

CONSERVATION OF ENERGY IN ATOMIC SYSTEMS

TRISTRAM DE PIRO

ABSTRACT.

Lemma 0.1. *Let $(\rho, \bar{J}, \bar{E}, \bar{B})$ be an electromagnetic system, satisfying Maxwell's equations on $\mathcal{R}^4$, then if the system is in thermal equilibrium and constant temperature, with U_{mech} the mechanical energy density, then;*

$$\nabla(U_{mech}) = \frac{\bar{f}}{3}$$

where $\bar{f} = \rho\bar{E} + \bar{J} \times \bar{B}$ is force density.

Proof. By the first law of thermodynamics;

$$dU_{mech} = dU = dQ - pdV = -pdxdydz$$

where p is electromagnetic pressure, dV is an element of volume and we have used the fact that temperature is constant and no heat is released to show that $dQ = 0$. It follows that;

$$\frac{\partial U_{mech}}{\partial x} = \frac{dU}{dx} = -pdydz$$

where p is the pressure on the wall $dydz$. By the definition of pressure as $p = \frac{F}{A}$, where F is the force exerted by the system and A is area, we have that $p = -\frac{\bar{f}_1}{3dydz}$, where 3 is included to model the fact that if the particles are in thermal equilibrium, they are in random motion like an ideal gas, and a third of the particles will exert a force on the wall $dydz$. It follows that;

$$\frac{\partial U_{mech}}{\partial x} = \frac{\bar{f}_1}{3dydz}dydz = \frac{\bar{f}_1}{3}$$

The argument for the partial derivatives $\{\frac{\partial U_{mech}}{\partial y}, \frac{\partial U_{mech}}{\partial z}\}$ is the same, which gives the result by the definition of $\nabla(U_{mech})$. $\square$

Definition 0.2. *By an atomic system, we mean charge and current $(\rho, \bar{J})$ on $\mathcal{R}^4$, satisfying the continuity equation;*

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \bar{J} = 0$$

confined to a bounded volume V , with $\bar{J}|_{\partial V} = \bar{0}$ in the base frame S . For the purposes of this paper, we restrict ourselves to the case when $V = B(\bar{0}, r)$, $r > 0$, with $\delta V = S(\bar{0}, r)$. We remark that we constructed such $(\rho, \bar{J})$ in [2] and [6], with $(\rho, \bar{J})$, satisfying additional wave like relations.

We say that the system is non radiating if for every volume $U_{t,\bar{v}}$ inside $V_{t,\bar{v}}$, travelling with velocity $-\bar{v}$ in an inertial frame $S_{\bar{v}}$, with velocity $\bar{v}$ relative to S , there exist electromagnetic fields $(\bar{E}_{\bar{v}}, \bar{B}_{\bar{v}})$ on $V_{t,\bar{v}}$, satisfying Maxwell's equations, with the property that;

$$\frac{d}{dt} \int_{U_{t,\bar{v}}} (U_{el} + U_{mech}) dV_{t,\bar{v}} = 0$$

where $U_{el} = \frac{1}{2}(\epsilon_0 E_{\bar{v}}^2 + \frac{1}{\mu_0} B_{\bar{v}}^2)$ is the energy stored in the fields and $E_{\bar{v}} = |\bar{E}_{\bar{v}}|$, $B_{\bar{v}} = |\bar{B}_{\bar{v}}|$.

Intuitively, the condition means, that energy is conserved and exchanged between the mechanical energy of electrons and the energy stored in the fields, with no flux of energy within the moving volume itself or to the outside.

Lemma 0.3. *A non radiating atomic system as in Definition 0.2 satisfies the wave like relations of [2], further noted at the beginning of [4].*

Proof. By the Reynolds transport theorem, Poynting's theorem, the divergence theorem and elementary vector identities, we have that;

$$\begin{aligned} \frac{d}{dt} \int_{U_{t,\bar{v}}} (U_{el} + U_{mech}) dV_{t,\bar{v}} &= \int_{U_{t,\bar{v}}} \left(\frac{\partial U_{el}}{\partial t} + \frac{\partial U_{mech}}{\partial t} \right) dV_{t,\bar{v}} \\ &+ \int_{\delta U_{t,\bar{v}}} \hat{n} \cdot (-\bar{v})(U_{el} + U_{mech}) d\delta U_{t,\bar{v}} \\ &= \int_{U_{t,\bar{v}}} -\frac{1}{\mu_0} (\nabla \cdot \bar{E}_{\bar{v}} \times \bar{B}_{\bar{v}}) dV_{t,\bar{v}} + \int_{U_{t,\bar{v}}} \nabla \cdot (-\bar{v}(U_{el} + U_{mech})) dV_{t,\bar{v}} \\ &= \int_{U_{t,\bar{v}}} -\frac{1}{\mu_0} (\nabla \cdot \bar{E}_{\bar{v}} \times \bar{B}_{\bar{v}}) - \bar{v} \cdot \nabla (U_{el} + U_{mech}) dV_{t,\bar{v}} \end{aligned}$$

$$= 0$$

so that, as $U_{t,\bar{v}}$ was arbitrary inside $V_{t,\bar{v}}$, we must have that, for every inertial frame $S_{\bar{v}}$, there exist $(\bar{E}_{\bar{v}}, \bar{B}_{\bar{v}})$ on $V_{t,\bar{v}}$ such that;

$$-\frac{1}{\mu_0}(\nabla \cdot \bar{E}_{\bar{v}} \times \bar{B}_{\bar{v}}) - \bar{v} \cdot \nabla(U_{el} + U_{mech}) = 0 \text{ (CC)}$$

so that, using results of [5] and [6], we have that in the frames $\{S_{1,\infty}, S_{2,\infty}, S_{3,\infty}\}$, there exist;

$$\{(\bar{E}_{1,\infty}, \bar{B}_{1,\infty}), (\bar{E}_{2,\infty}, \bar{B}_{2,\infty}), (\bar{E}_{3,\infty}, \bar{B}_{3,\infty})\}$$

with corresponding;

$$\{(\rho_{1,\infty}, \bar{J}_{1,\infty}), (\rho_{2,\infty}, \bar{J}_{2,\infty}), (\rho_{3,\infty}, \bar{J}_{3,\infty})\}$$

$$\{U_{el,1,\infty}, U_{mech,1,\infty}, U_{el,2,\infty}, U_{mech,2,\infty}, U_{el,3,\infty}, U_{mech,3,\infty}\}$$

on;

$$\{V_{t_{\infty},1,\infty}, V_{t_{\infty},2,\infty}, V_{t_{\infty},3,\infty}\}$$

such that;

$$-\frac{1}{\mu_0}(\nabla_{1,\infty} \cdot \bar{E}_{1,\infty} \times \bar{B}_{1,\infty}) - \bar{e}_{1,\infty} \cdot \nabla_{1,\infty}(U_{el,1,\infty} + U_{mech,1,\infty}) = 0$$

$$-\frac{1}{\mu_0}(\nabla_{2,\infty} \cdot \bar{E}_{2,\infty} \times \bar{B}_{2,\infty}) - \bar{e}_{2,\infty} \cdot \nabla_{2,\infty}(U_{el,2,\infty} + U_{mech,2,\infty}) = 0$$

$$-\frac{1}{\mu_0}(\nabla_{3,\infty} \cdot \bar{E}_{3,\infty} \times \bar{B}_{3,\infty}) - \bar{e}_{3,\infty} \cdot \nabla_{3,\infty}(U_{el,3,\infty} + U_{mech,3,\infty}) = 0$$

It follows that, noting that, in the above equations, the left hand term is finite, while the right hand term is infinite unless, in the frames $\{S_{1,\infty}, S_{2,\infty}, S_{3,\infty}\}$ respectively, we have that;

$$(\nabla_{1,\infty}(U_{el,1,\infty} + U_{mech,1,\infty}))_1 = 0$$

$$(\nabla_{2,\infty}(U_{el,2,\infty} + U_{mech,2,\infty}))_2 = 0$$

$$(\nabla_{3,\infty}(U_{el,3,\infty} + U_{mech,3,\infty}))_3 = 0$$

Using Lemma 0.1, we obtain, in the frames $\{S_{1,\infty}, S_{2,\infty}, S_{3,\infty}\}$ respectively;

$$\nabla_{1,\infty}(U_{el,1,\infty}) + \frac{1}{3}(\bar{f}_{1,\infty})_1 = 0$$

$$\nabla_{2,\infty}(U_{el,2,\infty}) + \frac{1}{3}(\bar{f}_{2,\infty})_2 = 0$$

$$\nabla_{3,\infty}(U_{el,3,\infty}) + \frac{1}{3}(\bar{f}_{3,\infty})_3 = 0$$

and, by the transformation rules for infinite boosts, given in Lemma 0.2 of [6], the transformation of $\bar{f}$ to $(\bar{E}, \bar{J})$ in the base frame S , given in Lemma 0.2 of [6], and noting the improvement in Lemma 0.5 of [6], we obtain, in the base frame S , $(\bar{E}, \bar{B})$ such that;

$$-\frac{i}{c}\frac{\partial}{\partial t}(\frac{1}{2}(\epsilon_0|(e_1, icb_3, -icb_2)|^2 + \frac{1}{\mu_0}|(b_1, -\frac{i}{c}e_3, \frac{i}{c}e_2)|^2)) + \frac{i}{3c}(\bar{E}, \bar{J}) = 0$$

$$-\frac{i}{c}\frac{\partial}{\partial t}(\frac{1}{2}(\epsilon_0|(-icb_3, e_2, icb_1)|^2 + \frac{1}{\mu_0}|(\frac{i}{c}e_3, b_2, -\frac{i}{c}e_1)|^2)) + \frac{i}{3c}(\bar{E}, \bar{J}) = 0$$

$$-\frac{i}{c}\frac{\partial}{\partial t}(\frac{1}{2}(\epsilon_0|(icb_2, -icb_1, e_3)|^2 + \frac{1}{\mu_0}|(-\frac{i}{c}e_2, \frac{i}{c}e_1, b_3)|^2)) + \frac{i}{3c}(\bar{E}, \bar{J}) = 0$$

and, expanding, we obtain;

$$-\frac{i}{c}\frac{\partial}{\partial t}(\frac{1}{2}(\epsilon_0(e_1^2 - c^2(b_2^2 + b_3^2)) + \frac{1}{\mu_0}(b_1^2 - \frac{1}{c^2}(e_2^2 + e_3^2)))) + \frac{i}{3c}(\bar{E}, \bar{J}) = 0$$

$$-\frac{i}{c}\frac{\partial}{\partial t}(\frac{1}{2}(\epsilon_0(e_2^2 - c^2(b_1^2 + b_3^2)) + \frac{1}{\mu_0}(b_2^2 - \frac{1}{c^2}(e_1^2 + e_3^2)))) + \frac{i}{3c}(\bar{E}, \bar{J}) = 0$$

$$-\frac{i}{c}\frac{\partial}{\partial t}(\frac{1}{2}(\epsilon_0(e_3^2 - c^2(b_1^2 + b_2^2)) + \frac{1}{\mu_0}(b_3^2 - \frac{1}{c^2}(e_1^2 + e_2^2)))) + \frac{i}{3c}(\bar{E}, \bar{J}) = 0$$

Adding the relations, we obtain that;

$$-\frac{i}{c}\frac{d}{dt}(U_{el} - 2U_{el} - U_{mech}) = 0$$

in the base frame S , so that;

$$\frac{d}{dt}(U_{el} + U_{mech}) = 0$$

and, using Poynting's theorem;

$$\nabla \cdot \bar{E} \times \bar{B} = 0$$

If the atomic system is non radiating, it is an exercise to show that it is non radiating in $S_{\bar{v}}$, travelling with velocity $\bar{v}$ relative to S . We, therefore, conclude that for any inertial frame $S_{\bar{v}}$, there exist electromagnetic fields $(\bar{E}_{\bar{v}}, \bar{B}_{\bar{v}})$, on the moving volume $V_{t,\bar{v}}$, such that;

$$\nabla_{\bar{v}} \cdot \bar{E}_{\bar{v}} \times \bar{B}_{\bar{v}} = 0$$

By the main result in [5], this implies there exists $(\bar{E}_{\bar{v}}, \bar{B}_{\bar{v}})$, on the moving volume $V_{t,\bar{v}}$, with $\bar{B}_{\bar{v}} = \bar{0}$. By the results of [2], this implies, that the original $(\rho, \bar{J})$ in the base frame S satisfy the wave like relations explained there and at the beginning of [4]. $\square$

Definition 0.4. We recall the example from [6], where $(\rho, \bar{J}, \bar{E}, \bar{B})$ constitute an atomic system, confined to the volume $B(r_0)$, with $\bar{J}|_{S(r_0)} = \bar{0}$;

$$\rho(r, \theta, \phi, t) = \frac{\sin(kr)}{r} e^{ikct}$$

$$\bar{J}(r, \theta, \phi, t) = \frac{ic}{k} \frac{kr \cos(kr) - \sin(kr)}{r^2} e^{ikct} \hat{r}$$

$$\bar{E} = \frac{i}{\epsilon_0 k c} \bar{J}$$

$$\bar{B} = \bar{0}$$

$$\text{and } \tan(kr_0) = kr_0$$

Lemma 0.5. The system given by $(\text{Re}(\rho), \text{Re}(\bar{J}), \text{Re}(\bar{E}), \text{Re}(\bar{B}))$ where $(\rho, \bar{J}, \bar{E}, \bar{B})$ is given by Definition 0.4 is non radiating.

Proof. By the method of [2], we can find $(\bar{E}_0, \bar{B}_0)$ such that if $(\text{Re}(\bar{E})_{\bar{v}}, \text{Re}(\bar{B})_{\bar{v}})$ is the transfer to the inertial frame $S_{\bar{v}}$, then $(\bar{E}_0, \bar{B}_0)$ is a solution to Maxwell's equation in vacuum on $S_{\bar{v}}$ and;

$$\nabla \times (\text{Re}(\bar{E})_{\bar{v}} - \bar{E}_0) = \bar{0}$$

on $S_{\bar{v}}$, so that $\text{Re}(\bar{B})_{\bar{v}} - \bar{B}_0$ is magnetostatic, by Maxwell's equations, and as;

$$\square^2(\text{Re}(\bar{B})_{\bar{v}}) = \square^2(\bar{B}_0) = \bar{0}$$

we must have that $Re(\overline{B})_{\overline{v}} - \overline{B}_0$ is harmonic on $\mathcal{R}^4$, not just on the moving volume $V_{t,\overline{v}}$, and, verifying the boundary conditions, see calculation below, $Re(\overline{B})_{\overline{v}} - \overline{B}_0 = \overline{0}$. This is enough to obtain the condition, that for every inertial frame $S_{\overline{v}}$, there exist real $(\overline{E}'_{\overline{v}}, \overline{B}'_{\overline{v}})$ such that $(Re(\rho)_{\overline{v}}, Re(\overline{J})_{\overline{v}}, \overline{E}'_{\overline{v}}, \overline{B}'_{\overline{v}})$ satisfy Maxwell's equations with;

$$\nabla_{\overline{v}}(\overline{E}'_{\overline{v}} \times \overline{B}'_{\overline{v}}) = \overline{0}.$$

We are now in the context of [5], and we use the example constructed there, without imposing the boundary conditions. By (CC) in Lemma 0.3, it is necessary to verify the condition that there exist $(\overline{E}''_{\overline{v}}, \overline{B}''_{\overline{v}})$ on $V_{t,\overline{v}}$ such that;

$$-\frac{1}{\mu_0}(\nabla_{\overline{v}} \cdot \overline{E}''_{\overline{v}} \times \overline{B}''_{\overline{v}}) - \overline{v} \cdot \nabla(U_{el} + U_{mech}) = 0 \quad (*)$$

We assume for now that $\overline{v} = v\overline{e}_2$ and let $S_{\overline{v}}$ be the base frame used in [5]. Then, we can find $\overline{B}''_{\overline{v}}$, not necessarily vanishing on $V_{t,\overline{v}}$ but with $\nabla \cdot (\overline{E}''_{\overline{v}} \times \overline{B}''_{\overline{v}}) = \overline{0}$, so that we just have to verify, from (*), dropping the subscript $\overline{v}$, that;

$$\overline{v} \cdot \nabla(U_{el} + U_{mech}) = 0$$

or equivalently;

$$(\nabla(U_{el} + U_{mech}))_2 = 0$$

and, by the Lemma 0.1, that;

$$(\nabla(U_{el}) + \frac{\overline{f}}{3})_2 = 0$$

This will follow from;

$$(\nabla(U_{el}))_2 = (\overline{f})_2 = 0$$

We calculate, using elementary vector identities, the product rule, $\frac{\partial}{\partial t}(\overline{E} \times \overline{B})_2 = \frac{\partial g_2}{\partial t} = 0$, $\frac{\partial p_{22}}{\partial y} = \frac{\partial p_{12}}{\partial x} = \frac{\partial p_{23}}{\partial z} = 0$, from [5];

$$\begin{aligned} (\nabla(U_{el}))_2 &= \frac{e_0}{2}(\nabla(e^2 + c^2 b^2))_2 \\ &= \frac{e_0}{2}(2\overline{E} \times (\nabla \times \overline{E}) + 2c^2 \overline{B} \times (\nabla \times \overline{B}) + 2(\overline{E} \cdot \nabla)\overline{E} + 2c^2(\overline{B} \cdot \nabla)\overline{B})_2 \end{aligned}$$

$$\begin{aligned}
 &= \frac{\epsilon_0}{2} (-2\bar{E} \times \frac{\partial \bar{B}}{\partial t} + 2c^2 \bar{B} \times (\mu_0 \bar{J} + \frac{1}{c^2} \frac{\partial \bar{E}}{\partial t})) + 2(\bar{E} \cdot \nabla) \bar{E} + 2c^2 (\bar{B} \cdot \nabla) \bar{B})_2 \\
 &= \frac{\epsilon_0}{2} (\frac{2}{\epsilon_0} (\bar{B} \times \bar{J}) + 2 \frac{\partial}{\partial t} (\bar{B} \times \bar{E}) + 2(\bar{E} \cdot \nabla) \bar{E} + 2c^2 (\bar{B} \cdot \nabla) \bar{B})_2 \\
 &= (\bar{B} \times \bar{J} + \epsilon_0 (\bar{E} \cdot \nabla) \bar{E} + \frac{1}{\mu_0} (\bar{B} \cdot \nabla) \bar{B})_2 \\
 &= (\bar{B} \times \bar{J})_2 + \epsilon_0 (e_1 e_{2x} + e_2 e_{2y} + e_3 e_{2z}) + \frac{1}{\mu_0} (b_1 b_{2x} + b_2 b_{2y} + b_3 b_{2z}) \\
 &= (\bar{B} \times \bar{J})_2 + \epsilon_0 (-e_1 e_{2x} + e_2 e_{2y} - e_3 e_{2z}) + \epsilon_0 ((e_1 e_2)_x + (e_2 e_3)_z) \\
 &\quad + \frac{1}{\mu_0} (-b_1 b_{2x} + b_2 b_{2y} - b_3 b_{2z}) + \frac{1}{\mu_0} ((b_1 b_2)_x + (b_2 b_3)_z) \\
 &= -(\bar{J} \times \bar{B})_2 - \epsilon_0 (e_2 e_{1x} + e_2 e_{2y} + e_2 e_{3z}) + \epsilon_0 ((e_1 e_2)_x + (e_2 e_3)_z) + \epsilon_0 (e_2^2)_y \\
 &\quad - \frac{1}{\mu_0} (b_2 b_{1x} + b_2 b_{2y} + b_2 b_{3z}) + \frac{1}{\mu_0} ((b_1 b_2)_x + (b_2 b_3)_z) + \frac{1}{\mu_0} (b_2^2)_y \\
 &= -(\bar{J} \times \bar{B})_2 - (\epsilon_0 (\nabla \cdot \bar{E}) \bar{E})_2 - \frac{1}{\mu_0} (\nabla \cdot \bar{B}) \bar{B}_2 + \frac{\epsilon_0}{2} (e^2)_y + \frac{1}{\mu_0} (b^2)_y \\
 &= -(\bar{J} \times \bar{B})_2 - (\rho \bar{E})_2 + \frac{\epsilon_0}{2} (e^2)_y + \frac{1}{\mu_0} (b^2)_y \\
 &= -\bar{f}_2 + (\nabla(U_{el}))_2
 \end{aligned}$$

so that $\bar{f}_2 = 0$, and $(\nabla(U_{el}))_2 = 0$ follows from $\frac{\partial \sigma}{\partial y} = 0$ in the paper [5].

..... We require the spectral representation of $(\rho, \bar{J})$, given in [1];

$$\rho(\bar{x}, t) = \int_{S(\bar{0}, k)} \gamma(k) e^{i\bar{k} \cdot \bar{x}} e^{ikct} dS(k)$$

$$\bar{J}(\bar{x}, t) = \int_{S(\bar{0}, k)} \beta(k) \bar{k} e^{i\bar{k} \cdot \bar{x}} e^{ikct} dS(k)$$

$$\text{where } \gamma(k) = \frac{1}{4\pi k} \text{ and } \beta(k) = -\frac{c}{k} \gamma(k) = -\frac{c}{4\pi k^2}.$$

The spectral representation of $\bar{E}$ is then given by;

$$\bar{E}(\bar{x}, t) = \int_{S(\bar{0}, k)} \delta(k) \bar{k} e^{i\bar{k} \cdot \bar{x}} e^{ikct} dS(k) \quad (*)$$

$$\text{where } \delta(k) = \frac{i}{\epsilon_0 k c} \beta(k) = -\frac{i}{4\pi \epsilon_0 k^3}$$

We find the spectral representation of $\overline{E}'_{\bar{v}}$, obtained as the transfer of $\overline{E}$ to the frame $S_{\bar{v}}$, where $\bar{v} = (v, 0, 0)$. By the transfer rules;

$$\begin{aligned}\overline{E}'_{\bar{v}} &= \overline{E}_{||} + \gamma_v(\overline{E}_{\perp} + \bar{v} \times \overline{B}) \\ &= \overline{E}_{||} + \gamma_v \overline{E}_{\perp} \\ &= (e_1, \gamma_v e_2, \gamma_v e_3) \\ &= (e'_1, e'_2, e'_3)\end{aligned}$$

so that, with;

$$t = \gamma_v(t' + \frac{vx'}{c^2}) \quad x = \gamma_v(x' + vt'), \quad y = y', \quad z = z'$$

and, using (*);

$$e'_1 = -\frac{i}{4\pi\epsilon_0 k^3} \int_{S(\bar{0}, k)} k_1 e^{ik_1 \gamma_v(x' + vt')} e^{ik_2 y'} e^{ik_3 z'} e^{ikc\gamma_v(t' + \frac{vx'}{c^2})} dS(k)$$

and, using the Doppler transformation;

$$k'_1 = k_1 \gamma_v + \frac{k\gamma_v v}{c}, \quad k' = \frac{k_1 \gamma_v v}{c} + k\gamma_v$$

with inverse;

$$\begin{aligned}k_1 &= k'_1 \gamma_v - \frac{k'\gamma_v v}{c}, \quad k = -\frac{k'_1 \gamma_v v}{c} + k'\gamma_v \\ e'_1 &= -\frac{i}{4\pi\epsilon_0 k^3} \int_{Ell(\bar{0}, k')} k_1 e^{i\vec{k}' \cdot \vec{x}'} e^{ik'ct'} \mu(\vec{k}) dEll(k') \quad (B)\end{aligned}$$

where $\mu(\vec{k})$ is the change in measure and $Ell(\bar{0}, k')$ is the image of $S(\bar{0}, k)$ under the Doppler transformation. To compute the change of measure, we use polar coordinates;

$$(k \sin(\theta) \cos(\phi), k \sin(\theta) \sin(\phi), k \cos(\theta))$$

on the sphere $S(\bar{0}, k)$, then, composing with the Doppler transformation;

$$(k \sin(\theta) \cos(\phi) \gamma_v + \frac{k\gamma_v v}{c}, k \sin(\theta) \sin(\phi), k \cos(\theta))$$

are coordinates on $Ell(\bar{0}, k')$. For these coordinates, we have;

$$\bar{r}'_\theta = (k \cos(\theta) \cos(\phi) \gamma_v, k \cos(\theta) \sin(\phi), -k \sin(\theta))$$

$$\bar{r}'_\phi = (-k \sin(\theta) \sin(\phi) \gamma_v, k \sin(\theta) \cos(\phi), 0)$$

$$|\bar{r}'_\theta \times \bar{r}'_\phi| = |(k^2 \sin^2(\theta) \cos(\phi), k^2 \sin^2(\theta) \sin(\phi) \gamma_v, k^2 \sin(\theta) \cos(\theta) \gamma_v)|$$

$$= |k^2 \sin(\theta)| \sqrt{\sin^2(\theta) \cos^2(\phi) + \sin^2(\theta) \sin^2(\phi) \gamma_v^2 + \cos^2(\theta) \gamma_v^2}$$

$$= |k^2 \sin(\theta)| \sqrt{\frac{k_1^2}{k^2} + \frac{k_2^2 \gamma_v^2}{k^2} + \frac{k_3^2 \gamma_v^2}{k^2}}$$

$$= |k \sin(\theta) \gamma_v| \sqrt{\frac{k_1^2}{\gamma_v^2} + k_2^2 + k_3^2}$$

$$= |k \sin(\theta) \gamma_v| \sqrt{k_1^2 (1 - \frac{v^2}{c^2}) + k_2^2 + k_3^2}$$

$$= |k \sin(\theta) \gamma_v| \sqrt{k^2 - \frac{k_1^2 v^2}{c^2}}$$

$$= |k^2 \sin(\theta) \gamma_v| \sqrt{1 - \frac{k_1^2 v^2}{k^2 c^2}}$$

so that $\mu(\bar{k}) = \frac{1}{\gamma_v \sqrt{1 - \frac{k_1^2 v^2}{k^2 c^2}}}$, and, from (B), we have that;

$$\begin{aligned} e'_1 &= -\frac{i}{4\pi\epsilon_0 k^3} \int_{Ell(\bar{0}, k')} k_1 e^{i\bar{k}' \cdot \bar{x}'} e^{ik' ct'} \frac{1}{\gamma_v \sqrt{1 - \frac{k_1^2 v^2}{k^2 c^2}}} dEll(k') \\ &= -\frac{i}{4\pi\epsilon_0 k^3} \int_{Ell(\bar{0}, k')} [k'_1 \gamma_v - \frac{k' \gamma_v v}{c}] e^{i\bar{k}' \cdot \bar{x}'} e^{ik' ct'} \frac{1}{\gamma_v \sqrt{1 - \frac{[k'_1 \gamma_v - \frac{k' \gamma_v v}{c}]^2 v^2}{[-\frac{k'_1 \gamma_v v}{c} + k' \gamma_v]^2 c^2}}} dEll(k') \end{aligned}$$

and similarly;

$$e'_2 = -\frac{i}{4\pi\epsilon_0 k^3} \int_{Ell(\bar{0}, k')} [k'_2 \gamma_v] e^{i\bar{k}' \cdot \bar{x}'} e^{ik' ct'} \frac{1}{\gamma_v \sqrt{1 - \frac{[k'_1 \gamma_v - \frac{k' \gamma_v v}{c}]^2 v^2}{[-\frac{k'_1 \gamma_v v}{c} + k' \gamma_v]^2 c^2}}} dEll(k')$$

$$e'_3 = -\frac{i}{4\pi\epsilon_0 k^3} \int_{Ell(\bar{0}, k')} [k'_3 \gamma_v] e^{i\bar{k}' \cdot \bar{x}'} e^{ik' ct'} \frac{1}{\gamma_v \sqrt{1 - \frac{[k'_1 \gamma_v - \frac{k' \gamma_v v}{c}]^2 v^2}{[-\frac{k'_1 \gamma_v v}{c} + k' \gamma_v]^2 c^2}}} dEll(k')$$

$$\bar{E}'_v = -\frac{i}{4\pi\epsilon_0 k^3} \int_{Ell(\bar{0}, k')} \bar{k}' e^{i\bar{k}' \cdot \bar{x}'} e^{ik' ct'} \frac{1}{\sqrt{1 - \frac{[k'_1 \gamma_v - \frac{k' \gamma_v v}{c}]^2 v^2}{[-\frac{k'_1 \gamma_v v}{c} + k' \gamma_v]^2 c^2}}} dEll(k')$$

$$-\frac{i}{4\pi\epsilon_0 k^3} \int_{Ell(\bar{0}, k')} \left(-\frac{k'\gamma_v v}{c}, 0, 0\right) e^{i\bar{k}' \cdot \bar{x}'} e^{ik' ct'} \frac{1}{\gamma_v \sqrt{1 - \frac{[k'_1 \gamma_v - \frac{k'\gamma_v v}{c}]^2 v^2}{[-\frac{k'_1}{c} \gamma_v v + k'\gamma_v]^2 c^2}}} dEll(k')$$

Then;

$$\begin{aligned} Re(\bar{E})'_{\bar{v}} &= Re(\bar{E}'_{\bar{v}}) \\ &= \frac{1}{4\pi\epsilon_0 k^3} \int_{Ell(\bar{0}, k')} \bar{k}' [\cos(\bar{k}' \cdot \bar{x}') \sin(k' ct') + \sin(\bar{k}' \cdot \bar{x}') \cos(k' ct')] \frac{1}{\sqrt{1 - \frac{[k'_1 \gamma_v - \frac{k'\gamma_v v}{c}]^2 v^2}{[-\frac{k'_1}{c} \gamma_v v + k'\gamma_v]^2 c^2}}} dEll(k') \\ &\quad + \frac{1}{4\pi\epsilon_0 k^3} \int_{Ell(\bar{0}, k')} \left(-\frac{k'\gamma_v v}{c}, 0, 0\right) [\cos(\bar{k}' \cdot \bar{x}') \sin(k' ct') + \sin(\bar{k}' \cdot \bar{x}') \cos(k' ct')] \\ &\quad \frac{1}{\gamma_v \sqrt{1 - \frac{[k'_1 \gamma_v - \frac{k'\gamma_v v}{c}]^2 v^2}{[-\frac{k'_1}{c} \gamma_v v + k'\gamma_v]^2 c^2}}} dEll(k') \\ &= \frac{1}{4\pi\epsilon_0 k^3} \int_{Ell(\bar{0}, k')} \bar{k}' \left[-\frac{i}{2} e^{-i\bar{k}' \cdot \bar{x}'} e^{ik' ct'} + \frac{i}{2} e^{i\bar{k}' \cdot \bar{x}'} e^{-ik' ct'}\right] \frac{1}{\sqrt{1 - \frac{[k'_1 \gamma_v - \frac{k'\gamma_v v}{c}]^2 v^2}{[-\frac{k'_1}{c} \gamma_v v + k'\gamma_v]^2 c^2}}} dEll(k') \\ &\quad + \frac{1}{4\pi\epsilon_0 k^3} \int_{Ell(\bar{0}, k')} \left(-\frac{k'\gamma_v v}{c}, 0, 0\right) \left[-\frac{i}{2} e^{-i\bar{k}' \cdot \bar{x}'} e^{ik' ct'} + \frac{i}{2} e^{i\bar{k}' \cdot \bar{x}'} e^{-ik' ct'}\right] \\ &\quad \frac{1}{\gamma_v \sqrt{1 - \frac{[k'_1 \gamma_v - \frac{k'\gamma_v v}{c}]^2 v^2}{[-\frac{k'_1}{c} \gamma_v v + k'\gamma_v]^2 c^2}}} dEll(k') \\ &= \frac{i}{8\pi\epsilon_0 k^3} \int_{Ell'(\bar{0}, k')} \bar{k}' e^{i\bar{k}' \cdot \bar{x}'} e^{ik' ct'} \frac{1}{\sqrt{1 - \frac{[-k'_1 \gamma_v - \frac{k'\gamma_v v}{c}]^2 v^2}{[\frac{k'_1}{c} \gamma_v v + k'\gamma_v]^2 c^2}}} dEll'(k') \\ &\quad + \frac{i}{8\pi\epsilon_0 k^3} \int_{Ell(\bar{0}, k')} \bar{k}' e^{i\bar{k}' \cdot \bar{x}'} e^{-ik' ct'} \frac{1}{\sqrt{1 - \frac{[k'_1 \gamma_v - \frac{k'\gamma_v v}{c}]^2 v^2}{[-\frac{k'_1}{c} \gamma_v v + k'\gamma_v]^2 c^2}}} dEll(k') \\ &\quad + \frac{i}{8\pi\epsilon_0 k^3} \int_{Ell'(\bar{0}, k')} \left(\frac{k'\gamma_v v}{c}, 0, 0\right) e^{i\bar{k}' \cdot \bar{x}'} e^{ik' ct'} \frac{1}{\gamma_v \sqrt{1 - \frac{[-k'_1 \gamma_v - \frac{k'\gamma_v v}{c}]^2 v^2}{[\frac{k'_1}{c} \gamma_v v + k'\gamma_v]^2 c^2}}} dEll'(k') \\ &\quad - \frac{i}{8\pi\epsilon_0 k^3} \int_{Ell(\bar{0}, k')} \left(\frac{k'\gamma_v v}{c}, 0, 0\right) e^{i\bar{k}' \cdot \bar{x}'} e^{-ik' ct'} \frac{1}{\gamma_v \sqrt{1 - \frac{[k'_1 \gamma_v - \frac{k'\gamma_v v}{c}]^2 v^2}{[-\frac{k'_1}{c} \gamma_v v + k'\gamma_v]^2 c^2}}} dEll(k') \end{aligned}$$

where $Ell'(\bar{0}, \bar{k}')$ is obtained from $Ell(\bar{0}, \bar{k}')$ by the coordinate transformation $\bar{k}'' = -\bar{k}'$.

We compute $Ell(\bar{0}, \bar{k}')$ and $Ell'(\bar{0}, \bar{k}')$. We have that;

$$Ell(\bar{0}, \bar{k}') = \{\bar{k}' : -\frac{k'_1 \gamma_v v}{c} + k' \gamma_v = k\}$$

where;

$$-\frac{k'_1 \gamma_v v}{c} + k' \gamma_v = k$$

$$\text{iff } k'^2 \gamma_v^2 = (k + \frac{k'_1 \gamma_v v}{c})^2$$

$$\text{iff } (k_1'^2 + k_2'^2 + k_3'^2) \gamma_v^2 = k^2 + \frac{k_1'^2 \gamma_v^2 v^2}{c^2} + \frac{2kk'_1 \gamma_v v}{c}$$

$$\text{iff } k_1'^2 - \frac{2kk'_1 \gamma_v v}{c} + (k_2'^2 + k_3'^2) \gamma_v^2 = k^2$$

$$\text{iff } (k'_1 - \frac{k \gamma_v v}{c})^2 + \gamma_v^2 (k_2'^2 + k_3'^2) = k^2 + \frac{k^2 \gamma_v^2 v^2}{c^2}$$

$$\text{iff } (k'_1 - \frac{k \gamma_v v}{c})^2 + \gamma_v^2 (k_2'^2 + k_3'^2) = \frac{k^2 c^2}{v^2}$$

which is an ellipsoid. Clearly then, $Ell'(\bar{0}, \bar{k}')$ is given by;

$$(k'_1 + \frac{k \gamma_v v}{c})^2 + \gamma_v^2 (k_2'^2 + k_3'^2) = \frac{k^2 c^2}{v^2}$$

Using the method of [2], we let;

$$\begin{aligned} Re(\bar{E})''_{\bar{v}} &= \frac{i}{8\pi\epsilon_0 k^3} (1 + \frac{k_1 \gamma_v v}{ck'}) \int_{Ell'(\bar{0}, k')} \bar{k}' e^{i\bar{k}' \cdot \bar{x}'} e^{ik' ct'} \frac{1}{\sqrt{1 - \frac{[-k'_1 \gamma_v - \frac{k' \gamma_v v}{c}]^2 v^2}{[\frac{k'_1 \gamma_v v}{c} + k' \gamma_v]^2 c^2}}} dEll'(k') \\ &+ \frac{i}{8\pi\epsilon_0 k^3} (1 - \frac{k_1 \gamma_v v}{ck'}) \int_{Ell(\bar{0}, k')} \bar{k}' e^{i\bar{k}' \cdot \bar{x}'} e^{-ik' ct'} \frac{1}{\sqrt{1 - \frac{[k'_1 \gamma_v - \frac{k' \gamma_v v}{c}]^2 v^2}{[-\frac{k'_1 \gamma_v v}{c} + k' \gamma_v]^2 c^2}}} dEll(k') \end{aligned}$$

Then $Re(\bar{E})''_{\bar{v}}$ is real and $\nabla \times Re(\bar{E})''_{\bar{v}} = \bar{0}$.

.....

□

.....(remove extra γ_v in calculation)

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FLAT 3, REDESDALE HOUSE, 85 THE PARK, CHELTENHAM, GL50 2RP
E-mail address: `t.depiro@curvalinea.net`