\epsilon}(\bar{k}) - \frac{\gamma_v v}{c^2} \bar{A}_{1,n,\epsilon}(\bar{k})] e^{ikc(\gamma_v t' + \frac{\gamma_v v x'_1}{c^2})} \\ &\quad + [\gamma_v B_{n,\epsilon}(\bar{k}) - \frac{\gamma_v v}{c^2} \bar{B}_{1,n,\epsilon}(\bar{k})] e^{-ikc(\gamma_v t' + \frac{\gamma_v v x'_1}{c^2})}) e^{ik_1(\gamma_v x'_1 + \gamma_v v t')} e^{ik_2 x'_2} e^{ik_3 x'_3} d\bar{k} \\ &= \frac{1}{(2\pi)^3} \int_{\mathcal{R}^3} [\gamma_v A_{n,\epsilon}(\bar{k}) - \frac{\gamma_v v}{c^2} \bar{A}_{1,n,\epsilon}(\bar{k})] e^{i(kc\gamma_v + k_1\gamma_v v)t'} e^{i(k_1\gamma_v + \frac{k_1\gamma_v v}{c})x'_1} e^{ik_2 x'_2} e^{ik_3 x'_3} d\bar{k} \\ &\quad + \frac{1}{(2\pi)^3} \int_{\mathcal{R}^3} [\gamma_v B_{n,\epsilon}(\bar{k}) - \frac{\gamma_v v}{c^2} \bar{B}_{1,n,\epsilon}(\bar{k})] e^{-i(kc\gamma_v - k_1\gamma_v v)t'} e^{i(k_1\gamma_v - \frac{k_1\gamma_v v}{c})x'_1} e^{ik_2 x'_2} e^{ik_3 x'_3} d\bar{k} \end{aligned}$$

(A)

We make the change of variables, in the first line of (A);

$$k'c = kc\gamma_v + k_1\gamma_v v$$

$$k' = k\gamma_v + \frac{k_1\gamma_v v}{c}$$

$$k'_1 = k_1 \gamma_v + \frac{k \gamma_v v}{c}$$

$$k'_2 = k_2$$

$$k'_3 = k_3$$

and, in the second line of (A);

$$k'c = kc\gamma_v - k_1\gamma_v v$$

$$k' = k\gamma_v - \frac{k_1\gamma_v v}{c}$$

$$k'_1 = k_1\gamma_v - \frac{k\gamma_v v}{c}$$

$$k'_2 = k_2$$

$$k'_3 = k_3$$

which are the Doppler shift transformations given in [7], for the velocity vector $(v, 0, 0)$. Calculating the Jacobian, and using the chain rule, we have that, for the first and second lines of (A) respectively;

$$dk_1 dk_2 dk_3 = \left(\gamma_v - \frac{\gamma_v k'_1 v}{k'c}\right) dk'_1 dk'_2 dk'_3$$

$$dk_1 dk_2 dk_3 = \left(\gamma_v + \frac{\gamma_v k'_1 v}{k'c}\right) dk'_1 dk'_2 dk'_3$$

so that, substituting in (A);

$$\begin{aligned} \rho'_{n,\epsilon}(\bar{x}', t') &= \frac{1}{(2\pi)^3} \int_{\mathcal{R}^3} [\gamma_v A_{n,\epsilon}(\bar{k}') - \frac{\gamma_v v}{c^2} \bar{A}_{1,n,\epsilon}(\bar{k}')] e^{ik'ct'} e^{ik'_1 x'_1} e^{ik'_2 x'_2} e^{ik'_3 x'_3} \\ &\quad \left(\gamma_v - \frac{\gamma_v k'_1 v}{k'c}\right) d\bar{k}' \\ &+ \frac{1}{(2\pi)^3} \int_{\mathcal{R}^3} [\gamma_v B_{n,\epsilon}(\bar{k}') - \frac{\gamma_v v}{c^2} \bar{B}_{1,n,\epsilon}(\bar{k}')] e^{-ik'ct'} e^{ik'_1 x'_1} e^{ik'_2 x'_2} e^{ik'_3 x'_3} \left(\gamma_v + \frac{\gamma_v k'_1 v}{k'c}\right) d\bar{k}' \\ &= \frac{1}{(2\pi)^3} \int_{\mathcal{R}^3} [\gamma_v A_{n,\epsilon}(\bar{k}') - \frac{\gamma_v v}{c^2} \bar{A}_{1,n,\epsilon}(\bar{k}')] e^{ik'ct'} e^{i\bar{k}' \cdot \bar{x}'} \left(\gamma_v - \frac{\gamma_v k'_1 v}{k'c}\right) d\bar{k}' \\ &+ \frac{1}{(2\pi)^3} \int_{\mathcal{R}^3} [\gamma_v B_{n,\epsilon}(\bar{k}') - \frac{\gamma_v v}{c^2} \bar{B}_{1,n,\epsilon}(\bar{k}')] e^{-ik'ct'} e^{i\bar{k}' \cdot \bar{x}'} \left(\gamma_v + \frac{\gamma_v k'_1 v}{k'c}\right) d\bar{k}' \end{aligned}$$

so that equating coefficients, and using the wave representation for $\rho'_{n,\epsilon}(\bar{x}', t')$, we have that;

$$A'_{n,\epsilon}(\bar{k}') = (\gamma_v - \frac{\gamma_v k'_1 v}{k'c}) [\gamma_v A_{n,\epsilon}(\bar{k}') - \frac{\gamma_v v}{c^2} \bar{A}_{1,n,\epsilon}(\bar{k}')]]$$

$$B'_{n,\epsilon}(\bar{k}') = (\gamma_v + \frac{\gamma_v k'_1 v}{k'c}) [\gamma_v B_{n,\epsilon}(\bar{k}') - \frac{\gamma_v v}{c^2} \bar{B}_{1,n,\epsilon}(\bar{k}')] \quad (E)$$

and, using (D), with the inverse relations for the first line;

$$k = k' \gamma_v - \frac{k'_1 \gamma_v v}{c}$$

$$k_1 = k'_1 \gamma_v - \frac{k'_1 \gamma_v v}{c}$$

$$k'_2 = k_2$$

$$k'_3 = k_3$$

$$\lim_{\epsilon \rightarrow 0} A_{n,\epsilon}(\bar{k}') = \frac{q}{2} + \frac{q\bar{p} \cdot \bar{k}}{2kc}$$

$$= \frac{q}{2} + \frac{qp_1(k'_1 \gamma_v - \frac{k'_1 \gamma_v v}{c}) + qp_2 k'_2}{2(k' \gamma_v - \frac{k'_1 \gamma_v v}{c})c}$$

$$\lim_{\epsilon \rightarrow 0} \bar{A}_{1,n,\epsilon}(\bar{k}') = [-\frac{cq}{2k} - \frac{q\bar{p} \cdot \bar{k}}{2k^2}] k_1$$

$$= -\frac{cq(k'_1 \gamma_v - \frac{k'_1 \gamma_v v}{c})}{2(k' \gamma_v - \frac{k'_1 \gamma_v v}{c})} - \frac{qp_1(k'_1 \gamma_v - \frac{k'_1 \gamma_v v}{c})^2 + qp_2 k'_2 (k'_1 \gamma_v - \frac{k'_1 \gamma_v v}{c})}{2(k' \gamma_v - \frac{k'_1 \gamma_v v}{c})^2}$$

and, substituting into the first line of (E);

$$\begin{aligned} \lim_{\epsilon \rightarrow 0} A'_{n,\epsilon}(\bar{k}') &= (\gamma_v - \frac{\gamma_v k'_1 v}{k'c}) [\gamma_v (\frac{q}{2} + \frac{qp_1(k'_1 \gamma_v - \frac{k'_1 \gamma_v v}{c}) + qp_2 k'_2}{2(k' \gamma_v - \frac{k'_1 \gamma_v v}{c})c}) - \frac{\gamma_v v}{c^2} (-\frac{cq(k'_1 \gamma_v - \frac{k'_1 \gamma_v v}{c})}{2(k' \gamma_v - \frac{k'_1 \gamma_v v}{c})} \\ &\quad - \frac{qp_1(k'_1 \gamma_v - \frac{k'_1 \gamma_v v}{c})^2 + qp_2 k'_2 (k'_1 \gamma_v - \frac{k'_1 \gamma_v v}{c})}{2(k' \gamma_v - \frac{k'_1 \gamma_v v}{c})^2})] \end{aligned}$$

Again, using (D), with the inverse relations for the second line;

$$k = k' \gamma_v + \frac{k'_1 \gamma_v v}{c}$$

$$k_1 = k'_1 \gamma_v + \frac{k'_1 \gamma_v v}{c}$$

$$k'_2 = k_2$$

$$k'_3 = k_3$$

$$\lim_{\epsilon \rightarrow 0} B_{n,\epsilon}(\bar{k}') = \frac{q}{2} - \frac{q\bar{p} \cdot \bar{k}}{2kc}$$

$$\begin{aligned}
&= \frac{q}{2} - \frac{qp_1(k'_1\gamma_v + \frac{k'\gamma_{vv}}{c}) + qp_2k'_2}{2(k'_1\gamma_v + \frac{k'_1\gamma_{vv}}{c})c} \\
\lim_{\epsilon \rightarrow 0} \overline{B}_{1,n,\epsilon}(\overline{k}') &= [\frac{cq}{2k} - \frac{q\overline{p}\cdot\overline{k}}{2k^2}]k_1 \\
&= \frac{cq(k'_1\gamma_v + \frac{k'\gamma_{vv}}{c})}{2(k'_1\gamma_v + \frac{k'_1\gamma_{vv}}{c})} - \frac{qp_1(k'_1\gamma_v + \frac{k'\gamma_{vv}}{c})^2 + qp_2k'_2(k'_1\gamma_v + \frac{k'\gamma_{vv}}{c})}{2(k'_1\gamma_v + \frac{k'_1\gamma_{vv}}{c})^2}
\end{aligned}$$

and, substituting into the second line of (E);

$$\begin{aligned}
\lim_{\epsilon \rightarrow 0} B'_{n,\epsilon}(\overline{k}') &= (\gamma_v + \frac{\gamma_v k'_1 v}{k'_1 c}) [\gamma_v (\frac{q}{2} - \frac{qp_1(k'_1\gamma_v + \frac{k'\gamma_{vv}}{c}) + qp_2k'_2}{2(k'_1\gamma_v + \frac{k'_1\gamma_{vv}}{c})c}) - \frac{\gamma_v v}{c^2} (\frac{cq(k'_1\gamma_v + \frac{k'\gamma_{vv}}{c})}{2(k'_1\gamma_v + \frac{k'_1\gamma_{vv}}{c})}) \\
&\quad - \frac{qp_1(k'_1\gamma_v + \frac{k'\gamma_{vv}}{c})^2 + qp_2k'_2(k'_1\gamma_v + \frac{k'\gamma_{vv}}{c})}{2(k'_1\gamma_v + \frac{k'_1\gamma_{vv}}{c})^2}]
\end{aligned}$$

Rearranging, we have that;

$$\begin{aligned}
\lim_{\epsilon \rightarrow 0} A'_{n,\epsilon}(\overline{k}') &= q\gamma_v^2 (1 - \frac{k'_1 v}{k'_1 c}) [(\frac{1}{2} + \frac{p_1 k'_1 (1 - \frac{k'_1 v}{k'_1 c}) + \frac{p_2}{\gamma_v} k'_2}{2k'(1 - \frac{k'_1 v}{k'_1 c})c}) - \frac{v}{c^2} (-\frac{ck'_1(1 - \frac{k'_1 v}{k'_1 c})}{2k'(1 - \frac{k'_1 v}{k'_1 c})} \\
&\quad - \frac{p_1 k_1'^2 (1 - \frac{k'_1 v}{k'_1 c})^2 + \frac{p_2}{\gamma_v} k'_1 k'_2 (1 - \frac{k'_1 v}{k'_1 c})}{2k'^2 (1 - \frac{k'_1 v}{k'_1 c})^2}] \\
&= q\gamma_v^2 (1 - \frac{k'_1 v}{k'_1 c}) [\frac{1}{2} + \frac{p_1 k'_1 (1 - \frac{k'_1 v}{k'_1 c})}{2k'(1 - \frac{k'_1 v}{k'_1 c})c} + \frac{v k'_1 (1 - \frac{k'_1 v}{k'_1 c})}{2k'(1 - \frac{k'_1 v}{k'_1 c})c} + \frac{vp_1 k_1'^2 (1 - \frac{k'_1 v}{k'_1 c})^2}{2k'^2 c^2 (1 - \frac{k'_1 v}{k'_1 c})^2}] \\
&\quad + q\gamma_v^2 (1 - \frac{k'_1 v}{k'_1 c}) [\frac{\frac{p_2}{\gamma_v} k'_2}{2k'(1 - \frac{k'_1 v}{k'_1 c})c} + \frac{\frac{p_2}{\gamma_v} k'_1 k'_2 v (1 - \frac{k'_1 v}{k'_1 c})}{2k'^2 c^2 (1 - \frac{k'_1 v}{k'_1 c})^2}] \\
&= \frac{q\gamma_v^2}{2} (1 - \frac{k'_1 v}{k'_1 c}) [1 + \frac{v k'_1 (1 - \frac{k'_1 v}{k'_1 c})}{k'(1 - \frac{k'_1 v}{k'_1 c})c}] [1 + \frac{p_1 k'_1 (1 - \frac{k'_1 v}{k'_1 c})}{k'(1 - \frac{k'_1 v}{k'_1 c})c}] + q\gamma_v^2 [\frac{\frac{p_2}{\gamma_v} k'_2}{2k'c} + \frac{\frac{p_2}{\gamma_v} k'_1 k'_2 v (1 - \frac{k'_1 v}{k'_1 c})}{2k'^2 c^2 (1 - \frac{k'_1 v}{k'_1 c})}] \\
&= \frac{q}{2} \frac{(1 - \frac{vp_1}{c^2}) [1 + \frac{k'_1}{k'_1 c} p'_1]}{(1 - \frac{k'_1 v}{k'_1 c})} + \frac{q\gamma_v^2}{2} \frac{(1 - \frac{vp_1}{c^2}) p'_2 k'_2}{k'c} [1 + \frac{v k'_1 (1 - \frac{k'_1 v}{k'_1 c})}{k'(1 - \frac{k'_1 v}{k'_1 c})c}] \\
&= \frac{q}{2} \frac{(1 - \frac{vp_1}{c^2}) [1 + \frac{k'_1}{k'_1 c} p'_1]}{(1 - \frac{k'_1 v}{k'_1 c})} + \frac{q\gamma_v^2}{2} \frac{(1 - \frac{vp_1}{c^2}) p'_2 k'_2}{k'c} [\frac{(1 - \frac{v^2}{c^2})}{(1 - \frac{k'_1 v}{k'_1 c})}] \\
&= \frac{q}{2} \frac{(1 - \frac{vp_1}{c^2}) [1 + \frac{k'_1}{k'_1 c} p'_1]}{(1 - \frac{k'_1 v}{k'_1 c})} + \frac{q}{2} \frac{(1 - \frac{vp_1}{c^2}) \frac{k'_2}{k'_1 c} p'_2}{(1 - \frac{k'_1 v}{k'_1 c})} \\
&= \frac{q}{2} \frac{(1 - \frac{vp_1}{c^2}) [1 + \frac{\overline{p}' \cdot \overline{k}'}{k'_1 c}]}{(1 - \frac{k'_1 v}{k'_1 c})} \\
&= \frac{q}{2} \frac{\frac{\gamma - \overline{p} \cdot \overline{v}}{\gamma p \gamma v} \lim_{\epsilon \rightarrow 0} A''_{n,\epsilon}}{(1 - \frac{k'_1 v}{k'_1 c})} \\
&= \frac{q}{2} \frac{\lim_{\epsilon \rightarrow 0} A''_{n,\epsilon}(\overline{k}')}{\gamma_v (1 - \frac{k'_1 v}{k'_1 c})}
\end{aligned}$$

$$= \frac{q'}{2} \lim_{\epsilon \rightarrow 0} A''_{n,\epsilon}(\bar{k}') \frac{k'}{k}$$

and, using the symmetries;

$$\lim_{\epsilon \rightarrow 0} A''_{n,\epsilon}(\bar{k}')(\bar{p}) = \lim_{\epsilon \rightarrow 0} B''_{n,\epsilon}(\bar{k}')(-\bar{p})$$

$$\lim_{\epsilon \rightarrow 0} A'_{n,\epsilon}(\bar{k}')(v) = \lim_{\epsilon \rightarrow 0} B'_{n,\epsilon}(\bar{k}')(-v)$$

we have;

$$\lim_{\epsilon \rightarrow 0} B'_{n,\epsilon}(\bar{k}') = \frac{q'}{2} \lim_{\epsilon \rightarrow 0} B''_{n,\epsilon}(\bar{k}') \frac{k'}{k}$$

where $q' = q \frac{\gamma_{\bar{p}^* - \bar{v}}}{\gamma_p}$ is the new charge in S' due to the change in mass of the electron, $m' = \frac{m \gamma_{\bar{p}^* - \bar{v}}}{\gamma_w}$, $\{A''_{n,\epsilon}(\bar{k}'), B''_{n,\epsilon}(\bar{k}')\}$ are the factors in the wave equation for an electron moving with the transformed velocity $\bar{p}' = (\frac{p_1 - v}{1 - \frac{vp_1}{c^2}}, \frac{p_2}{\gamma_v(1 - \frac{vp_1}{c^2})}, 0)$ in S_2 and $\frac{k'}{k}$ is the background Doppler shift in amplitude, transferring from S_1 to S_2 . Now complete the proof back to S' using Lemma 0.14.

□

REFERENCES

- [1] Electromagnetic Fields and Waves, Third Edition, Dale Corson, Francois Lorrain and Paul Lorrain, W.H Freeman and Company, (1988).
- [2] Partial Differential Equations, 2nd Edition, L. Evans, AMS.
- [3] Some Arguments for the Wave Equation in Quantum Theory, Tristram de Piro, Open Journal of Mathematical Sciences, also available at <http://www.curvalinea.net> (58) or Accademia, (2021).
- [4] Some Arguments for the Wave Equation in Quantum Theory 6: Waves, Current and Charge, Tristram de Piro, available at <http://www.curvalinea.net> (70) or Accademia, (2022).
- [5] Some Arguments for the Wave Equation in Quantum Theory 10, Tristram de Piro, available at <http://www.curvalinea.net> (83), (2026).
- [6] Some Arguments for the Wave Equation in Quantum Theory 2, Tristram de Piro, available at <http://www.curvalinea.net> (59), (2022).
- [7] Introduction to Special Relativity, Wolfgang Rindler, Oxford Science Publications, (1991).

FLAT 3, REDESDALE HOUSE, 85 THE PARK, CHELTENHAM, GL50 2RP
E-mail address: `t.depiro@curvalinea.net`