A STOCHASTIC MODEL OF THE ELECTRON AND THE WAVE EQUATION

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Abstract. We show that the wave equation for charge and current supports the idea that an electron can simultaneously exhibit wave and particle properties. The transformations between frames of charge and current obeying the wave equation is consistent with the transformation for particles in special relativity, up to a Doppler shift, provided we allow for reverse particle paths. The Doppler shift is well known in the literature and we use the reverse particle paths to develop a stochastic theory of the motion of an electron, for observers travelling at a velocity relative to the source. This theory can explain noise received in radio signals and can even predict the velocity of the observer given knowledge of the autocorrelations of the noise.

Lemma 0.1. *We have that;*

$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} cos^n(y)dy = \frac{n!\pi}{2^n[(\frac{n}{2})!]^2}$, *if n is even*

$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} cos^n(y)dy = \frac{[\frac{n-1}{2}!]^2 2^n}{n!}$, *if n is odd*

Proof. Let $I_n = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} cos^n(y)dy$, then for $n \geq 2$, we have that, using integration by parts;

$I_n = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} cos^n(y)dy$

$= [cos^{n-1}(y)sin(y)]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} + \int -\frac{\pi}{2}^{\frac{\pi}{2}}(n-1)cos^{n-2}(y)sin^2(y)dy$

$= \int -\frac{\pi}{2}^{\frac{\pi}{2}}(n-1)cos^{n-2}(y)(1-cos^2(y))dy$

$= (n-1)I_{n-2} - (n-1)I_n$

so that, rearranging;

$I_n = \frac{n-1}{n} I_{n-2}$

and, using the fact $I_0=\pi$, $I_1=2$, we have that, for n even;

$I_n=\frac{n!}{2^n[(\frac{n}{2})!]^2}\pi$

and, for n odd;

$I_n=\frac{[\frac{n-1}{2}!]^2 2^n}{n!}$

□

Lemma 0.2. *Let $n\in\mathcal{N}$, $\epsilon>0$, and let $\gamma_{n,\epsilon}$ be defined by;*

$\gamma_{n,\epsilon}(x)=\frac{1}{\epsilon}cos^n(\frac{\pi x}{2\epsilon})$, *for* $x\in[-\epsilon,\epsilon]$

$\gamma_{n,\epsilon}(x)=0$, *otherwise*

Then $\gamma_{n,\epsilon}$ has the following properties;

(i). $\gamma_{n,\epsilon}\in C^{n-1}(\mathcal{R})$.

(ii). $\gamma_{n,\epsilon}\geq 0$.

(iii). $\int_{\mathcal{R}}\gamma_{n,\epsilon}(x)dx=\frac{n!}{2^{n-1}[(\frac{n}{2})!]^2}$, *$n$ even*

$\int_{\mathcal{R}}\gamma_{n,\epsilon}(x)dx=\frac{[\frac{n-1}{2}!]^2 2^{n+1}}{\pi n!}$, *$n$ odd*

(iv) $\gamma_{n,\epsilon}$ is supported on $[-\epsilon,\epsilon]$.

Proof. (ii) is clear as $cos(y)\geq 0$ for $y\in[-\frac{\pi}{2},\frac{\pi}{2}]$, (iv) is clear by the definition of $\gamma_{n,\epsilon}$. To prove (i), it is sufficient to show that;

$cos^n(\frac{\pi x}{2\epsilon})^{(m)}(\epsilon)=cos^n(\frac{\pi x}{2\epsilon})^{(m)}(-\epsilon)=0$

for $0\leq m\leq n-1$. We can prove this by induction on n, as for $n=1$, we have that;

$cos(\frac{\pi x}{2\epsilon})(\epsilon)=cos(\frac{\pi}{2})=cos(\frac{\pi x}{2\epsilon})(-\epsilon)=cos(-\frac{\pi}{2})=0$

and, if the inductive hypothesis holds for $n\in\mathcal{N}$, then, for $1\leq m\leq n$;

$cos^{n+1}(\frac{\pi x}{2\epsilon})^{(m)}(\epsilon)$

$= -[\frac{\pi(n+1)}{2\epsilon}cos^{n}(\frac{\pi x}{2\epsilon})sin(\frac{\pi x}{2\epsilon})]^{(m-1)}(\epsilon)$

$= -\frac{\pi(n+1)}{2\epsilon}[\sum_{k=0}^{m-1} C_{k}^{m-1}cos^{n}(\frac{\pi x}{2\epsilon})^{(m-1-k)}sin(\frac{\pi x}{2\epsilon})^{(k)}](\epsilon)$

$= 0$

and similarly;

$cos^{n+1}(\frac{\pi x}{2\epsilon})^{(m)}(-\epsilon) = 0$

while, clearly;

$cos^{n+1}(\frac{\pi x}{2\epsilon})(\epsilon) = cos^{n+1}(\frac{\pi x}{2\epsilon})(-\epsilon) = 0$

To prove (iii), we have that, for $n \in \mathcal{N}$;

$\int_{\mathcal{R}} \gamma_{n,\epsilon}(x)dx$

$= \frac{1}{\epsilon}\int_{-\epsilon}^{\epsilon} cos^{n}(\frac{\pi x}{2\epsilon})$

$= \frac{1}{\epsilon}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} cos^{n}(y)\frac{2\epsilon}{\pi}dy$, $(y = \frac{\pi x}{2\epsilon})$

$= \frac{2}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} cos^{n}(y)dy$

so that, using Lemma 0.1, for n even;

$\int_{\mathcal{R}} \gamma_{n,\epsilon}(x)dx = \frac{2}{\pi}\frac{n!}{2^{n}[(\frac{n}{2})!]^{2}}\pi$

$= \frac{n!}{2^{n-1}[(\frac{n}{2})!]^{2}}$

and, for n odd;

$\int_{\mathcal{R}} \gamma_{n,\epsilon}(x)dx = \frac{2}{\pi}\frac{[\frac{n-1}{2}!]^{2}2^{n}}{n!}$

$= \frac{[\frac{n-1}{2}!]^{2}2^{n+1}}{\pi n!}$

□

Lemma 0.3. *Let* $\delta_{n,\epsilon}(x)$ *be defined by;*

$\delta_{n,\epsilon}(x) = \frac{2^{n-1}[\frac{n}{2}!]^2}{n!}\gamma_{n,\epsilon}$, *for n even*

and by;

$\delta_{n,\epsilon}(x) = \frac{\pi n!}{[(\frac{n-1}{2})!]^2 2^{n+1}}\gamma_{n,\epsilon}$, *for n odd*

Then the properties $(i),(ii),(iv)$ *of Lemma 0.2 hold, with* (iii) *changed to;*

$(iii)'$. $\int_{\mathcal{R}} \gamma_{n,\epsilon}(x)dx = 1$, $n \in \mathcal{N}$

and, for $n \in \mathcal{N}$*;*

$lim_{\epsilon \to 0}\delta_{n,\epsilon} = \delta$

in the sense of distributions, where δ *is the Dirac delta function on* $\mathcal{R}$*.*

Proof. The first claim is clear as we have just normalised $\gamma_{n,\epsilon}$. For the remaining claim, let $f \in C_c^\infty(\mathcal{R})$, and write;

$f = f^+ + f^-$

where;

$f^+(x) = f(x)$, if $f(x) \geq 0$

$f^+(x) = 0$ otherwise

$f^-(x) = f(x)$, if $f(x) \leq 0$

$f^-(x) = 0$ otherwise

Then, using properties $(ii), (iii)', (iv)$ of $\delta_{n,\epsilon}$ and continuity of f;

$min_{[-\epsilon,\epsilon]}f^+ + min_{[-\epsilon,\epsilon]}f^- \leq \delta_{n,\epsilon}(f) = \int_{-\epsilon}^{\epsilon}\delta_{n,\epsilon}(x)f^+(x)dx + \int_{-\epsilon}^{\epsilon}\delta_{n,\epsilon}(x)f^-(x)dx$

$\leq max_{[-\epsilon,\epsilon]}f^+ + max_{[-\epsilon,\epsilon]}f^-$

with;

$lim_{\epsilon\rightarrow 0}min_{[-\epsilon,\epsilon]}f^{+}+min_{[-\epsilon,\epsilon]}f^{-}=lim_{\epsilon\rightarrow 0}max_{[-\epsilon,\epsilon]}f^{+}+max_{[-\epsilon,\epsilon]}f^{-}$

$=f(0)$

so that $lim_{\epsilon\rightarrow 0}\delta_{n,\epsilon}(f)=f(0)=\delta(f)$, as required.

□

Lemma 0.4. *We define the reverse time derivative δ'_t of the delta function δ to be;*

$\frac{d}{dt}\delta(x+vt)$

where v is the velocity, so that;

$\delta'_t=v\delta'$

in the sense of distributions. Similarly, we define the reverse time derivative $\delta'_{n,\epsilon,t}$ of the approximations by;

$\frac{d}{dt}\delta_{n,\epsilon}(x+vt)$

so that, by the chain rule;

$\delta'_{n,\epsilon,t}(x)=v\delta'_{n,\epsilon}(x)$

Then;

$lim_{\epsilon\rightarrow 0}\delta'_{n,\epsilon,t}=\delta'_t$

in the sense of distributions.

Moreover, for n even, $n\geq 2$;

$\delta'_{n,\epsilon,t}(x)=-\frac{v2^{n-1}\pi[\frac{n}{2}!]^2}{2\epsilon(n-1)!}\gamma_{n-1,\epsilon}sin(\frac{\pi x}{2\epsilon})$

and, for n odd, $n\geq 3$;

$\delta'_{n,\epsilon,t}(x)=-\frac{v\pi^2 n!n}{2\epsilon[(\frac{n-1}{2})!]^2 2^{n+1}}\gamma_{n-1,\epsilon}sin(\frac{\pi x}{2\epsilon})$

In particularly, for $n\geq 2$;

$\delta'_{n,\epsilon,t} \in C_c^{n-2}(\mathcal{R})$

Proof. For the first claim, let $f \in C_c^\infty(\mathcal{R})$, then, using integration by parts, (iv) of Lemma 0.3;

$\delta'_{n,\epsilon,t}(f) = v\int_{-\epsilon}^{\epsilon} \delta'_{n,\epsilon}(x)f(x)dx$

$= v([\delta'_{n,\epsilon}(x)f(x)]_{-\epsilon}^{\epsilon} - \int_{-\epsilon}^{\epsilon} \delta_{n,\epsilon}(x)f'(x)dx)$

$= -v\int_{-\epsilon}^{\epsilon} \delta_{n,\epsilon}(x)f'(x)dx$

so that, using the main result of Lemma 0.3;

$lim_{\epsilon\rightarrow 0}\delta'_{n,\epsilon,t}(f)$

$= -vlim_{\epsilon\rightarrow 0}\int_{-\epsilon}^{\epsilon} \delta_{n,\epsilon}(x)f'(x)dx$

$= -vf'(0)$

$= v\delta'(f)$

$= \delta'_t(f)$

as required. For the second claim, we have that, for n even, $n \geq 2$;

$\delta'_{n,\epsilon,t}(x) = v\delta'_{n,\epsilon}(x)$

$= v\frac{2^{n-1}[\frac{n}{2}!]^2}{n!}\gamma'_{n,\epsilon}(x)$

$= v\frac{2^{n-1}[\frac{n}{2}!]^2}{n!}(-ncos^{n-1}(\frac{\pi x}{2\epsilon})sin(\frac{\pi x}{2\epsilon})\frac{\pi}{2\epsilon^2})$

$= -v\frac{2^{n-1}\pi[\frac{n}{2}!]^2}{2\epsilon(n-1)!}(\frac{1}{\epsilon}cos^{n-1}(\frac{\pi x}{2\epsilon})sin(\frac{\pi x}{2\epsilon}))$

$= -\frac{v2^{n-1}\pi[\frac{n}{2}!]^2}{2\epsilon(n-1)!}\gamma_{n-1,\epsilon}(x)sin(\frac{\pi x}{2\epsilon})$

and, for n odd, $n \geq 3$;

$\delta'_{n,\epsilon,t}(x) = v\delta'_{n,\epsilon}(x)$

$= v\frac{\pi n!}{[(\frac{n-1}{2})!]^2 2^{n+1}}\gamma'_{n,\epsilon}(x)$

$= v\frac{\pi n!}{[(\frac{n-1}{2})!]^2 2^{n+1}}(-ncos^{n-1}(\frac{\pi x}{2\epsilon})sin(\frac{\pi x}{2\epsilon})\frac{\pi}{2\epsilon^2})$

$= -v\frac{\pi^2 n!n}{2\epsilon[(\frac{n-1}{2})!]^2 2^{n+1}}(\frac{1}{\epsilon}cos^{n-1}(\frac{\pi x}{2\epsilon})sin(\frac{\pi x}{2\epsilon}))$

$= -\frac{v\pi^2 n!n}{2\epsilon[(\frac{n-1}{2})!]^2 2^{n+1}}\gamma_{n-1,\epsilon}(x)sin(\frac{\pi x}{2\epsilon})$

For the final claim, we have that, by the product rule, for $0 \leq m \leq n-2$;

$(\gamma_{n-1,\epsilon}(x)sin(\frac{\pi x}{2\epsilon}))^{(m)}$

$= \sum_{k=0}^{m}\gamma_{n-1,\epsilon}^{(m-k)}(sin(\frac{\pi x}{2\epsilon}))^k$

and use the fact that $\gamma_{n-1,\epsilon} \in C_c^{n-2}(\mathcal{R})$ □

Lemma 0.5. *Let D denote the 3-dimensional Dirac delta function. For $n \in \mathcal{N}, \epsilon > 0$, let $D_{n,\epsilon}$ be defined by;*

$D_{n,\epsilon}(x,y,z) = \delta_{n,\epsilon}(x)\delta_{n,\epsilon}(y)\delta_{n,\epsilon}(z)$

Then $D_{n,\epsilon}$ has the following properties;

(i). $D_{n,\epsilon} \in C^{n-1}(\mathcal{R}^3)$.

(ii). $D_{n,\epsilon} \geq 0$.

(iii). $\int_{\mathcal{R}^3} D_{n,\epsilon}(x,y,z)dxdydz = 1$

(iv) $D_{n,\epsilon}$ is supported on $[-\epsilon,\epsilon]^3$.

Moreover, for $n \in \mathcal{N}$, $lim_{\epsilon\rightarrow 0}D_{n,\epsilon} = D$ in the sense of distributions, $(*)$.

We define the reverse time derivative D'_t of the delta function D to be;

$\frac{d}{dt}D(\overline{x}+\overline{v}t)$

where $\overline{v}$ is the velocity vector, so that;

$D'_t = v_1 D_x + v_2 D_y + v_3 D_z$

in the sense of distributions. Similarly, we define the reverse time derivative $D'_{n,\epsilon,t}$ of the approximations by;

$\frac{d}{dt}D_{n,\epsilon}(\overline{x}+\overline{v}t)$

so that, by the chain rule;

$D'_{n,\epsilon,t}(\overline{x})=v_1D_{n,\epsilon,x}(\overline{x})+v_2D_{n,\epsilon,y}(\overline{x})+v_3D_{n,\epsilon,z}(\overline{x})$

$=v_1\delta'_{n,\epsilon}(x)\delta_{n,\epsilon}(y)\delta_{n,\epsilon}(z)+v_2\delta_{n,\epsilon}(x)\delta'_{n,\epsilon}(y)\delta_{n,\epsilon}(z)+v_3\delta_{n,\epsilon}(x)\delta_{n,\epsilon}(y)\delta'_{n,\epsilon}(z)$

Then;

$lim_{\epsilon\rightarrow 0}D'_{n,\epsilon,t}=D'_t$, $(**)$

in the sense of distributions.

Finally, we have that;

$D_{n,\epsilon}(x,y,z)=(\frac{2^{n-1}[\frac{n}{2}!]^2}{n!})^3\gamma_{n,\epsilon}(x)\gamma_{n,\epsilon}(y)\gamma_{n,\epsilon}(z)$, *for n even*

and;

$D_{n,\epsilon}(x,y,z)=(\frac{\pi n!}{[(\frac{n-1}{2})!]^2 2^{n+1}})^3\gamma_{n,\epsilon}(x)\gamma_{n,\epsilon}(y)\gamma_{n,\epsilon}(z)$, *for n odd*

We have that;

$D'_{n,\epsilon,t}=-\frac{2^{n-1}\pi[\frac{n}{2}!]^2}{2\epsilon(n-1)!}(\frac{2^{n-1}[\frac{n}{2}!]^2}{n!})^2[v_1\gamma_{n-1,\epsilon}(x)\gamma_{n,\epsilon}(y)\gamma_{n,\epsilon}(z)sin(\frac{\pi x}{2\epsilon})$

$+v_2\gamma_{n,\epsilon}(x)\gamma_{n-1,\epsilon}(y)\gamma_{n,\epsilon}(z)sin(\frac{\pi y}{2\epsilon})+v_3\gamma_{n,\epsilon}(x)\gamma_{n,\epsilon}(y)\gamma_{n-1,\epsilon}(z)sin(\frac{\pi z}{2\epsilon})]$

for n even.

$D'_{n,\epsilon,t}=-\frac{\pi^2 n!n}{2\epsilon[(\frac{n-1}{2})!]^2 2^{n+1}}(\frac{\pi n!}{[(\frac{n-1}{2})!]^2 2^{n+1}})^2[v_1\gamma_{n-1,\epsilon}(x)\gamma_{n,\epsilon}(y)\gamma_{n,\epsilon}(z)sin(\frac{\pi x}{2\epsilon})$

$+v_2\gamma_{n,\epsilon}(x)\gamma_{n-1,\epsilon}(y)\gamma_{n,\epsilon}(z)sin(\frac{\pi y}{2\epsilon})+v_3\gamma_{n,\epsilon}(x)\gamma_{n,\epsilon}(y)\gamma_{n-1,\epsilon}(z)sin(\frac{\pi z}{2\epsilon})]$

for n odd.

In particularly, $D'_{n,\epsilon,t}\in C_c^{n-2}(\mathcal{R}^3)$.

Proof. For the first claim, (i) follows from the fact that $\delta_{n,\epsilon}\in C^{n-1}(\mathcal{R})$ and the fact that if $i+j+k=n-1$;

$$\frac{\partial^{i+j+k}D_{n,\epsilon}}{\partial x^i\partial y^j\partial z^k}(x,y,z)=\delta_{n,\epsilon}^{(i)}(x)\delta_{n,\epsilon}^{(j)}(y)\delta_{n,\epsilon}^{(k)}(z)$$

(ii) is trivial from the corresponding property of the $\delta_{n,\epsilon}$. (iii) follows form Fubini's theorem;

$$\int_{[-\epsilon,\epsilon]}\int_{[-\epsilon,\epsilon]}\int_{[-\epsilon,\epsilon]}D_{n,\epsilon}(x,y,z)dxdydz=\int_{[-\epsilon,\epsilon]}\delta_{n,\epsilon}(x)dx\int_{[-\epsilon,\epsilon]}\delta_{n,\epsilon}(y)dy\int_{[-\epsilon,\epsilon]}\delta_{n,\epsilon}(z)dz$$

and the corresponding property (iii) of $\delta_{n,\epsilon}$. (iv) is again trivial from the the corresponding property of $\delta_{n,\epsilon}$. The claim $(*)$ is a consequence of properties $(ii),(iii),(iv)$, see the proof above for the 1-dimensional case. For the claim $(**)$, we have that, for $f\in C_c^\infty(\mathcal{R}^3)$, using integration by parts and Tonelli's theorem;

$$D'_{n,\epsilon,t}(f)=\int_{[-\epsilon,\epsilon]}\int_{[-\epsilon,\epsilon]}\int_{[-\epsilon,\epsilon]}[v_1\delta'_{n,\epsilon}(x)\delta_{n,\epsilon}(y)\delta_{n,\epsilon}(z)+v_2\delta_{n,\epsilon}(x)\delta'_{n,\epsilon}(y)\delta_{n,\epsilon}(z)$$

$$+v_3\delta_{n,\epsilon}(x)\delta_{n,\epsilon}(y)\delta'_{n,\epsilon}(z)]f dxdydz$$

$$=\int_{[-\epsilon,\epsilon]}\int_{[-\epsilon,\epsilon]}\int_{[-\epsilon,\epsilon]}[-v_1\delta_{n,\epsilon}(x)\delta_{n,\epsilon}(y)\delta_{n,\epsilon}(z)f_x-v_2\delta_{n,\epsilon}(x)\delta_{n,\epsilon}(y)\delta_{n,\epsilon}(z)f_y$$

$$-v_3\delta_{n,\epsilon}(x)\delta_{n,\epsilon}(y)\delta_{n,\epsilon}(z)]f_z dxdydz$$

so that, by the claim $(*)$;

$$lim_{\epsilon\rightarrow 0}D'_{n,\epsilon,t}(f)=-v_1f_x(0)-v_2f_y(0)-v_3f_z(0)$$

$$=(v_1D_x+v_2D_y+v_3D_z)(f)$$

$$=D'_t(f)$$

The first computational claim follows from Lemma 0.3 and the definition of $D_{n,\epsilon}$. For the second computational claim;

$$D'_{n,\epsilon,t}(\overline{x})=v_1\delta'_{n,\epsilon}(x)\delta_{n,\epsilon}(y)\delta_{n,\epsilon}(z)+v_2\delta_{n,\epsilon}(x)\delta'_{n,\epsilon}(y)\delta_{n,\epsilon}(z)+v_3\delta_{n,\epsilon}(x)\delta_{n,\epsilon}(y)\delta'_{n,\epsilon}(z)$$

so that, for n even;

$$D'_{n,\epsilon,t}(\overline{x})=v_1[-\tfrac{2^{n-1}\pi[\frac{n}{2}!]^2}{2\epsilon(n-1)!}\gamma_{n-1,\epsilon}(x)sin(\tfrac{\pi x}{2\epsilon})][(\tfrac{2^{n-1}[\frac{n}{2}!]^2}{n!})\gamma_{n,\epsilon}(y)][(\tfrac{2^{n-1}[\frac{n}{2}!]^2}{n!})\gamma_{n,\epsilon}(z)]$$

$+v_2[(\frac{2^{n-1}[\frac{n}{2}!]^2}{n!})\gamma_{n,\epsilon}(x)][-\frac{2^{n-1}\pi[\frac{n}{2}!]^2}{2\epsilon(n-1)!}\gamma_{n-1,\epsilon}(y)sin(\frac{\pi y}{2\epsilon})][(\frac{2^{n-1}[\frac{n}{2}!]^2}{n!})\gamma_{n,\epsilon}(z)]$

$+v_3[(\frac{2^{n-1}[\frac{n}{2}!]^2}{n!})\gamma_{n,\epsilon}(x)][(\frac{2^{n-1}[\frac{n}{2}!]^2}{n!})\gamma_{n,\epsilon}(y)][-\frac{2^{n-1}\pi[\frac{n}{2}!]^2}{2\epsilon(n-1)!}\gamma_{n-1,\epsilon}(z)sin(\frac{\pi z}{2\epsilon})]$

$= -v_1\frac{2^{n-1}\pi[\frac{n}{2}!]^2}{2\epsilon(n-1)!}(\frac{2^{n-1}[\frac{n}{2}!]^2}{n!})^2\gamma_{n-1,\epsilon}(x)\gamma_{n,\epsilon}(y)\gamma_{n,\epsilon}(z)sin(\frac{\pi x}{2\epsilon})$

$-v_2\frac{2^{n-1}\pi[\frac{n}{2}!]^2}{2\epsilon(n-1)!}(\frac{2^{n-1}[\frac{n}{2}!]^2}{n!})^2\gamma_{n,\epsilon}(x)\gamma_{n-1,\epsilon}(y)\gamma_{n,\epsilon}(z)sin(\frac{\pi y}{2\epsilon})$

$-v_3\frac{2^{n-1}\pi[\frac{n}{2}!]^2}{2\epsilon(n-1)!}(\frac{2^{n-1}[\frac{n}{2}!]^2}{n!})^2\gamma_{n,\epsilon}(x)\gamma_{n,\epsilon}(y)\gamma_{n-1,\epsilon}(z)sin(\frac{\pi z}{2\epsilon})$

and, for n odd;

$D'_{n,\epsilon,t}(\overline{x}) = v_1[-\frac{\pi^2 n!n}{2\epsilon[(\frac{n-1}{2})!]^2 2^{n+1}}\gamma_{n-1,\epsilon}(x)sin(\frac{\pi x}{2\epsilon})][(\frac{\pi n!}{[(\frac{n-1}{2})!]^2 2^{n+1}})\gamma_{n,\epsilon}(y)][(\frac{\pi n!}{[(\frac{n-1}{2})!]^2 2^{n+1}})\gamma_{n,\epsilon}(z)]$

$+v_2[(\frac{\pi n!}{[(\frac{n-1}{2})!]^2 2^{n+1}})\gamma_{n,\epsilon}(x)][-\frac{\pi^2 n!n}{2\epsilon[(\frac{n-1}{2})!]^2 2^{n+1}}\gamma_{n-1,\epsilon}(y)sin(\frac{\pi y}{2\epsilon})][(\frac{\pi n!}{[(\frac{n-1}{2})!]^2 2^{n+1}})\gamma_{n,\epsilon}(z)]$

$+v_3[(\frac{\pi n!}{[(\frac{n-1}{2})!]^2 2^{n+1}})\gamma_{n,\epsilon}(x)][(\frac{\pi n!}{[(\frac{n-1}{2})!]^2 2^{n+1}})\gamma_{n,\epsilon}(y)][-\frac{\pi^2 n!n}{2\epsilon[(\frac{n-1}{2})!]^2 2^{n+1}}\gamma_{n-1,\epsilon}(z)sin(\frac{\pi z}{2\epsilon})]$

$= -v_1\frac{\pi^2 n!n}{2\epsilon[(\frac{n-1}{2})!]^2 2^{n+1}}(\frac{\pi n!}{[(\frac{n-1}{2})!]^2 2^{n+1}})^2\gamma_{n-1,\epsilon}(x)\gamma_{n,\epsilon}(y)\gamma_{n,\epsilon}(z)sin(\frac{\pi x}{2\epsilon})$

$-v_2\frac{\pi^2 n!n}{2\epsilon[(\frac{n-1}{2})!]^2 2^{n+1}}(\frac{\pi n!}{[(\frac{n-1}{2})!]^2 2^{n+1}})^2\gamma_{n,\epsilon}(x)\gamma_{n-1,\epsilon}(y)\gamma_{n,\epsilon}(z)sin(\frac{\pi y}{2\epsilon})$

$-v_3\frac{\pi^2 n!n}{2\epsilon[(\frac{n-1}{2})!]^2 2^{n+1}}(\frac{\pi n!}{[(\frac{n-1}{2})!]^2 2^{n+1}})^2\gamma_{n,\epsilon}(x)\gamma_{n,\epsilon}(y)\gamma_{n-1,\epsilon}(z)sin(\frac{\pi z}{2\epsilon})$

The final claim follows as above, by repeated application of the product rule, using the fact that $\gamma_{n-1,\epsilon} \in C_c^{n-2}(\mathcal{R})$ and $\gamma_{n,\epsilon} \in C_c^{n-1}(\mathcal{R})$ $\square$

Definition 0.6. *For $n \geq 4$, we let $\rho_{n,\epsilon}$ be the unique charge distribution on $\mathcal{R}^3 \times \mathcal{R}_{\geq 0}$ defined by the initial conditions $\{qD_{n,\epsilon}, qD'_{n,\epsilon,t}\}$, satisfying the wave equation with velocity c, $\square^2(\rho_{n,\epsilon}) = 0$, on $\mathcal{R}^3 \times \mathcal{R}_{\geq 0}$, where q is the total charge.*

Lemma 0.7. *For $n \geq 6$, we have that;*

$\rho_{n,\epsilon}(\overline{x}, t) = \frac{1}{(2\pi)^3}\int_{\mathcal{R}^3}(A(\overline{k})e^{ikct} + B(\overline{k})e^{-ikct})e^{i\overline{k}.\overline{x}}d\overline{k}$

where;

$A(\overline{k}) = \frac{\mathcal{F}(qD_{n,\epsilon})}{2} + \frac{\mathcal{F}(qD'_{n,\epsilon,t})}{2ikc}$

$B(\overline{k}) = \frac{\mathcal{F}(qD_{n,\epsilon})}{2} - \frac{\mathcal{F}(qD'_{n,\epsilon,t})}{2ikc}$

and $\mathcal{F}$ *denotes the usual* 3 *dimensional Fourier transform;*

$\mathcal{F}(g)=\int_{\mathcal{R}^3}g(\overline{x})e^{-i\overline{k}\centerdot\overline{x}}d\overline{x}$

for $g\in L^1(\mathcal{R}^3)$.

Proof. We have that the existence of $\rho_{n,\epsilon}$ is guaranteed by Kirchoff's formula, for $n\geq 4$, as the initial conditions $D_{n,\epsilon}\in C_c^3(\mathcal{R}^3$ and $D'_{n,\epsilon,t}\in C_c^2(\mathcal{R}^4)$ for $n\geq 4$, see [4]. Using the fact that the initial conditions $\{qD_{n,\epsilon},qD'_{n,\epsilon,t}\}$ have compact support, $\rho_{n,\epsilon,t}$ has compact support as a process, in particularly $\rho_{n,\epsilon,t}\in L^1(\mathcal{R}^3)$, for $t\geq 0$. Using Kirchoff's formula, see [4], we have that, for $t>0$;

$\rho_{n,\epsilon}(\overline{x},t)=\frac{1}{4\pi c^2t^2}\int_{\delta B(\overline{x},ct)}[ctqD'_{n,\epsilon,t}(\overline{y})+qD_{n,\epsilon}(\overline{y})+DqD_{n,\epsilon}(\overline{y})\centerdot(\overline{y}$

$-\overline{x})]dS(\overline{y})$

Then, we have that, using the substitution $\overline{z}=\overline{y}-(h,0,0)$, $d\overline{z}=d\overline{y}$ and interchanging limits;

$\frac{\partial\rho_{n,\epsilon}}{\partial x}=lim_{h\rightarrow 0}\frac{1}{h}(\frac{1}{4\pi c^2t^2}\int_{\delta B(\overline{x}+(h,0,0),ct)}[ctqD'_{n,\epsilon,t}(\overline{y})+qD_{n,\epsilon}(\overline{y})+DqD_{n,\epsilon}(\overline{y})\centerdot$

$(\overline{y}-(\overline{x}+(h,0,0)))]dS(\overline{y})-\frac{1}{4\pi c^2t^2}\int_{\delta B(\overline{x},ct)}[ctqD'_{n,\epsilon,t}(\overline{y})+qD_{n,\epsilon}(\overline{y})+$

$DqD_{n,\epsilon}(\overline{y})\centerdot(\overline{y}-\overline{x})]dS(\overline{y}))$

$=lim_{h\rightarrow 0}\frac{1}{h}(\frac{1}{4\pi c^2t^2}\int_{\delta B(\overline{x},ct)}[ctqD'_{n,\epsilon,t}(\overline{y}+(h,0,0))+qD_{n,\epsilon}(\overline{y}+(h,0,0))$

$+DqD_{n,\epsilon}(\overline{y}+(h,0,0))\centerdot(\overline{y}+(h,0,0)-(\overline{x}+(h,0,0)))]dS(\overline{y})-\frac{1}{4\pi c^2t^2}\int_{\delta B(\overline{x},ct)}[ctqD'_{n,\epsilon,t}(\overline{y})$

$+qD_{n,\epsilon}(\overline{y})+DqD_{n,\epsilon}(\overline{y})\centerdot(\overline{y}-\overline{x})]dS(\overline{y}))$

$=\frac{1}{4\pi c^2t^2}\int_{\delta B(\overline{x},ct)}[lim_{h\rightarrow 0}\frac{1}{h}(ctqD'_{n,\epsilon,t}(\overline{y}+(h,0,0))-ctqD'_{n,\epsilon,t}(\overline{y}))+lim_{h\rightarrow 0}\frac{1}{h}(qD_{n,\epsilon}(\overline{y}+$

$(h,0,0))-qD_{n,\epsilon}(\overline{y}))+lim_{h\rightarrow 0}\frac{1}{h}(DqD_{n,\epsilon}(\overline{y}+(h,0,0))-DqD_{n,\epsilon}(\overline{y}))\centerdot(\overline{y}-$

$\overline{x})]dS(\overline{y})$

$=\frac{1}{4\pi c^2t^2}\int_{\delta B(\overline{x},ct)}[ctqD'_{n,\epsilon,t,x}(\overline{y})+qD_{n,\epsilon,x}(\overline{y})+DqD_{n,\epsilon,x}(\overline{y})\centerdot(\overline{y}-\overline{x})]dS(\overline{y})$

so that, as the initial conditions $\{qD_{n,\epsilon}, qD'_{n,\epsilon,t}\}$ are in $C_c^{n-1}(\mathcal{R}^3)$ and $C_c^{n-2}(\mathcal{R}^3)$ respectively, it follows that for $t > 0$, $\rho_{n,\epsilon,t} \in C_c^{n-3}(\mathcal{R}^3)$. In particularly, for $n \geq 6$, $\rho_{n,\epsilon,t} \in C_c^3(\mathcal{R}^3)$, $(*)$. We have that;

$$\Box^2(\rho_{n,\epsilon,t}) = \bigtriangledown^2(\rho_{n,\epsilon,t}) - \frac{1}{c^2}\frac{\partial^2\rho_{n,\epsilon,t}}{\partial t^2} = 0$$

so we can apply $\mathcal{F}$ to both sides and obtain, and differentiating under the integral sign;

$$\mathcal{F}(\bigtriangledown^2(\rho_{n,\epsilon,t}))(\overline{k},t) - \frac{1}{c^2}\frac{\partial^2\mathcal{F}(\rho_{n,\epsilon,t})(\overline{k},t)}{\partial t^2} = 0$$

As $\rho_{n,\epsilon,t} \in C_c^2(\mathcal{R}^3)$, by $(*)$, we have, using integration by parts, that;

$$\mathcal{F}(\bigtriangledown^2(\rho_{n,\epsilon,t}))(\overline{k},t) = -k^2\mathcal{F}(\rho_{n,\epsilon,t})(\overline{k},t)$$

so that;

$$-k^2\mathcal{F}(\rho_{n,\epsilon,t})(\overline{k},t) - \frac{1}{c^2}\frac{\partial^2\mathcal{F}(\rho_{n,\epsilon,t})(\overline{k},t)}{\partial t^2} = 0$$

and we can use Peano's theorem to solve the ODE in time, to obtain;

$$\mathcal{F}(\rho_{n,\epsilon,t})(\overline{k},t) = A(\overline{k})e^{ikct} + B(\overline{k})e^{-ikct}$$

where, at $t = 0$;

$$A(\overline{k}) + B(\overline{k}) = \mathcal{F}(qD_{n,\epsilon})$$

and, taking the time derivative at $t = 0$;

$$ikcA(\overline{k}) - ikcB(\overline{k}) = \mathcal{F}(qD'_{n,\epsilon,t})$$

Now we can solve the simultaneous equations to get the expressions for $\{A(\overline{k}), B(\overline{k})\}$, We have that $\mathcal{F}(qD_{n,\epsilon}) \in C^\infty(\mathcal{R}^3)$ as $qD_{n,\epsilon}$ has compact support and similarly for $\mathcal{F}(qD'_{n,\epsilon,t})$. Moreover, we have that, using integration by parts, for $k_1 \neq 0$;

$$\mathcal{F}(qD_{n,\epsilon,x}) = ik_1\mathcal{F}(qD_{n,\epsilon})$$

so that, for $k \neq 0$, using the fact that for $n \geq 6$, $qD_{n,\epsilon} \in C_c^5(\mathcal{R}^3)$, $qD'_{n,\epsilon,t} \in C^4(\mathcal{R}^3)$;

$\mathcal{F}((\bigtriangledown^2)^2 qD_{n,\epsilon}) = k^4\mathcal{F}(qD_{n,\epsilon})$

$|\mathcal{F}(qD_{n,\epsilon})| \leq \frac{|\mathcal{F}((\bigtriangledown^2)^2 qD_{n,\epsilon})|}{k^4}$

$\leq \frac{C}{k^4}$

$\mathcal{F}((\bigtriangledown^2)^2 qD'_{n,\epsilon,t}) = k^4\mathcal{F}(qD'_{n,\epsilon,t})$

$|\mathcal{F}(qD'_{n,\epsilon,t})| \leq \frac{|\mathcal{F}((\bigtriangledown^2)^2 qD'_{n,\epsilon,t})|}{k^4}$

$\leq \frac{D}{k^4}$ (†)

Converting to polar coordinates, this proves that $\{\mathcal{F}(qD_{n,\epsilon}), \mathcal{F}(qD'_{n,\epsilon,t})\} \subset L^1(\mathcal{R}^3)$. Now we can use the fact that, for $k \neq 0$;

$|A(\overline{k})| \leq \frac{|\mathcal{F}(qD_{n,\epsilon})|}{2} + \frac{|\mathcal{F}(qD'_{n,\epsilon,t})|}{2kc}$

$|B(\overline{k})| \leq \frac{|\mathcal{F}(qD_{n,\epsilon})|}{2} + \frac{|\mathcal{F}(qD'_{n,\epsilon,t})|}{2kc}$

the fact that $\{\mathcal{F}(qD_{n,\epsilon}), \mathcal{F}(qD'_{n,\epsilon,t}) \subset C^\infty(\mathcal{R}^3)$, and (†), using polar coordinates again, to prove that $\{A(\overline{k}), B(\overline{k})\} \subset L^1(\mathcal{R}^3)$. Finally, for $t > 0$;

$|\mathcal{F}(\rho_{n,\epsilon})| = |A(\overline{k})e^{ikct} + B(\overline{k}e^{-ikct}|$

$\leq |A(\overline{k})| + |B(\overline{k})|$

so that $\mathcal{F}(\rho_{n,\epsilon}) \in L^1(\mathcal{R}^3)$ for $t > 0$. It follows that we can apply the inversion theorem in the last step.

□

Lemma 0.8. *Let $\mathcal{F}$ be the 1-dimensional Fourier transform, then, for n even, $k \in \mathcal{R}$, $k \neq \frac{\pi}{\epsilon}(\frac{n}{2} - j)$, $0 \leq j \leq n$;*

$\mathcal{F}(\gamma_{n,\epsilon})(k) = -\frac{1}{2^{n-1}} \sum_{j=0}^{n} C_j^n \frac{1}{\pi(\frac{n}{2}-j)-\epsilon k}[(-1)^{\frac{n}{2}-j} sin(\epsilon k)]$

and, for $0 \leq j_0 \leq n$;

$\mathcal{F}(\gamma_{n,\epsilon})(\frac{\pi}{\epsilon}(\frac{n}{2} - j_0)) = \frac{1}{2^{n-1}} C_{j_0}^n$

In particularly;

$lim_{\epsilon\rightarrow 0}\mathcal{F}(\gamma_{n,\epsilon})=\frac{n!}{2^{n-1}(\frac{n}{2}!)^2}$

uniformly on compact subsets of $\mathcal{R}$. *For* n *even;*

$\mathcal{F}(\delta_{n,\epsilon})=\frac{2^{n-1}[\frac{n}{2}!]^2}{n!}\mathcal{F}(\gamma_{n,\epsilon})$

in particularly;

$lim_{\epsilon\rightarrow 0}\mathcal{F}(\delta_{n,\epsilon})=1$

uniformly on compact subsets of $\mathcal{R}$. *For* n *even;*

$\mathcal{F}(\delta'_{n,\epsilon,t})(k)=ivk\mathcal{F}(\delta_{n,\epsilon})$

in particularly;

$lim_{\epsilon\rightarrow 0}\mathcal{F}(\delta'_{n,\epsilon,t})(k)=ivk$

uniformly on compact subsets of $\mathcal{R}$.

Proof. We have that, for $k\in\mathcal{R}$, n even, $k\neq\frac{\pi}{\epsilon}(\frac{n}{2}-j)$, $0\leq j\leq n$, by the definition of $\gamma_{n,\epsilon}$ and $\mathcal{F}$;

$\mathcal{F}(\gamma_{n,\epsilon})=\frac{1}{\epsilon}\int_{-\epsilon}^{\epsilon}cos^n(\frac{\pi x}{2\epsilon})e^{-ixk}dx$

$=\frac{1}{\epsilon}\int_{-\epsilon}^{\epsilon}(\frac{e^{\frac{i\pi x}{2\epsilon}}+e^{\frac{-i\pi x}{2\epsilon}}}{2})^n e^{-ixk}dx$

$=\frac{1}{2^n\epsilon}\int_{-\epsilon}^{\epsilon}\sum_{j=0}^{n}C_j^n e^{\frac{i\pi x(n-j)}{2\epsilon}}e^{\frac{-i\pi xj}{2\epsilon}}e^{-ixk}dx$

$=\frac{1}{2^n\epsilon}\sum_{j=0}^{n}C_j^n\int_{-\epsilon}^{\epsilon}e^{\frac{i\pi x(n-2j-\frac{2\epsilon k}{\pi})}{2\epsilon}}dx$

$=\frac{1}{2^n\epsilon}\sum_{j=0}^{n}C_j^n[\frac{e^{\frac{i\pi x(n-2j-\frac{2\epsilon k}{\pi})}{2\epsilon}}}{\frac{i\pi}{2\epsilon}(n-2j-\frac{2\epsilon k}{\pi})}]_{-\epsilon}^{\epsilon}$

$=\frac{1}{2^n\epsilon}\sum_{j=0}^{n}C_j^n\frac{1}{\frac{i\pi}{2\epsilon}(n-2j-\frac{2\epsilon k}{\pi})}[e^{\frac{i\pi x(n-2j-\frac{2\epsilon k}{\pi})}{2\epsilon}}]_{-\epsilon}^{\epsilon}$

$=\frac{1}{2^n\epsilon}\sum_{j=0}^{n}C_j^n\frac{1}{\frac{i\pi}{2\epsilon}(n-2j)-ik}[i^{n-2j-\frac{2\epsilon k}{\pi}}-(-i)^{n-2j-\frac{2\epsilon k}{\pi}}]$

$= \frac{1}{2^n\epsilon}\sum_{j=0}^{n} C_j^n \frac{1}{\frac{i\pi}{2\epsilon}(n-2j)-ik}[i^{n-2j}e^{-i\epsilon k}-(-i)^{n-2j}e^{i\epsilon k}]$

$= \frac{1}{2^n\epsilon}\sum_{j=0}^{n} C_j^n \frac{1}{\frac{i\pi}{2\epsilon}(n-2j)-ik}[i^{n-2j}e^{-i\epsilon k}-i^{n-2j}e^{i\epsilon k}]$, $(n-2j)$ even

$= \frac{1}{2^n\epsilon}\sum_{j=0}^{n} C_j^n \frac{1}{\frac{i\pi}{2\epsilon}(n-2j)-ik}[-2i^{n-2j+1}sin(\epsilon k)]$

$= \frac{1}{2^n\epsilon}\sum_{j=0}^{n} C_j^n \frac{1}{\frac{\pi}{2\epsilon}(n-2j)-k}[-2i^{n-2j}sin(\epsilon k)]$

$= -\frac{1}{2^{n-1}\epsilon}\sum_{j=0}^{n} C_j^n \frac{1}{\frac{\pi}{2\epsilon}(n-2j)-k}[(-1)^{\frac{n}{2}-j}sin(\epsilon k)]$

$= -\frac{1}{2^{n-1}}\sum_{j=0}^{n} C_j^n \frac{1}{\pi(\frac{n}{2}-j)-\epsilon k}[(-1)^{\frac{n}{2}-j}sin(\epsilon k)]$

We know that $\mathcal{F}(\gamma_{n,\epsilon})$ is continuous, so that, for fixed ϵ, $0\leq j_0\leq n$, using L'Hopital's rule;

$\mathcal{F}(\gamma_{n,\epsilon})(\frac{\pi}{\epsilon}(\frac{n}{2}-j_0)) = lim_{k\rightarrow\frac{\pi}{\epsilon}(\frac{n}{2}-j_0)}\mathcal{F}(\gamma_{n,\epsilon})(k)$

$= -\frac{1}{2^{n-1}}(\sum_{j=0,j\neq j_0}^{n} C_j^n \frac{1}{\pi(j_0-j)}[(-1)^{\frac{n}{2}-j}sin(\pi(\frac{n}{2}-j_0))]$

$-\frac{1}{2^{n-1}}C_{j_0}^n\frac{1}{-\epsilon}[(-1)^{\frac{n}{2}-j_0}\epsilon cos(\pi(\frac{n}{2}-j_0))]$

$= \frac{1}{2^{n-1}}C_{j_0}^n[(-1)^{\frac{n}{2}-j_0}(-1)^{\frac{n}{2}-j_0}]$

$= \frac{1}{2^{n-1}}C_{j_0}^n$

In particular;

$\mathcal{F}(\gamma_{n,\epsilon})(0) = \frac{1}{2^{n-1}}C_{\frac{n}{2}}^n$

$= \frac{n!}{2^{n-1}(\frac{n}{2}!)^2}$, $(*)$

independently of ϵ.

It follows that, for $k\neq 0$, using L'Hopital's rule;

$lim_{\epsilon\rightarrow 0}\mathcal{F}(\gamma_{n,\epsilon})(k) = lim_{\epsilon\rightarrow 0}\frac{1}{2^{n-1}}C_{\frac{n}{2}}^n\frac{1}{\epsilon k}sin(\epsilon k)$

$= \frac{1}{2^{n-1}}\frac{n!}{(\frac{n}{2}!)^2}lim_{\epsilon\rightarrow 0}\frac{1}{k}cos(\epsilon k)k$

$= \frac{n!}{2^{n-1}(\frac{n}{2}!)^2}lim_{\epsilon\rightarrow 0}cos(\epsilon k)$

$= \frac{n!}{2^{n-1}(\frac{n}{2}!)^2}$

and clearly;

$lim_{\epsilon \rightarrow 0}\mathcal{F}(\gamma_{n,\epsilon})(0) = \frac{n!}{2^{n-1}(\frac{n}{2}!)^2}$

by $(*)$.

It follows that;

$lim_{\epsilon \rightarrow 0}\mathcal{F}(\gamma_{n,\epsilon}) = \frac{n!}{2^{n-1}(\frac{n}{2}!)^2}$ (pointwise convergence)

To obtain uniform convergence on compact subsets, note that;

$\frac{1}{\epsilon k}sin(\epsilon k)$

converges uniformly to 1 on compact subsets as, for $|k| \leq K$;

$|\frac{1}{\epsilon k}sin(\epsilon k) - 1| = \frac{|sin(\epsilon k)-\epsilon k|}{|\epsilon k|}$

$= \frac{|\epsilon k + O((\epsilon k)^3) - \epsilon k|}{|\epsilon k|}$

$\leq C|\epsilon k|^2$

$\leq C\epsilon^2 K^2$

and $sin(\epsilon k)$

converges uniformly to 0 on compact subsets as , for $|k| \leq K$;

$|sin(\epsilon k)| \leq |\epsilon k|$

$\leq \epsilon K$

By Lemma 0.3, we have that, for n even;

$\mathcal{F}(\delta_{n,\epsilon}) = \frac{2^{n-1}[\frac{n}{2}!]^2}{n!}\mathcal{F}(\gamma_{n,\epsilon})$

so that, by the previous claim in this lemma;

$lim_{\epsilon\rightarrow 0}\mathcal{F}(\delta_{n,\epsilon}) = \frac{2^{n-1}[\frac{n}{2}!]^2}{n!}\frac{n!}{2^{n-1}(\frac{n}{2}!)^2} = 1$

on compact subsets of $\mathcal{R}$. We have that, using integration by parts and the definition of $\delta'_{n,\epsilon,t}$;

$\mathcal{F}(\delta'_{n,\epsilon,t})(k) = \mathcal{F}(v\delta'_{n,\epsilon})$

$= v(ik)\mathcal{F}(\delta_{n,\epsilon})$

$= ivk\mathcal{F}(\delta_{n,\epsilon})$

so that, by the previous claim;

$lim_{\epsilon\rightarrow 0}\mathcal{F}(\delta_{n,\epsilon}) = ivklim_{\epsilon\rightarrow 0}\mathcal{F}(\delta_{n,\epsilon})$

$= ivk$

uniformly on compact subsets of $\mathcal{R}$.

□

Lemma 0.9. *Let $\mathcal{F}$ be the three dimensional Fourier transform, then;*

$\mathcal{F}(D_{n,\epsilon}) = \mathcal{F}_1(\delta_{n,\epsilon})(k_1)\mathcal{F}_1(\delta_{n,\epsilon})(k_2)\mathcal{F}_1(\delta_{n,\epsilon})(k_3)$

where $\mathcal{F}_1$ is the 1-dimensional Fourier transform.

In particularly;

$lim_{\epsilon\rightarrow 0}\mathcal{F}(D_{n,\epsilon}) = 1$

uniformly on compact subsets of $\mathcal{R}^3$.

$\mathcal{F}(D'_{n,\epsilon,t})(k_1,k_2,k_3) = v_1\mathcal{F}_1(\delta'_{n,\epsilon})(k_1)\mathcal{F}_1(\delta_{n,\epsilon})(k_2)\mathcal{F}_1(\delta_{n,\epsilon})(k_3)$

$+v_2\mathcal{F}_1(\delta_{n,\epsilon})(k_1)\mathcal{F}_1(\delta'_{n,\epsilon})(k_2)\mathcal{F}_1(\delta_{n,\epsilon})(k_3)+v_3\mathcal{F}_1(\delta_{n,\epsilon})(k_1)\mathcal{F}_1(\delta_{n,\epsilon})(k_2)\mathcal{F}_1(\delta'_{n,\epsilon})(k_3)$

In particularly;

$lim_{\epsilon\rightarrow 0}\mathcal{F}(D'_{n,\epsilon,t})(\overline{k}) = i\overline{v}\cdot\overline{k}$

uniformly on compact subsets of $\mathcal{R}^3$.

Proof. By Definition 0.5 and the theorem of Fubini;

$$\mathcal{F}(D_{n\epsilon})(k_1,k_2,k_3)=\mathcal{F}_1(\delta_{n,\epsilon})(k_1)\mathcal{F}_1(\delta_{n,\epsilon})(k_2)\mathcal{F}_1(\delta_{n,\epsilon})(k_3)$$

where $\mathcal{F}_1$ is the 1-dimensional Fourier transform. In particular, as the projections of a compact subset are compact, and using the result of Lemma 0.8;

$$lim_{\epsilon\rightarrow 0}\mathcal{F}(D_{n\epsilon})(\overline{k})=1.1.1$$

$$=1$$

uniformly on compact subsets of $\mathcal{R}^3$.

By definition 0.5 again and Fubini's theorem;

$$\mathcal{F}(D'_{n,\epsilon,t})(k_1,k_2,k_3)=v_1\mathcal{F}_1(\delta'_{n,\epsilon})(k_1)\mathcal{F}_1(\delta_{n,\epsilon})(k_2)\mathcal{F}_1(\delta_{n,\epsilon})(k_3)$$

$$v_2\mathcal{F}_1(\delta_{n,\epsilon})(k_1)\mathcal{F}_1(\delta'_{n,\epsilon})(k_2)\mathcal{F}_1(\delta_{n,\epsilon})(k_3)+v_3\mathcal{F}_1(\delta_{n,\epsilon})(k_1)\mathcal{F}_1(\delta_{n,\epsilon})(k_2)\mathcal{F}_1(\delta'_{n,\epsilon})(k_3)$$

so that, using the result of Lemma 0.8, integration by parts and the previous observation;

$$lim_{\epsilon\rightarrow 0}\mathcal{F}(D'_{n,\epsilon,t})(k_1,k_2,k_3)=v_1(ik_1).1.1+v_2(ik_2).1.1+v_3(ik_3).1.1$$

$$=i\overline{v}\centerdot\overline{k}$$

uniformly on compact subsets of $\mathcal{R}^3$.

□

Lemma 0.10. *For the charge distribution* $\rho_{n,\epsilon}$ *of Lemma 0.7, we have that;*

$$lim_{\epsilon\rightarrow 0}A_{n,\epsilon}(\overline{k})=\frac{q}{2}+\frac{q\overline{v}\centerdot\overline{k}}{2kc}$$

$$lim_{\epsilon\rightarrow 0}B_{n,\epsilon}(\overline{k})=\frac{q}{2}-\frac{q\overline{v}\centerdot\overline{k}}{2kc}$$

uniformly on compact subsets of $\mathcal{R}^3\setminus\{0\}$.

Proof. We have that, by Lemma 0.7 and the results of Lemma 0.9;

$lim_{\epsilon\rightarrow 0}A_{n,\epsilon}(\overline{k})=lim_{\epsilon\rightarrow 0}(\frac{\mathcal{F}(qD_{n,\epsilon})}{2}+\frac{\mathcal{F}(qD'_{n,\epsilon,t})}{2ikc})$

$=\frac{q}{2}+\frac{qi\overline{v}.\overline{k}}{2ikc}$

$=\frac{q}{2}+\frac{q\overline{v}.\overline{k}}{2kc}$

$lim_{\epsilon\rightarrow 0}B_{n,\epsilon}(\overline{k})=lim_{\epsilon\rightarrow 0}(\frac{\mathcal{F}(qD_{n,\epsilon})}{2}-\frac{\mathcal{F}(qD'_{n,\epsilon,t})}{2ikc})$

$=\frac{q}{2}-\frac{qi\overline{v}.\overline{k}}{2ikc}$

$=\frac{q}{2}-\frac{q\overline{v}.\overline{k}}{2kc}$

uniformly on compact subsets of $\mathcal{R}^3\setminus\{0\}$.

□

Lemma 0.11. *For $n\geq 6$, for $k\in\mathcal{R}_{>0}$, we define the intensity $I_{n,\epsilon}(\overline{x},t,k)$ at k to be;*

$I_{n,\epsilon}(\overline{x},t,k)=\frac{1}{(2\pi)^3}\int_{S_k}(A(\overline{k})e^{ikct}+B(\overline{k})e^{-ikct})e^{i\overline{k}.\overline{x}}dS(\overline{k})$

so that, by Lemma 0.7;

$\rho_{n,\epsilon}(\overline{x},t)=\int_{\mathcal{R}}I(\overline{x},t,k)dk$

Then;

$lim_{\epsilon\rightarrow 0}I_{n,\epsilon}(s\overline{v},t,k)=\frac{q}{(2\pi)^3}(\frac{4\pi k}{sv}sin(svk)cos(ctk)-\frac{4\pi}{s^2vc}sin(svk)sin(ctk)$

$+\frac{4\pi k}{cs}cos(svk)sin(ctk))$

For the electron at time t moving a distance d, we obtain local maxima in the wave number when;

$2kd=-Dtan(2kd)$

where $D=\frac{c-v}{c+v}$

Proof. By Lemma 0.10 and the definition of $I_{n,\epsilon}$, we have that;

$lim_{\epsilon\rightarrow 0}I_{n,\epsilon}(s\overline{v},t,k)=\frac{1}{(2\pi)^3}\int_{S_k}(A_{n,\epsilon}(\overline{k})e^{ikct}+B_{n,\epsilon}(\overline{k})e^{-ikct})e^{i\overline{k}.s\overline{v}}dS(\overline{k})$

$=\frac{q}{(2\pi)^3}e^{ikct}\int_{S_k}(\frac{1}{2}+\frac{\overline{v}.\overline{k}}{2kc})e^{is\overline{v}.\overline{k}}dS(\overline{k})+\frac{q}{(2\pi)^3}e^{-ikct}\int_{S_k}(\frac{1}{2}-\frac{\overline{v}.\overline{k}}{2kc})e^{is\overline{v}.\overline{k}}dS(\overline{k})$,
$(*)$

Switching to polar coordinates, letting θ be the angle between $\overline{v}$ and $\overline{k}\in S_k$, we have that;

$\int_{S_k}\frac{1}{2}e^{is\overline{v}.\overline{k}}dS(\overline{k})=\frac{1}{2}\int_0^{\pi}\int_{-\pi}^{\pi}e^{isvkcos(\theta)}k^2sin(\theta)d\theta d\phi$

$=\pi k^2\int_0^{\pi}e^{isvkcos(\theta)}sin(\theta)d\theta$

$=\pi k^2\int_{-1}^{1}e^{isvku}du$, $(u=cos(\theta),du=-sin(\theta)d\theta)$

$=\pi k^2[\frac{e^{isvku}}{isvk}]_{-1}^{1}$

$=\frac{\pi k^2}{isvk}(e^{isvk}-e^{-isvk})$

$=\frac{\pi k^2}{isvk}2isin(svk)$

$=\frac{2\pi ksin(svk)}{sv}$

and;

$\int_{S_k}\frac{\overline{v}.\overline{k}}{2kc}e^{is\overline{v}.\overline{k}}dS(\overline{k})=\frac{1}{2}\int_0^{\pi}\int_{-\pi}^{\pi}\frac{vkcos(\theta)}{2kc}e^{isvkcos(\theta)}k^2sin(\theta)d\theta d\phi$

$=\frac{\pi vk^2}{c}\int_0^{\pi}e^{isvkcos(\theta)}cos(\theta)sin(\theta)d\theta$

$=\frac{\pi vk^2}{c}\int_{-1}^{1}ue^{isvku}du$, $(u=cos(\theta),du=-sin(\theta)d\theta)$

$=\frac{\pi vk^2}{c}([\frac{ue^{isvku}}{isvk}]_{-1}^{1}-\int_{-1}^{1}\frac{e^{isvku}}{isvk}du)$

$=\frac{\pi vk^2}{c}(\frac{e^{isvk}+e^{-isvk}}{isvk}-[\frac{e^{isvku}}{(isvk)^2}]_{-1}^{1})$

$=\frac{\pi vk^2}{cisvk}(2cos(svk))-\frac{\pi vk^2}{c(isvk)^2}2isin(svk)$

$=\frac{2\pi isin(svk)}{s^2vc}-\frac{2\pi ikcos(svk)}{cs}$

It follows from $(*)$, that;

$$lim_{\epsilon\rightarrow 0}I_{n,\epsilon}(s\overline{v},t,k)=\frac{q}{(2\pi)^3}e^{ikct}(\frac{2\pi ksin(svk)}{sv}+\frac{2\pi isin(svk)}{s^2vc}-\frac{2\pi ikcos(svk)}{cs})$$

$$+\frac{q}{(2\pi)^3}e^{-ikct}(\frac{2\pi ksin(svk)}{sv}-\frac{2\pi isin(svk)}{s^2vc}+\frac{2\pi ikcos(svk)}{cs})$$

$$=\frac{q}{(2\pi)^3}(\frac{2\pi k}{sv}sin(svk)2cos(kct)+\frac{2\pi i^2}{s^2vc}sin(svk)2sin(kct)-\frac{2\pi ik}{cs}cos(svk)2isin(kct))$$

$$=\frac{q}{(2\pi)^3}(\frac{4\pi k}{sv}sin(svk)cos(kct)-\frac{4\pi}{s^2vc}sin(svk)sin(kct)+\frac{4\pi k}{cs}cos(svk)sin(kct))$$

(†)

The electron wave propagates at speed c, so we are interested in the case when $|s\overline{v}|=ct$, so that $s=\frac{ct}{v}$. Making this substitution in (†), we obtain that;

$$lim_{\epsilon\rightarrow 0}I_{n,\epsilon}(\frac{ct}{v}\overline{v},t,k)=\frac{q}{(2\pi)^3}(\frac{4\pi k}{ct}sin(kct)cos(kct)-\frac{4\pi v}{c^3t^2}sin^2(kct)+\frac{4\pi kv}{c^2t}cos(kct)sin(kct))$$

$$=\frac{q}{(2\pi)^3}([\frac{4\pi k}{ct}+\frac{4\pi kv}{c^2t}]sin(kct)cos(kct)-\frac{4\pi v}{c^3t^2}sin^2(kct))$$

$$=\frac{q}{(2\pi)^3}([\frac{2\pi k}{ct}+\frac{2\pi kv}{c^2t}]sin(2kct)-\frac{4\pi v}{c^3t^2}(\frac{1-cos(2kct)}{2}))$$

$$=\frac{q}{(2\pi)^3}([\frac{2\pi k}{ct}+\frac{2\pi kv}{c^2t}]sin(2kct)+\frac{2\pi v}{c^3t^2}cos(2kct)-\frac{2\pi v}{c^3t^2})$$

We look for a local maximum in k, so that;

$$\frac{d}{dk}lim_{\epsilon\rightarrow 0}I_{n,\epsilon}(\frac{ct}{v}\overline{v},t,k)=\frac{d}{dk}(\frac{q}{(2\pi)^3}([\frac{2\pi k}{ct}+\frac{2\pi kv}{c^2t}]sin(2kct)+\frac{2\pi v}{c^3t^2}cos(2kct)$$

$$-\frac{2\pi v}{c^3t^2}))$$

$$=\frac{q}{(2\pi)^3}([\frac{2\pi}{ct}+\frac{2\pi v}{c^2t}]sin(2kct)+2ct[\frac{2\pi k}{ct}+\frac{2\pi kv}{c^2t}]cos(2kct)-2ct\frac{2\pi v}{c^3t^2}sin(2kct))$$

$$=\frac{q}{(2\pi)^3}([\frac{2\pi}{ct}+\frac{2\pi v}{c^2t}]sin(2kct)+[4\pi k+\frac{4\pi kv}{c}]cos(2kct)-\frac{4\pi v}{c^2t}sin(2kct))$$

$$=\frac{q}{(2\pi)^3}([\frac{2\pi}{ct}-\frac{2\pi v}{c^2t}]sin(2kct)+[4\pi k+\frac{4\pi kv}{c}]cos(2kct))$$

$$=0$$

so that, rearranging;

$$k=-Dtan(2kct)$$

where $D=\frac{\frac{2\pi}{ct}-\frac{2\pi v}{c^2t}}{4\pi(1+\frac{v}{c})}=\frac{c-v}{2ct(c+v)}$

Now use the fact that the distance at time t is $d=ct$.

□

Lemma 0.12. *For an electron moving at velocity $(v,0,0)$ in the laboratory frame S, the amplitudes in the spectrum of the stationary electron are shifted by the Doppler factor $\frac{k'}{k}$ in the frame S' of the electron, moving at velocity $(v,0,0)$ relative to S.*

Proof. We consider a frame S' moving with velocity $(v,0,0)$ relative to the base frame S. For $n\geq 6$, by results of [7], [8] and [9], given $\rho_{n,\epsilon}$ in the base frame S, there exists a unique current $\overline{J}_{n,\epsilon}$ with compact support, such that the standard relations are satisfied for $(\rho_{n,\epsilon},\overline{J}_{n,\epsilon})$. By results of [7], we have that $\square^2(\overline{J}_{n,\epsilon})=\overline{0}$ and moreover, $\overline{J}_{n,\epsilon}$ has the wave representaion;

$\overline{J}_{n,\epsilon}=\frac{1}{(2\pi)^3}\int_{\mathcal{R}^3}(\overline{A}_{n,\epsilon}(\overline{k})e^{ikct}+\overline{B}_{n,\epsilon}(\overline{k})e^{-ikct})e^{i\overline{k}\cdot\overline{x}}d\overline{k}$ (†)

where, for $k\neq 0$;

$\overline{A}_{n,\epsilon}(\overline{k})=-\frac{cA_{n,\epsilon}(\overline{k})}{k}\overline{k}$

$\overline{B}_{n,\epsilon}(\overline{k})=\frac{cB_{n,\epsilon}(\overline{k})}{k}\overline{k}$ $(*)$

If the electron travels with velocity vector $(v,0,0)$ in the base frame S, then by Lemma 0.10, we have that;

$lim_{\epsilon\rightarrow 0}A_{n,\epsilon}(\overline{k})=\frac{q}{2}+\frac{qvk_1}{2kc}$

$lim_{\epsilon\rightarrow 0}B_{n,\epsilon}(\overline{k})=\frac{q}{2}-\frac{qvk_1}{2kc}$

so that, by $(*)$, uniformly on compact subsets of $\mathcal{R}^3\setminus\{\overline{0}\}$;

$lim_{\epsilon\rightarrow 0}\overline{A}_{n,\epsilon}(\overline{k})=[-\frac{cq}{2k}-\frac{qvk_1}{2k^2}]\overline{k}$

$lim_{\epsilon\rightarrow 0}\overline{B}_{n,\epsilon}(\overline{k})=[\frac{cq}{2k}-\frac{qvk_1}{2k^2}]\overline{k}$ (D)

We have that $\rho_{n,\epsilon}$ transforms to S' as;

$\rho'_{n,\epsilon}(\overline{x}',t')=\gamma_v(\rho_{n,\epsilon}-\frac{vj_{1,n,\epsilon}}{c^2})(\overline{x},t)$

see [1], so that, using (†) and the wave representation for $\rho_{n,\epsilon}$;

$\rho'_{n,\epsilon}(\overline{x}',t')=\frac{1}{(2\pi)^3}\int_{\mathcal{R}^3}(\gamma_v A_{n,\epsilon}(\overline{k})e^{ikct}+\gamma_v B_{n,\epsilon}(\overline{k})e^{-ikct})e^{i\overline{k}.\overline{x}}d\overline{k}$

$-\frac{1}{(2\pi)^3}\int_{\mathcal{R}^3}(\frac{\gamma_v v}{c^2}\overline{A}_{1,n,\epsilon}(\overline{k})e^{ikct}+\frac{\gamma_v v}{c^2}\overline{B}_{1,n,\epsilon}(\overline{k})e^{-ikct})e^{i\overline{k}.\overline{x}}d\overline{k}$

$=\frac{1}{(2\pi)^3}\int_{\mathcal{R}^3}([\gamma_v A_{n,\epsilon}(\overline{k})-\frac{\gamma_v v}{c^2}\overline{A}_{1,n,\epsilon}(\overline{k})]e^{ikct}+[\gamma_v B_{n,\epsilon}(\overline{k})-\frac{\gamma_v v}{c^2}\overline{B}_{1,n,\epsilon}(\overline{k})]e^{-ikct})e^{i\overline{k}.\overline{x}}d\overline{k}$
(††)

We have that, using the inverse Lorentz transformation;

$t=\gamma_v t'+\frac{\gamma_v v x'_1}{c^2}$

$x_1=\gamma_v x'_1+\gamma_v v t'$

$x_2=x'_2$

$x_3=x'_3$

so that, substituting into (††);

$\rho'_{n,\epsilon}(\overline{x}',t')=\frac{1}{(2\pi)^3}\int_{\mathcal{R}^3}([\gamma_v A_{n,\epsilon}(\overline{k})-\frac{\gamma_v v}{c^2}\overline{A}_{1,n,\epsilon}(\overline{k})]e^{ikc(\gamma_v t'+\frac{\gamma_v v x'_1}{c^2})}$

$+[\gamma_v B_{n,\epsilon}(\overline{k})-\frac{\gamma_v v}{c^2}\overline{B}_{1,n,\epsilon}(\overline{k})]e^{-ikc(\gamma_v t'+\frac{\gamma_v v x'_1}{c^2})})e^{ik_1(\gamma_v x'_1+\gamma_v v t')}e^{ik_2x'_2}e^{ik_3x'_3}d\overline{k}$

$=\frac{1}{(2\pi)^3}\int_{\mathcal{R}^3}[\gamma_v A_{n,\epsilon}(\overline{k})-\frac{\gamma_v v}{c^2}\overline{A}_{1,n,\epsilon}(\overline{k})]e^{i(kc\gamma_v+k_1\gamma_v v)t'}e^{i(k_1\gamma_v+\frac{k\gamma_v v}{c})x'_1}e^{ik_2x'_2}e^{ik_3x'_3}d\overline{k}$

$+\frac{1}{(2\pi)^3}\int_{\mathcal{R}^3}[\gamma_v B_{n,\epsilon}(\overline{k})-\frac{\gamma_v v}{c^2}\overline{B}_{1,n,\epsilon}(\overline{k})]e^{-i(kc\gamma_v-k_1\gamma_v v)t'}e^{i(k_1\gamma_v-\frac{k\gamma_v v}{c})x'_1}e^{ik_2x'_2}e^{ik_3x'_3}d\overline{k}$

(A)

We make the change of variables, in the first line of (A);

$k'c=kc\gamma_v+k_1\gamma_v v$

$k'=k\gamma_v+\frac{k_1\gamma_v v}{c}$

$k'_1=k_1\gamma_v+\frac{k\gamma_v v}{c}$

$k'_2 = k_2$

$k'_3 = k_3$

and, in the second line of (A);

$k'c = kc\gamma_v - k_1\gamma_v v$

$k' = k\gamma_v - \frac{k_1\gamma_v v}{c}$

$k'_1 = k_1\gamma_v - \frac{k\gamma_v v}{c}$

$k'_2 = k_2$

$k'_3 = k_3$

which are the Doppler shift transformations given in [14], for the velocity vector $(v, 0, 0)$. Calculating the Jacobian, and using the chain rule, we have that, for the first and second lines of (A) respectively;

$dk_1dk_2dk_3 = (\gamma_v - \frac{\gamma_v k'_1 v}{k'c})dk'_1dk'_2dk'_3$

$dk_1dk_2dk_3 = (\gamma_v + \frac{\gamma_v k'_1 v}{k'c})dk'_1dk'_2dk'_3$

so that, substituting in (A);

$\rho'_{n,\epsilon}(\overline{x}', t') = \frac{1}{(2\pi)^3}\int_{\mathcal{R}^3}[\gamma_v A_{n,\epsilon}(\overline{k}') - \frac{\gamma_v v}{c^2}\overline{A}_{1,n,\epsilon}(\overline{k}')]e^{ik'ct'}e^{ik'_1x'_1}e^{ik'_2x'_2}e^{ik'_3x'_3}$

$(\gamma_v - \frac{\gamma_v k'_1 v}{k'c})d\overline{k}'$

$+\frac{1}{(2\pi)^3}\int_{\mathcal{R}^3}[\gamma_v B_{n,\epsilon}(\overline{k}') - \frac{\gamma_v v}{c^2}\overline{B}_{1,n,\epsilon}(\overline{k}')]e^{-ik'ct'}e^{ik'_1x'_1}e^{ik_2x'_2}e^{ik_3x'_3}(\gamma_v + \frac{\gamma_v k'_1 v}{k'c})d\overline{k}'$

$= \frac{1}{(2\pi)^3}\int_{\mathcal{R}^3}[\gamma_v A_{n,\epsilon}(\overline{k}') - \frac{\gamma_v v}{c^2}\overline{A}_{1,n,\epsilon}(\overline{k}')]e^{ik'ct'}e^{i\overline{k}'\cdot\overline{x}'}(\gamma_v - \frac{\gamma_v k'_1 v}{k'c})d\overline{k}'$

$+\frac{1}{(2\pi)^3}\int_{\mathcal{R}^3}[\gamma_v B_{n,\epsilon}(\overline{k}') - \frac{\gamma_v v}{c^2}\overline{B}_{1,n,\epsilon}(\overline{k}')]e^{-ik'ct'}e^{i\overline{k}'\cdot\overline{x}'}(\gamma_v + \frac{\gamma_v k'_1 v}{k'c})d\overline{k}'$

so that equating coefficients, and using the wave representation for $\rho'_{n,\epsilon}(\overline{x}', t')$, we have that;

$A'_{n,\epsilon}(\overline{k}') = (\gamma_v - \frac{\gamma_v k'_1 v}{k'c})[\gamma_v A_{n,\epsilon}(\overline{k}') - \frac{\gamma_v v}{c^2}\overline{A}_{1,n,\epsilon}(\overline{k}')]$

$B'_{n,\epsilon}(\overline{k}') = (\gamma_v + \frac{\gamma_v k'_1 v}{k'c})[\gamma_v B_{n,\epsilon}(\overline{k}') - \frac{\gamma_v v}{c^2}\overline{B}_{1,n,\epsilon}(\overline{k}')]$ (E)

and, using (D), with the inverse relations for the first line;

$k = k'\gamma_v - \frac{k'_1\gamma_v v}{c}$

$k_1 = k'_1\gamma_v - \frac{k'\gamma_v v}{c}$

$k'_2 = k_2$

$k'_3 = k_3$

$lim_{\epsilon\rightarrow 0}A_{n,\epsilon}(\overline{k}') = \frac{q}{2} + \frac{qvk_1}{2kc}$

$= \frac{q}{2} + \frac{qv(k'_1\gamma_v - \frac{k'\gamma_v v}{c})}{2(k'\gamma_v - \frac{k'_1\gamma_v v}{c})c}$

$lim_{\epsilon\rightarrow 0}\overline{A}_{1,n,\epsilon}(\overline{k}') = [-\frac{cq}{2k} - \frac{qvk_1}{2k^2}]k_1$

$= -\frac{cq(k'_1\gamma_v - \frac{k'\gamma_v v}{c})}{2(k'\gamma_v - \frac{k'_1\gamma_v v}{c})} - \frac{qv(k'_1\gamma_v - \frac{k'\gamma_v v}{c})^2}{2(k'\gamma_v - \frac{k'_1\gamma_v v}{c})^2}$

and, substituting into the first line of (E);

$\lim_{\epsilon\rightarrow 0} A'_{n,\epsilon}(\overline{k}') = (\gamma_v - \frac{\gamma_v k'_1 v}{k'c})[\gamma_v(\frac{q}{2} + \frac{qv(k'_1\gamma_v - \frac{k'\gamma_v v}{c})}{2(k'\gamma_v - \frac{k'_1\gamma_v v}{c})c}) - \frac{\gamma_v v}{c^2}(-\frac{cq(k'_1\gamma_v - \frac{k'\gamma_v v}{c})}{2(k'\gamma_v - \frac{k'_1\gamma_v v}{c})}$
$-\frac{qv(k'_1\gamma_v - \frac{k'\gamma_v v}{c})^2}{2(k'\gamma_v - \frac{k'_1\gamma_v v}{c})^2})]$

Again, using (D), with the inverse relations for the second line;

$k = k'\gamma_v + \frac{k'_1\gamma_v v}{c}$

$k_1 = k'_1\gamma_v + \frac{k'\gamma_v v}{c}$

$k'_2 = k_2$

$k'_3 = k_3$

$lim_{\epsilon\rightarrow 0}B_{n,\epsilon}(\overline{k}') = \frac{q}{2} - \frac{qvk_1}{2kc}$

$= \frac{q}{2} - \frac{qv(k'_1\gamma_v + \frac{k'\gamma_v v}{c})}{2(k'\gamma_v + \frac{k'_1\gamma_v v}{c})c}$

$lim_{\epsilon\rightarrow 0}\overline{B}_{1,n,\epsilon}(\overline{k}')=[\frac{cq}{2k}-\frac{qvk_1}{2k^2}]k_1$

$=\frac{cq(k_1'\gamma_v+\frac{k'\gamma_v v}{c})}{2(k'\gamma_v+\frac{k_1'\gamma_v v}{c})}-\frac{qv(k_1'\gamma_v+\frac{k'\gamma_v v}{c})^2}{2(k'\gamma_v+\frac{k_1'\gamma_v v}{c})^2}$

and, substituting into the second line of (E);

$\lim_{\epsilon\rightarrow 0}B'_{n,\epsilon}(\overline{k}')=(\gamma_v+\frac{\gamma_v k_1' v}{k'c})[\gamma_v(\frac{q}{2}-\frac{qv(k_1'\gamma_v+\frac{k'\gamma_v v}{c})}{2(k'\gamma_v+\frac{k_1'\gamma_v v}{c})c})-\frac{\gamma_v v}{c^2}(\frac{cq(k_1'\gamma_v+\frac{k'\gamma_v v}{c})}{2(k'\gamma_v+\frac{k_1'\gamma_v v}{c})}$

$-\frac{qv(k_1'\gamma_v+\frac{k'\gamma_v v}{c})^2}{2(k'\gamma_v+\frac{k_1'\gamma_v v}{c})^2})]$

Rearranging, we have that;

$\lim_{\epsilon\rightarrow 0}A'_{n,\epsilon}(\overline{k}')=q\gamma_v^2(1-\frac{k_1'v}{k'c})[(\frac{1}{2}+\frac{vk_1'(1-\frac{k'v}{k_1'c})}{2k'(1-\frac{k_1'v}{k'c})c})-\frac{v}{c^2}(-\frac{ck_1'(1-\frac{k'v}{k_1'c})}{2k'(1-\frac{k_1'v}{k'c})}$

$-\frac{vk_1'^2(1-\frac{k'v}{k_1'c})^2}{2k'^2(1-\frac{k_1'v}{k'c})^2})]$

$=q\gamma_v^2(1-\frac{k_1'v}{k'c})[\frac{1}{2}+\frac{vk_1'(1-\frac{k'v}{k_1'c})}{k'(1-\frac{k_1'v}{k'c})c}+\frac{v^2k_1'^2(1-\frac{k'v}{k_1'c})^2}{2k'^2c^2(1-\frac{k_1'v}{k'c})^2}]$

$=\frac{q\gamma_v^2}{2}(1-\frac{k_1'v}{k'c})[1+\frac{vk_1'(1-\frac{k'v}{k_1'c})}{k'(1-\frac{k_1'v}{k'c})c}]^2$

$=\frac{q\gamma_v^2}{2}(1-\frac{k_1'v}{k'c})[\frac{k'(1-\frac{k_1'v}{k'c})c+vk_1'(1-\frac{k'v}{k_1'c})}{k'(1-\frac{k_1'v}{k'c})c}]^2$

$=\frac{q\gamma_v^2}{2}(1-\frac{k_1'v}{k'c})[\frac{k'c-k_1'v+k_1'v-\frac{v^2k'}{c}}{k'(1-\frac{k_1'v}{k'c})c}]^2$

$=\frac{q\gamma_v^2}{2}(1-\frac{k_1'v}{k'c})[\frac{k'c(1-\frac{v^2}{c^2})}{k'c(1-\frac{k_1'v}{k'c})}]^2$

$=\frac{q\gamma_v^2}{2}(1-\frac{k_1'v}{k'c})[\frac{(1-\frac{v^2}{c^2})}{(1-\frac{k_1'v}{k'c})}]^2$

$=\frac{q}{2}[\frac{(1-\frac{v^2}{c^2})}{(1-\frac{k_1'v}{k'c})}]$

$=\frac{q}{2\gamma_v}[\frac{1}{\gamma_v(1-\frac{k_1'v}{k'c})}]$

$=\frac{q'}{2}\frac{k'}{k}$

and, using the symmetry;

$\lim_{\epsilon\rightarrow 0} A'_{n,\epsilon}(\overline{k}')(v)=\lim_{\epsilon\rightarrow 0} B'_{n,\epsilon}(\overline{k}')(-v)$

we have;

$\lim_{\epsilon\rightarrow 0} B'_{n,\epsilon}(\overline{k}')=\frac{q\gamma_v^2}{2}(1+\frac{k_1'v}{k'c})[1-\frac{vk_1'(1+\frac{k'v}{k_1'c})}{k'(1+\frac{k_1'v}{k'c})c}]^2$

$=\frac{q'}{2}\frac{k'}{k}$

where $q'=\frac{q}{\gamma_v}$ is the new charge due to the change in mass of the electron, $m'=\frac{m}{\gamma_v}$ and $\frac{k'}{k}$ is the Doppler shift in amplitude of the spectrum of the stationary electron.

□

Lemma 0.13. *Let an electron move at velocity $(w,0,0)$ in the base frame S, and let S' move at velocity $(v,0,0)$ relative to S, then the amplitudes in the spectrum of the electron in S' are shifted by the background Doppler factor $\frac{k'}{k}$.*

Proof. The proof is almost the same as Lemma 0.12. We consider a frame S' moving with velocity $(v,0,0)$ relative to the base frame S. For $n\geq 6$, by results of [7], [8] and [9], given $\rho_{n,\epsilon}$ in the base frame S, there exists a unique current $\overline{J}_{n,\epsilon}$ with compact support, such that the standard relations are satisfied for $(\rho_{n,\epsilon},\overline{J}_{n,\epsilon})$. By results of [7], we have that $\square^2(\overline{J}_{n,\epsilon})=\overline{0}$ and moreover, $\overline{J}_{n,\epsilon}$ has the wave representaion;

$\overline{J}_{n,\epsilon}=\frac{1}{(2\pi)^3}\int_{\mathcal{R}^3}(\overline{A}_{n,\epsilon}(\overline{k})e^{ikct}+\overline{B}_{n,\epsilon}(\overline{k})e^{-ikct})e^{i\overline{k}\centerdot\overline{x}}d\overline{k}$ (†)

where, for $k\neq 0$;

$\overline{A}_{n,\epsilon}(\overline{k})=-\frac{cA_{n,\epsilon}(\overline{k})}{k}\overline{k}$

$\overline{B}_{n,\epsilon}(\overline{k})=\frac{cB_{n,\epsilon}(\overline{k})}{k}\overline{k}$ (∗)

If the electron travels with velocity vector $(w,0,0)$ in the base frame S, then by Lemma 0.10, we have that;

$lim_{\epsilon\rightarrow 0}A_{n,\epsilon}(\overline{k})=\frac{q}{2}+\frac{qwk_1}{2kc}$

$lim_{\epsilon\rightarrow 0}B_{n,\epsilon}(\overline{k})=\frac{q}{2}-\frac{qwk_1}{2kc}$

so that, by $(*)$, uniformly on compact subsets of $\mathcal{R}^3 \setminus \{\overline{0}\}$;

$lim_{\epsilon\rightarrow 0}\overline{A}_{n,\epsilon}(\overline{k}) = [-\frac{cq}{2k} - \frac{qwk_1}{2k^2}]\overline{k}$

$lim_{\epsilon\rightarrow 0}\overline{B}_{n,\epsilon}(\overline{k}) = [\frac{cq}{2k} - \frac{qwk_1}{2k^2}]\overline{k}$ (D)

We have that $\rho_{n,\epsilon}$ transforms to S' as;

$\rho'_{n,\epsilon}(\overline{x}',t') = \gamma_v(\rho_{n,\epsilon} - \frac{vj_{1,n,\epsilon}}{c^2})(\overline{x},t)$

see [1], so that, using $(\dagger)$ and the wave representation for $\rho_{n,\epsilon}$;

$\rho'_{n,\epsilon}(\overline{x}',t') = \frac{1}{(2\pi)^3}\int_{\mathcal{R}^3}(\gamma_v A_{n,\epsilon}(\overline{k})e^{ikct} + \gamma_v B_{n,\epsilon}(\overline{k})e^{-ikct})e^{i\overline{k}.\overline{x}}d\overline{k}$

$-\frac{1}{(2\pi)^3}\int_{\mathcal{R}^3}(\frac{\gamma_v v}{c^2}\overline{A}_{1,n,\epsilon}(\overline{k})e^{ikct} + \frac{\gamma_v v}{c^2}\overline{B}_{1,n,\epsilon}(\overline{k})e^{-ikct})e^{i\overline{k}.\overline{x}}d\overline{k}$

$= \frac{1}{(2\pi)^3}\int_{\mathcal{R}^3}([\gamma_v A_{n,\epsilon}(\overline{k}) - \frac{\gamma_v v}{c^2}\overline{A}_{1,n,\epsilon}(\overline{k})]e^{ikct} + [\gamma_v B_{n,\epsilon}(\overline{k}) - \frac{\gamma_v v}{c^2}\overline{B}_{1,n,\epsilon}(\overline{k})]e^{-ikct})e^{i\overline{k}.\overline{x}}d\overline{k}$
$(\dagger\dagger)$

We have that, using the inverse Lorentz transformation;

$t = \gamma_v t' + \frac{\gamma_v v x'_1}{c^2}$

$x_1 = \gamma_v x'_1 + \gamma_v v t'$

$x_2 = x'_2$

$x_3 = x'_3$

so that, substituting into $(\dagger\dagger)$;

$\rho'_{n,\epsilon}(\overline{x}',t') = \frac{1}{(2\pi)^3}\int_{\mathcal{R}^3}([\gamma_v A_{n,\epsilon}(\overline{k}) - \frac{\gamma_v v}{c^2}\overline{A}_{1,n,\epsilon}(\overline{k})]e^{ikc(\gamma_v t' + \frac{\gamma_v v x'_1}{c^2})}$

$+[\gamma_v B_{n,\epsilon}(\overline{k}) - \frac{\gamma_v v}{c^2}\overline{B}_{1,n,\epsilon}(\overline{k})]e^{-ikc(\gamma_v t' + \frac{\gamma_v v x'_1}{c^2})})e^{ik_1(\gamma_v x'_1 + \gamma_v v t')}e^{ik_2 x'_2}e^{ik_3 x'_3}d\overline{k}$

$= \frac{1}{(2\pi)^3}\int_{\mathcal{R}^3}[\gamma_v A_{n,\epsilon}(\overline{k}) - \frac{\gamma_v v}{c^2}\overline{A}_{1,n,\epsilon}(\overline{k})]e^{i(kc\gamma_v + k_1\gamma_v v)t'}e^{i(k_1\gamma_v + \frac{k\gamma_v v}{c})x'_1}e^{ik_2 x'_2}e^{ik_3 x'_3}d\overline{k}$

$+\frac{1}{(2\pi)^3}\int_{\mathcal{R}^3}[\gamma_v B_{n,\epsilon}(\overline{k}) - \frac{\gamma_v v}{c^2}\overline{B}_{1,n,\epsilon}(\overline{k})]e^{-i(kc\gamma_v - k_1\gamma_v v)t'}e^{i(k_1\gamma_v - \frac{k\gamma_v v}{c})x'_1}e^{ik_2 x'_2}e^{ik_3 x'_3}d\overline{k}$

(A)

We make the change of variables, in the first line of (A);

$k'c = kc\gamma_v + k_1\gamma_v v$

$k' = k\gamma_v + \frac{k_1\gamma_v v}{c}$

$k'_1 = k_1\gamma_v + \frac{k\gamma_v v}{c}$

$k'_2 = k_2$

$k'_3 = k_3$

and, in the second line of (A);

$k'c = kc\gamma_v - k_1\gamma_v v$

$k' = k\gamma_v - \frac{k_1\gamma_v v}{c}$

$k'_1 = k_1\gamma_v - \frac{k\gamma_v v}{c}$

$k'_2 = k_2$

$k'_3 = k_3$

which are the Doppler shift transformations given in [14], for the velocity vector $(v, 0, 0)$. Calculating the Jacobian, and using the chain rule, we have that, for the first and second lines of (A) respectively;

$dk_1 dk_2 dk_3 = (\gamma_v - \frac{\gamma_v k'_1 v}{k'c})dk'_1 dk'_2 dk'_3$

$dk_1 dk_2 dk_3 = (\gamma_v + \frac{\gamma_v k'_1 v}{k'c})dk'_1 dk'_2 dk'_3$

so that, substituting in (A);

$\rho'_{n,\epsilon}(\overline{x}', t') = \frac{1}{(2\pi)^3}\int_{\mathcal{R}^3}[\gamma_v A_{n,\epsilon}(\overline{k}') - \frac{\gamma_v v}{c^2}\overline{A}_{1,n,\epsilon}(\overline{k}')]e^{ik'ct'}e^{ik'_1 x'_1}e^{ik'_2 x'_2}e^{ik'_3 x'_3}$

$(\gamma_v - \frac{\gamma_v k'_1 v}{k'c})d\overline{k}'$

$+\frac{1}{(2\pi)^3}\int_{\mathcal{R}^3}[\gamma_v B_{n,\epsilon}(\overline{k}') - \frac{\gamma_v v}{c^2}\overline{B}_{1,n,\epsilon}(\overline{k}')]e^{-ik'ct'}e^{ik'_1 x'_1}e^{ik_2 x'_2}e^{ik_3 x'_3}(\gamma_v + \frac{\gamma_v k'_1 v}{k'c})d\overline{k}'$

$= \frac{1}{(2\pi)^3}\int_{\mathcal{R}^3}[\gamma_v A_{n,\epsilon}(\overline{k}') - \frac{\gamma_v v}{c^2}\overline{A}_{1,n,\epsilon}(\overline{k}')]e^{ik'ct'}e^{i\overline{k}'\cdot\overline{x}'}(\gamma_v - \frac{\gamma_v k'_1 v}{k'c})d\overline{k}'$

$+\frac{1}{(2\pi)^3}\int_{\mathcal{R}^3}[\gamma_v B_{n,\epsilon}(\overline{k}') - \frac{\gamma_v v}{c^2}\overline{B}_{1,n,\epsilon}(\overline{k}')]e^{-ik'ct'}e^{i\overline{k}'.\overline{x}'}(\gamma_v + \frac{\gamma_v k_1' v}{k'c})d\overline{k}'$

so that equating coefficients, and using the wave representation for $\rho'_{n,\epsilon}(\overline{x}', t')$, we have that;

$A'_{n,\epsilon}(\overline{k}') = (\gamma_v - \frac{\gamma_v k_1' v}{k'c})[\gamma_v A_{n,\epsilon}(\overline{k}') - \frac{\gamma_v v}{c^2}\overline{A}_{1,n,\epsilon}(\overline{k}')]$

$B'_{n,\epsilon}(\overline{k}') = (\gamma_v + \frac{\gamma_v k_1' v}{k'c})[\gamma_v B_{n,\epsilon}(\overline{k}') - \frac{\gamma_v v}{c^2}\overline{B}_{1,n,\epsilon}(\overline{k}')]$ (E)

and, using (D), with the inverse relations for the first line;

$k = k'\gamma_v - \frac{k_1'\gamma_v v}{c}$

$k_1 = k_1'\gamma_v - \frac{k'\gamma_v v}{c}$

$k_2' = k_2$

$k_3' = k_3$

$lim_{\epsilon\rightarrow 0}A_{n,\epsilon}(\overline{k}') = \frac{q}{2} + \frac{qwk_1}{2kc}$

$= \frac{q}{2} + \frac{qw(k_1'\gamma_v - \frac{k'\gamma_v v}{c})}{2(k'\gamma_v - \frac{k_1'\gamma_v v}{c})c}$

$lim_{\epsilon\rightarrow 0}\overline{A}_{1,n,\epsilon}(\overline{k}') = [-\frac{cq}{2k} - \frac{qwk_1}{2k^2}]k_1$

$= -\frac{cq(k_1'\gamma_v - \frac{k'\gamma_v v}{c})}{2(k'\gamma_v - \frac{k_1'\gamma_v v}{c})} - \frac{qw(k_1'\gamma_v - \frac{k'\gamma_v v}{c})^2}{2(k'\gamma_v - \frac{k_1'\gamma_v v}{c})^2}$

and, substituting into the first line of (E);

$\lim_{\epsilon\rightarrow 0} A'_{n,\epsilon}(\overline{k}') = (\gamma_v - \frac{\gamma_v k_1' v}{k'c})[\gamma_v(\frac{q}{2} + \frac{qw(k_1'\gamma_v - \frac{k'\gamma_v v}{c})}{2(k'\gamma_v - \frac{k_1'\gamma_v v}{c})c}) - \frac{\gamma_v v}{c^2}(-\frac{cq(k_1'\gamma_v - \frac{k'\gamma_v v}{c})}{2(k'\gamma_v - \frac{k_1'\gamma_v v}{c})}$

$-\frac{qw(k_1'\gamma_v - \frac{k'\gamma_v v}{c})^2}{2(k'\gamma_v - \frac{k_1'\gamma_v v}{c})^2})]$

Again, using (D), with the inverse relations for the second line;

$k = k'\gamma_v + \frac{k_1'\gamma_v v}{c}$

$k_1 = k_1'\gamma_v + \frac{k'\gamma_v v}{c}$

$k_2'=k_2$

$k_3'=k_3$

$lim_{\epsilon\rightarrow 0}B_{n,\epsilon}(\overline{k}')=\frac{q}{2}-\frac{qwk_1}{2kc}$

$=\frac{q}{2}-\frac{qw(k_1'\gamma_v+\frac{k'\gamma_v v}{c})}{2(k'\gamma_v+\frac{k_1'\gamma_v v}{c})c}$

$lim_{\epsilon\rightarrow 0}\overline{B}_{1,n,\epsilon}(\overline{k}')=[\frac{cq}{2k}-\frac{qwk_1}{2k^2}]k_1$

$=\frac{cq(k_1'\gamma_v+\frac{k'\gamma_v v}{c})}{2(k'\gamma_v+\frac{k_1'\gamma_v v}{c})}-\frac{qw(k_1'\gamma_v+\frac{k'\gamma_v v}{c})^2}{2(k'\gamma_v+\frac{k_1'\gamma_v v}{c})^2}$

and, substituting into the second line of (E);

$\lim_{\epsilon\rightarrow 0}B'_{n,\epsilon}(\overline{k}')=(\gamma_v+\frac{\gamma_v k_1' v}{k'c})[\gamma_v(\frac{q}{2}-\frac{qw(k_1'\gamma_v+\frac{k'\gamma_v v}{c})}{2(k'\gamma_v+\frac{k_1'\gamma_v v}{c})c})-\frac{\gamma_v v}{c^2}(\frac{cq(k_1'\gamma_v+\frac{k'\gamma_v v}{c})}{2(k'\gamma_v+\frac{k_1'\gamma_v v}{c})}$
$-\frac{qw(k_1'\gamma_v+\frac{k'\gamma_v v}{c})^2}{2(k'\gamma_v+\frac{k_1'\gamma_v v}{c})^2})]$

Rearranging, we have that;

$\lim_{\epsilon\rightarrow 0}A'_{n,\epsilon}(\overline{k}')=q\gamma_v^2(1-\frac{k_1'v}{k'c})[(\frac{1}{2}+\frac{wk_1'(1-\frac{k'v}{k_1'c})}{2k'(1-\frac{k_1'v}{k'c})c})-\frac{v}{c^2}(-\frac{ck_1'(1-\frac{k'v}{k_1'c})}{2k'(1-\frac{k_1'v}{k'c})}$
$-\frac{wk_1'^2(1-\frac{k'v}{k_1'c})^2}{2k'^2(1-\frac{k_1'v}{k'c})^2})]$

$=q\gamma_v^2(1-\frac{k_1'v}{k'c})[\frac{1}{2}+\frac{wk_1'(1-\frac{k'v}{k_1'c})}{2k'(1-\frac{k_1'v}{k'c})c}+\frac{vk_1'(1-\frac{k'v}{k_1'c})}{2k'(1-\frac{k_1'v}{k'c})c}+\frac{vwk_1'^2(1-\frac{k'v}{k_1'c})^2}{2k'^2c^2(1-\frac{k_1'v}{k'c})^2}]$

$=\frac{q\gamma_v^2}{2}(1-\frac{k_1'v}{k'c})[1+\frac{vk_1'(1-\frac{k'v}{k_1'c})}{k'(1-\frac{k_1'v}{k'c})c}][1+\frac{wk_1'(1-\frac{k'v}{k_1'c})}{k'(1-\frac{k_1'v}{k'c})c}]$

$=\frac{q\gamma_v^2}{2}(1-\frac{k_1'v}{k'c})[\frac{(1-\frac{v^2}{c^2})}{(1-\frac{k_1'v}{k'c})}][1+\frac{wk_1'(1-\frac{k'v}{k_1'c})}{k'(1-\frac{k_1'v}{k'c})c}]$

$=\frac{q}{2}[1+\frac{wk_1'(1-\frac{k'v}{k_1'c})}{k'(1-\frac{k_1'v}{k'c})c}]$

$=\frac{q}{2}[\frac{k'c-k_1'v+k_1'w-\frac{k'vw}{c}}{k'(1-\frac{k_1'v}{k'c})c}]$

$=\frac{q}{2}[\frac{k'c(1-\frac{vw}{c^2})+k_1'(w-v)}{k'(1-\frac{k_1'v}{k'c})c}]$

$= \frac{q}{2}\frac{[k'c(1-\frac{vw}{c^2})][1+\frac{k'_1}{k'c}(\frac{w-v}{1-\frac{vw}{c^2}})]}{k'(1-\frac{k'_1v}{k'c})c}$

$= \frac{q}{2}\frac{(1-\frac{vw}{c^2})[1+\frac{k'_1}{k'c}w']}{(1-\frac{k'_1v}{k'c})}$

$= \frac{q}{2}\frac{(1-\frac{vw}{c^2})[1+\frac{k'_1}{k'c}w']}{(1-\frac{k'_1v}{k'c})}$

$= \frac{q}{2}\frac{\frac{\gamma_{-\overline{w}*\overline{v}}}{\gamma_w\gamma_v}lim_{\epsilon\rightarrow 0}A''_{n,\epsilon}}{(1-\frac{k'_1v}{k'c})}$

$= \frac{q'}{2}\frac{lim_{\epsilon\rightarrow 0}A''_{n,\epsilon}(\overline{k}')}{\gamma_v(1-\frac{k'_1v}{k'c})}$

$= \frac{q'}{2}lim_{\epsilon\rightarrow 0}A''_{n,\epsilon}(\overline{k}')\frac{k'}{k}$

and, using the symmetries;

$\lim_{\epsilon\rightarrow 0} A''_{n,\epsilon}(\overline{k}')(w) = \lim_{\epsilon\rightarrow 0} B''_{n,\epsilon}(\overline{k}')(-w)$

$\lim_{\epsilon\rightarrow 0} A'_{n,\epsilon}(\overline{k}')(v) = \lim_{\epsilon\rightarrow 0} B'_{n,\epsilon}(\overline{k}')(-v)$

we have;

$\lim_{\epsilon\rightarrow 0} B'_{n,\epsilon}(\overline{k}') = \frac{q'}{2}lim_{\epsilon\rightarrow 0}B''_{n,\epsilon}(\overline{k}')\frac{k'}{k}$

where $q' = q\frac{\gamma_{\overline{w}*-\overline{v}}}{\gamma_w}$ is the new charge in S' due to the change in mass of the electron, $m' = \frac{m\gamma_{\overline{w}*-\overline{v}}}{\gamma_v}$, $\{A''_{n,\epsilon}(\overline{k}'), B''_{n,\epsilon}(\overline{k}')\}$ are the factors in the wave equation for an electron moving with the transformed velocity $w' = \frac{w-v}{1-\frac{vw}{c^2}}$ in S' and $\frac{k'}{k}$ is the background Doppler shift in amplitude, transferring from S to S'.

□

Lemma 0.14. *Let an electron move at velocity $\overline{w}$ in the base frame S, and let S' be a rotation of S by $g \in O(3)$, then there is no Doppler shift in the amplitudes of the spectrum of the electron in S', and the electron moves with velocity $\overline{w}^g$ in S'.*

Proof. We consider a rotated frame S' relative to the base frame S. For $n \geq 6$,if the electron travels with velocity vector $\overline{w}$ in the base frame S, then by Lemma 0.10, we have that;

$lim_{\epsilon\rightarrow 0}A_{n,\epsilon}(\overline{k})=\frac{q}{2}+\frac{q\overline{w}.\overline{k}}{2kc}$

$lim_{\epsilon\rightarrow 0}B_{n,\epsilon}(\overline{k})=\frac{q}{2}-\frac{q\overline{w}.\overline{k}}{2kc}$ (D)

We have that $\rho_{n,\epsilon}$ transforms to S' as;

$\rho'_{n,\epsilon}(\overline{x}',t')=\rho_{n,\epsilon}(\overline{x},t)$, where $\overline{x}'=\overline{x}^g$

see [10], so that, using (†) and the wave representation for $\rho_{n,\epsilon}$;

$\rho'_{n,\epsilon}(\overline{x}',t')=\frac{1}{(2\pi)^3}\int_{\mathcal{R}^3}(A_{n,\epsilon}(\overline{k})e^{ikct}+B_{n,\epsilon}(\overline{k})e^{-ikct})e^{i\overline{k}.\overline{x}}d\overline{k}$

(††)

We have that, using the inverse rotation, $\overline{x}=\overline{x}'^{g^{-1}}$ and substituting into (††), using the fact that $g\in O(3)$;

$\rho'_{n,\epsilon}(\overline{x}',t')=\frac{1}{(2\pi)^3}\int_{\mathcal{R}^3}\int_{\mathcal{R}^3}(A_{n,\epsilon}(\overline{k})e^{ikct}+B_{n,\epsilon}(\overline{k})e^{-ikct})e^{i\overline{k}.\overline{x}'^{g^{-1}}}d\overline{k}$

$=\frac{1}{(2\pi)^3}\int_{\mathcal{R}^3}\int_{\mathcal{R}^3}(A_{n,\epsilon}(\overline{k})e^{ikct}+B_{n,\epsilon}(\overline{k})e^{-ikct})e^{i\overline{k}^g.\overline{x}'}d\overline{k}$, (A)

We make the change of variables $\overline{k}'=\overline{k}^g$. Calculating the Jacobian, and using the fact that $g\in O(3)$, we have that;

$d\overline{k}=d\overline{k}'$

so that, substituting in (A);

$\rho'_{n,\epsilon}(\overline{x}',t')=\frac{1}{(2\pi)^3}\int_{\mathcal{R}^3}\int_{\mathcal{R}^3}(A_{n,\epsilon}(\overline{k}'^{g^{-1}})e^{ik'ct}+B_{n,\epsilon}(\overline{k}'^{g^{-1}})e^{-ik'ct})e^{i\overline{k}'.\overline{x}'}d\overline{k}'$

and equating coefficients, using the wave representation for $\rho'_{n,\epsilon}(\overline{x}',t')$, we have that;

$A'_{n,\epsilon}(\overline{k}')=A_{n,\epsilon}(\overline{k}'^{g^{-1}})$
$B'_{n,\epsilon}(\overline{k}')=B_{n,\epsilon}(\overline{k}'^{g^{-1}})$ (E)

so that, using (D) and the fact that $g\in O(3)$, $k=k'$;

$\lim_{\epsilon\rightarrow 0}A'_{n,\epsilon}(\overline{k}')=\frac{q}{2}+\frac{q\overline{w}.\overline{k}'^{g^{-1}}}{2k'c}$

$= \frac{q}{2} + \frac{q\overline{w}^g.\overline{k}'}{2k'c}$

$\lim_{\epsilon\rightarrow 0} B'_{n,\epsilon}(\overline{k}') = \frac{q}{2} - \frac{q\overline{w}.\overline{k}'^{g^{-1}}}{2k'c}$

$= \frac{q}{2} - \frac{q\overline{w}^g.\overline{k}'}{2k'c}$

which represents an electron with charge q and velocity vector $\overline{w}^g$ moving in S'. There is no mass change and no Doppler effect as $k' = k$.

$\square$

Lemma 0.15. *Let an electron move at velocity $\overline{w}$ in the base frame S, and let S' move at velocity $\overline{v}$ relative to S, then the amplitudes in the spectrum of the electron in S' are shifted by the background Doppler factor $\frac{k'}{k}$.*

Proof. The proof generalises Lemma 0.13. We consider a frame S' moving with velocity $\overline{v}$ relative to the base frame S and let $\overline{w}$ be the velocity of the electron is S. Choose $g \in SO(3)$, with $g(\overline{v}) = (v, 0, 0)$ and $g(\overline{w}) = \overline{p} = (p_1, p_2, 0)$, with $w = \sqrt{p_1^2 + p_2^2}$. Then, by a result in [10], we have that;

$R_g B_{\overline{v}} = B_{g(\overline{v})} R_g = B_{(v,0,0)} R_g$

and the electron travels at velocity $(p_1, p_2, 0)$ in the rotated by g frame S_1 with no Doppler shift. Let S_2 be the frame connected to S_1 by the boost $B_{(v,0,0)}$. For $n \geq 6$, by results of [7], [8] and [9] again, given $\rho_{n,\epsilon}$ in the base frame S_1, there exists a unique current $\overline{J}_{n,\epsilon}$ with compact support, such that the standard relations are satisfied for $(\rho_{n,\epsilon}, \overline{J}_{n,\epsilon})$. By results of [7], we have that $\square^2(\overline{J}_{n,\epsilon}) = \overline{0}$ and moreover, $\overline{J}_{n,\epsilon}$ has the wave representation;

$\overline{J}_{n,\epsilon} = \frac{1}{(2\pi)^3} \int_{\mathcal{R}^3} (\overline{A}_{n,\epsilon}(\overline{k}) e^{ikct} + \overline{B}_{n,\epsilon}(\overline{k}) e^{-ikct}) e^{i\overline{k}.\overline{x}} d\overline{k}$ (†)

where, for $k \neq 0$;

$\overline{A}_{n,\epsilon}(\overline{k}) = -\frac{cA_{n,\epsilon}(\overline{k})}{k} \overline{k}$

$\overline{B}_{n,\epsilon}(\overline{k}) = \frac{cB_{n,\epsilon}(\overline{k})}{k} \overline{k}$ (∗)

If the electron travels with velocity vector $\overline{p}$ in the frame S_1, then by Lemma 0.10, we have that;

$lim_{\epsilon\rightarrow 0}A_{n,\epsilon}(\overline{k})=\frac{q}{2}+\frac{q\overline{p}\centerdot\overline{k}}{2kc}$

$lim_{\epsilon\rightarrow 0}B_{n,\epsilon}(\overline{k})=\frac{q}{2}-\frac{q\overline{p}\centerdot\overline{k}}{2kc}$

so that, by $(*)$, uniformly on compact subsets of $\mathcal{R}^3\setminus\{\overline{0}\}$;

$lim_{\epsilon\rightarrow 0}\overline{A}_{n,\epsilon}(\overline{k})=[-\frac{cq}{2k}-\frac{q\overline{p}\centerdot\overline{k}}{2k^2}]\overline{k}$

$lim_{\epsilon\rightarrow 0}\overline{B}_{n,\epsilon}(\overline{k})=[\frac{cq}{2k}-\frac{q\overline{p}\centerdot\overline{k}}{2k^2}]\overline{k}$ (D)

We have that $\rho_{n,\epsilon}$ transforms to S_2 as;

$\rho'_{n,\epsilon}(\overline{x}',t')=\gamma_v(\rho_{n,\epsilon}-\frac{vj_{1,n,\epsilon}}{c^2})(\overline{x},t)$

see [1], so that, using $(\dagger)$ and the wave representation for $\rho_{n,\epsilon}$;

$\rho'_{n,\epsilon}(\overline{x}',t')=\frac{1}{(2\pi)^3}\int_{\mathcal{R}^3}(\gamma_v A_{n,\epsilon}(\overline{k})e^{ikct}+\gamma_v B_{n,\epsilon}(\overline{k})e^{-ikct})e^{i\overline{k}\centerdot\overline{x}}d\overline{k}$

$-\frac{1}{(2\pi)^3}\int_{\mathcal{R}^3}(\frac{\gamma_v v}{c^2}\overline{A}_{1,n,\epsilon}(\overline{k})e^{ikct}+\frac{\gamma_v v}{c^2}\overline{B}_{1,n,\epsilon}(\overline{k})e^{-ikct})e^{i\overline{k}\centerdot\overline{x}}d\overline{k}$

$=\frac{1}{(2\pi)^3}\int_{\mathcal{R}^3}([\gamma_v A_{n,\epsilon}(\overline{k})-\frac{\gamma_v v}{c^2}\overline{A}_{1,n,\epsilon}(\overline{k})]e^{ikct}+[\gamma_v B_{n,\epsilon}(\overline{k})-\frac{\gamma_v v}{c^2}\overline{B}_{1,n,\epsilon}(\overline{k})]e^{-ikct})e^{i\overline{k}\centerdot\overline{x}}d\overline{k}$
$(\dagger\dagger)$

We have that, using the inverse Lorentz transformation;

$t=\gamma_v t'+\frac{\gamma_v v x'_1}{c^2}$

$x_1=\gamma_v x'_1+\gamma_v v t'$

$x_2=x'_2$

$x_3=x'_3$

so that, substituting into $(\dagger\dagger)$;

$\rho'_{n,\epsilon}(\overline{x}',t')=\frac{1}{(2\pi)^3}\int_{\mathcal{R}^3}([\gamma_v A_{n,\epsilon}(\overline{k})-\frac{\gamma_v v}{c^2}\overline{A}_{1,n,\epsilon}(\overline{k})]e^{ikc(\gamma_v t'+\frac{\gamma_v v x'_1}{c^2})}$

$+[\gamma_v B_{n,\epsilon}(\overline{k})-\frac{\gamma_v v}{c^2}\overline{B}_{1,n,\epsilon}(\overline{k})]e^{-ikc(\gamma_v t'+\frac{\gamma_v v x'_1}{c^2})})e^{ik_1(\gamma_v x'_1+\gamma_v v t')}e^{ik_2 x'_2}e^{ik_3 x'_3}d\overline{k}$

$=\frac{1}{(2\pi)^3}\int_{\mathcal{R}^3}[\gamma_v A_{n,\epsilon}(\overline{k})-\frac{\gamma_v v}{c^2}\overline{A}_{1,n,\epsilon}(\overline{k})]e^{i(kc\gamma_v+k_1\gamma_v v)t'}e^{i(k_1\gamma_v+\frac{k\gamma_v v}{c})x'_1}e^{ik_2 x'_2}e^{ik_3 x'_3}d\overline{k}$

$+\frac{1}{(2\pi)^3}\int_{\mathcal{R}^3}[\gamma_v B_{n,\epsilon}(\overline{k})-\frac{\gamma_v v}{c^2}\overline{B}_{1,n,\epsilon}(\overline{k})]e^{-i(kc\gamma_v-k_1\gamma_v v)t'}e^{i(k_1\gamma_v-\frac{k\gamma_v v}{c})x_1'}e^{ik_2x_2'}e^{ik_3x_3'}d\overline{k}$

(A)

We make the change of variables, in the first line of (A);

$k'c=kc\gamma_v+k_1\gamma_v v$

$k'=k\gamma_v+\frac{k_1\gamma_v v}{c}$

$k_1'=k_1\gamma_v+\frac{k\gamma_v v}{c}$

$k_2'=k_2$

$k_3'=k_3$

and, in the second line of (A);

$k'c=kc\gamma_v-k_1\gamma_v v$

$k'=k\gamma_v-\frac{k_1\gamma_v v}{c}$

$k_1'=k_1\gamma_v-\frac{k\gamma_v v}{c}$

$k_2'=k_2$

$k_3'=k_3$

which are the Doppler shift transformations given in [14], for the velocity vector $(v,0,0)$. Calculating the Jacobian, and using the chain rule, we have that, for the first and second lines of (A) respectively;

$dk_1dk_2dk_3=(\gamma_v-\frac{\gamma_v k_1'v}{k'c})dk_1'dk_2'dk_3'$

$dk_1dk_2dk_3=(\gamma_v+\frac{\gamma_v k_1'v}{k'c})dk_1'dk_2'dk_3'$

so that, substituting in (A);

$\rho_{n,\epsilon}'(\overline{x}',t')=\frac{1}{(2\pi)^3}\int_{\mathcal{R}^3}[\gamma_v A_{n,\epsilon}(\overline{k}')-\frac{\gamma_v v}{c^2}\overline{A}_{1,n,\epsilon}(\overline{k}')]e^{ik'ct'}e^{ik_1'x_1'}e^{ik_2'x_2'}e^{ik_3'x_3'}$

$(\gamma_v-\frac{\gamma_v k_1' v}{k'c})d\overline{k}'$

$+\frac{1}{(2\pi)^3}\int_{\mathcal{R}^3}[\gamma_v B_{n,\epsilon}(\overline{k}')-\frac{\gamma_v v}{c^2}\overline{B}_{1,n,\epsilon}(\overline{k}')]e^{-ik'ct'}e^{ik_1'x_1'}e^{ik_2x_2'}e^{ik_3x_3'}(\gamma_v+\frac{\gamma_v k_1' v}{k'c})d\overline{k}'$

$=\frac{1}{(2\pi)^3}\int_{\mathcal{R}^3}[\gamma_v A_{n,\epsilon}(\overline{k}')-\frac{\gamma_v v}{c^2}\overline{A}_{1,n,\epsilon}(\overline{k}')]e^{ik'ct'}e^{i\overline{k}'.\overline{x}'}(\gamma_v-\frac{\gamma_v k_1' v}{k'c})d\overline{k}'$

$+\frac{1}{(2\pi)^3}\int_{\mathcal{R}^3}[\gamma_v B_{n,\epsilon}(\overline{k}')-\frac{\gamma_v v}{c^2}\overline{B}_{1,n,\epsilon}(\overline{k}')]e^{-ik'ct'}e^{i\overline{k}'.\overline{x}'}(\gamma_v+\frac{\gamma_v k_1' v}{k'c})d\overline{k}'$

so that equating coefficients, and using the wave representation for $\rho'_{n,\epsilon}(\overline{x}',t')$, we have that;

$A'_{n,\epsilon}(\overline{k}')=(\gamma_v-\frac{\gamma_v k_1' v}{k'c})[\gamma_v A_{n,\epsilon}(\overline{k}')-\frac{\gamma_v v}{c^2}\overline{A}_{1,n,\epsilon}(\overline{k}')]$

$B'_{n,\epsilon}(\overline{k}')=(\gamma_v+\frac{\gamma_v k_1' v}{k'c})[\gamma_v B_{n,\epsilon}(\overline{k}')-\frac{\gamma_v v}{c^2}\overline{B}_{1,n,\epsilon}(\overline{k}')]$ (E)

and, using (D), with the inverse relations for the first line;

$k=k'\gamma_v-\frac{k_1'\gamma_v v}{c}$

$k_1=k_1'\gamma_v-\frac{k'\gamma_v v}{c}$

$k_2'=k_2$

$k_3'=k_3$

$lim_{\epsilon\rightarrow 0}A_{n,\epsilon}(\overline{k}')=\frac{q}{2}+\frac{q\overline{p}.\overline{k}}{2kc}$

$=\frac{q}{2}+\frac{qp_1(k_1'\gamma_v-\frac{k'\gamma_v v}{c})+qp_2k_2'}{2(k'\gamma_v-\frac{k_1'\gamma_v v}{c})c}$

$lim_{\epsilon\rightarrow 0}\overline{A}_{1,n,\epsilon}(\overline{k}')=[-\frac{cq}{2k}-\frac{q\overline{p}.\overline{k}}{2k^2}]k_1$

$=-\frac{cq(k_1'\gamma_v-\frac{k'\gamma_v v}{c})}{2(k'\gamma_v-\frac{k_1'\gamma_v v}{c})}-\frac{qp_1(k_1'\gamma_v-\frac{k'\gamma_v v}{c})^2+qp_2k_2'(k_1'\gamma_v-\frac{k'\gamma_v v}{c})}{2(k'\gamma_v-\frac{k_1'\gamma_v v}{c})^2}$

and, substituting into the first line of (E);

$\lim_{\epsilon\rightarrow 0}A'_{n,\epsilon}(\overline{k}')=(\gamma_v-\frac{\gamma_v k_1' v}{k'c})[\gamma_v(\frac{q}{2}+\frac{qp_1(k_1'\gamma_v-\frac{k'\gamma_v v}{c})+qp_2k_2'}{2(k'\gamma_v-\frac{k_1'\gamma_v v}{c})c})-\frac{\gamma_v v}{c^2}(-\frac{cq(k_1'\gamma_v-\frac{k'\gamma_v v}{c})}{2(k'\gamma_v-\frac{k_1'\gamma_v v}{c})}$

$-\frac{qp_1(k_1'\gamma_v-\frac{k'\gamma_v v}{c})^2+qp_2k_2'(k_1'\gamma_v-\frac{k'\gamma_v v}{c})}{2(k'\gamma_v-\frac{k_1'\gamma_v v}{c})^2})]$

Again, using (D), with the inverse relations for the second line;

$k=k'\gamma_v+\frac{k_1'\gamma_v v}{c}$

$k_1=k_1'\gamma_v+\frac{k'\gamma_v v}{c}$

$k_2'=k_2$

$k_3'=k_3$

$lim_{\epsilon\rightarrow 0}B_{n,\epsilon}(\overline{k}')=\frac{q}{2}-\frac{q\overline{p}\centerdot\overline{k}}{2kc}$

$=\frac{q}{2}-\frac{qp_1(k_1'\gamma_v+\frac{k'\gamma_v v}{c})+qp_2k_2'}{2(k'\gamma_v+\frac{k_1'\gamma_v v}{c})c}$

$lim_{\epsilon\rightarrow 0}\overline{B}_{1,n,\epsilon}(\overline{k}')=[\frac{cq}{2k}-\frac{q\overline{p}\centerdot\overline{k}}{2k^2}]k_1$

$=\frac{cq(k_1'\gamma_v+\frac{k'\gamma_v v}{c})}{2(k'\gamma_v+\frac{k_1'\gamma_v v}{c})}-\frac{qp_1(k_1'\gamma_v+\frac{k'\gamma_v v}{c})^2+qp_2k_2'(k_1'\gamma_v+\frac{k'\gamma_v v}{c})}{2(k'\gamma_v+\frac{k_1'\gamma_v v}{c})^2}$

and, substituting into the second line of (E);

$\lim_{\epsilon\rightarrow 0}B'_{n,\epsilon}(\overline{k}')=(\gamma_v+\frac{\gamma_v k_1' v}{k'c})[\gamma_v(\frac{q}{2}-\frac{qp_1(k_1'\gamma_v+\frac{k'\gamma_v v}{c})+qp_2k_2'}{2(k'\gamma_v+\frac{k_1'\gamma_v v}{c})c})-\frac{\gamma_v v}{c^2}(\frac{cq(k_1'\gamma_v+\frac{k'\gamma_v v}{c})}{2(k'\gamma_v+\frac{k_1'\gamma_v v}{c})}$

$-\frac{qp_1(k_1'\gamma_v+\frac{k'\gamma_v v}{c})^2+qp_2k_2'(k_1'\gamma_v+\frac{k'\gamma_v v}{c})}{2(k'\gamma_v+\frac{k_1'\gamma_v v}{c})^2})]$

Rearranging, we have that;

$\lim_{\epsilon\rightarrow 0}A'_{n,\epsilon}(\overline{k}')=q\gamma_v^2(1-\frac{k_1'v}{k'c})[(\frac{1}{2}+\frac{p_1k_1'(1-\frac{k'v}{k_1'c})+\frac{p_2}{\gamma_v}k_2'}{2k'(1-\frac{k_1'v}{k'c})c})-\frac{v}{c^2}(-\frac{ck_1'(1-\frac{k'v}{k_1'c})}{2k'(1-\frac{k_1'v}{k'c})}$

$-\frac{p_1k_1'^2(1-\frac{k'v}{k_1'c})^2+\frac{p_2}{\gamma_v}k_1'k_2'(1-\frac{k'v}{k_1'c})}{2k'^2(1-\frac{k_1'v}{k'c})^2})]$

$=q\gamma_v^2(1-\frac{k_1'v}{k'c})[\frac{1}{2}+\frac{p_1k_1'(1-\frac{k'v}{k_1'c})}{2k'(1-\frac{k_1'v}{k'c})c}+\frac{vk_1'(1-\frac{k'v}{k_1'c})}{2k'(1-\frac{k_1'v}{k'c})c}+\frac{vp_1k_1'^2(1-\frac{k'v}{k_1'c})^2}{2k'^2c^2(1-\frac{k_1'v}{k'c})^2}]$

$+q\gamma_v^2(1-\frac{k_1'v}{k'c})[\frac{\frac{p_2}{\gamma_v}k_2'}{2k'(1-\frac{k_1'v}{k'c})c}+\frac{\frac{p_2}{\gamma_v}k_1'k_2'v(1-\frac{k'v}{k_1'c})}{2k'^2c^2(1-\frac{k_1'v}{k'c})^2}]$

$=\frac{q\gamma_v^2}{2}(1-\frac{k_1'v}{k'c})[1+\frac{vk_1'(1-\frac{k'v}{k_1'c})}{k'(1-\frac{k_1'v}{k'c})c}][1+\frac{p_1k_1'(1-\frac{k'v}{k_1'c})}{k'(1-\frac{k_1'v}{k'c})c}]+q\gamma_v^2[\frac{\frac{p_2}{\gamma_v}k_2'}{2k'c}+\frac{\frac{p_2}{\gamma_v}k_1'k_2'v(1-\frac{k'v}{k_1'c})}{2k'^2c^2(1-\frac{k_1'v}{k'c})}]$

$=\frac{q}{2}\frac{(1-\frac{vp_1}{c^2})[1+\frac{k_1'}{k'c}p_1']}{(1-\frac{k_1'v}{k'c})}+\frac{q\gamma_v^2}{2}\frac{(1-\frac{vp_1}{c^2})p_2'k_2'}{k'c}[1+\frac{vk_1'(1-\frac{k'v}{k_1'c})}{k'(1-\frac{k_1'v}{k'c})c}]$

$= \frac{q}{2}\frac{(1-\frac{vp_1}{c^2})[1+\frac{k'_1}{k'c}p'_1]}{(1-\frac{k'_1v}{k'c})} + \frac{q\gamma_v^2}{2}\frac{(1-\frac{vp_1}{c^2})p'_2k'_2}{k'c}[\frac{(1-\frac{v^2}{c^2})}{(1-\frac{k'_1v}{k'c})}]$

$= \frac{q}{2}\frac{(1-\frac{vp_1}{c^2})[1+\frac{k'_1}{k'c}p'_1]}{(1-\frac{k'_1v}{k'c})} + \frac{q}{2}\frac{(1-\frac{vp_1}{c^2})\frac{k'_2}{k'c}p'_2}{(1-\frac{k'_1v}{k'c})}$

$= \frac{q}{2}\frac{(1-\frac{vp_1}{c^2})[1+\frac{\overline{p}'\cdot\overline{k}'}{k'c}]}{(1-\frac{k'_1v}{k'c})}$

$= \frac{q}{2}\frac{\frac{\gamma_{-\overline{p}*\overline{v}}}{\gamma_p\gamma_v}lim_{\epsilon\rightarrow 0}A''_{n,\epsilon}}{(1-\frac{k'_1v}{k'c})}$

$= \frac{q'}{2}\frac{lim_{\epsilon\rightarrow 0}A''_{n,\epsilon}(\overline{k}')}{\gamma_v(1-\frac{k'_1v}{k'c})}$

$= \frac{q'}{2}lim_{\epsilon\rightarrow 0}A''_{n,\epsilon}(\overline{k}')\frac{k'}{k}$

and, using the symmetries;

$\lim_{\epsilon\rightarrow 0} A''_{n,\epsilon}(\overline{k}')(\overline{p}) = \lim_{\epsilon\rightarrow 0} B''_{n,\epsilon}(\overline{k}')(-\overline{p})$

$\lim_{\epsilon\rightarrow 0} A'_{n,\epsilon}(\overline{k}')(v) = \lim_{\epsilon\rightarrow 0} B'_{n,\epsilon}(\overline{k}')(-v)$

we have;

$\lim_{\epsilon\rightarrow 0} B'_{n,\epsilon}(\overline{k}') = \frac{q'}{2}lim_{\epsilon\rightarrow 0}B''_{n,\epsilon}(\overline{k}')\frac{k'}{k}$

where $q' = q\frac{\gamma_{\overline{p}*-\overline{v}}}{\gamma_p}$ is the new charge in S' due to the change in mass of the electron, $m' = \frac{m\gamma_{\overline{p}*-\overline{v}}}{\gamma_w}$, $\{A''_{n,\epsilon}(\overline{k}'), B''_{n,\epsilon}(\overline{k}')\}$ are the factors in the wave equation for an electron moving with the transformed velocity $\overline{p}' = (\frac{p_1-v}{1-\frac{vp_1}{c^2}}, \frac{p_2}{\gamma_v(1-\frac{vp_1}{c^2})}, 0)$ in S_2 and $\frac{k'}{k}$ is the background Doppler shift in amplitude, transferring from S_1 to S_2. Now complete the proof back to S' using Lemma 0.14.

□

Lemma 0.16. *Let an electron move at velocity* $(w, 0, 0)$ *in the base frame* S*, and let* S' *move at velocity* $(v, 0, 0)$ *relative to* S*, then the amplitudes in the spectrum of the current for the electron in* S' *are shifted by the background Doppler factor* $\frac{k'}{k}$*.*

Proof. We consider a frame S' moving with velocity $(v, 0, 0)$ relative to the base frame S. For $n \geq 6$, by results of [7], [8] and [9], given $\rho_{n,\epsilon}$ in the base frame S, there exists a unique current $\overline{J}_{n,\epsilon}$ with compact support, such that the standard relations are satisfied for $(\rho_{n,\epsilon}, \overline{J}_{n,\epsilon})$. By results of [7], we have that $\square^2(\overline{J}_{n,\epsilon}) = \overline{0}$ and moreover, $\overline{J}_{n,\epsilon}$ has the wave representaion;

$\overline{J}_{n,\epsilon} = \frac{1}{(2\pi)^3}\int_{\mathcal{R}^3}(\overline{A}_{n,\epsilon}(\overline{k})e^{ikct} + \overline{B}_{n,\epsilon}(\overline{k})e^{-ikct})e^{i\overline{k}.\overline{x}}d\overline{k}$ (†)

where, for $k \neq 0$;

$\overline{A}_{n,\epsilon}(\overline{k}) = -\frac{cA_{n,\epsilon}(\overline{k})}{k}\overline{k}$

$\overline{B}_{n,\epsilon}(\overline{k}) = \frac{cB_{n,\epsilon}(\overline{k})}{k}\overline{k}$ $(*)$

If the electron travels with velocity vector $(w, 0, 0)$ in the base frame S, then by Lemma 0.10, we have that;

$lim_{\epsilon\rightarrow 0}A_{n,\epsilon}(\overline{k}) = \frac{q}{2} + \frac{qwk_1}{2kc}$

$lim_{\epsilon\rightarrow 0}B_{n,\epsilon}(\overline{k}) = \frac{q}{2} - \frac{qwk_1}{2kc}$

so that, by $(*)$, uniformly on compact subsets of $\mathcal{R}^3 \setminus \{\overline{0}\}$;

$lim_{\epsilon\rightarrow 0}\overline{A}_{n,\epsilon}(\overline{k}) = [-\frac{cq}{2k} - \frac{qwk_1}{2k^2}]\overline{k}$

$lim_{\epsilon\rightarrow 0}\overline{B}_{n,\epsilon}(\overline{k}) = [\frac{cq}{2k} - \frac{qwk_1}{2k^2}]\overline{k}$ (D)

We have that $\overline{J}_{n,\epsilon}$ transforms to S' as;

$\overline{J}'_{n,\epsilon}(\overline{x}', t') = (\gamma_v(\overline{J}_{n,\epsilon,||} - \rho_{n,\epsilon}\overline{v}) + \overline{J}_{n,\epsilon,\perp})(\overline{x}, t)$

$= (\gamma_v j_{1,n,\epsilon} - \gamma_v v\rho_{n,\epsilon}, j_{2,n,\epsilon}, j_{3,n,\epsilon})$

see [1], so that, using (†) and the wave representations for $(\rho_{n,\epsilon}, \overline{J}_{n,\epsilon})$;

$\overline{J}'_{n,\epsilon}(\overline{x}', t') = \frac{1}{(2\pi)^3}(\int_{\mathcal{R}^3}([\gamma_v\overline{A}_{1,n,\epsilon}(\overline{k}) - \gamma_v vA_{n,\epsilon}(\overline{k})]e^{ikct}$

$+[\gamma_v\overline{B}_{1,n,\epsilon}(\overline{k}) - \gamma_v vB_{n,\epsilon}(\overline{k})]e^{-ikct})e^{i\overline{k}.\overline{x}}d\overline{k}, \int_{\mathcal{R}^3}(\overline{A}_{2,n,\epsilon}(\overline{k})e^{ikct} + \overline{B}_{2,n,\epsilon}(\overline{k})e^{-ikct})e^{i\overline{k}.\overline{x}}d\overline{k},$

$\int_{\mathcal{R}^3}(\overline{A}_{3,n,\epsilon}(\overline{k})e^{ikct} + \overline{B}_{3,n,\epsilon}(\overline{k})e^{-ikct})e^{i\overline{k}.\overline{x}}d\overline{k})$

(††)

We have that, using the inverse Lorentz transformation;

$t=\gamma_v t'+\frac{\gamma_v v x_1'}{c^2}$

$x_1=\gamma_v x_1'+\gamma_v v t'$

$x_2=x_2'$

$x_3=x_3'$

so that, substituting into (††);

$\overline{J}'_{n,\epsilon}(\overline{x}',t')=\frac{1}{(2\pi)^3}(\int_{\mathcal{R}^3}([\gamma_v\overline{A}_{1,n,\epsilon}(\overline{k})-\gamma_v v A_{n,\epsilon}(\overline{k})]e^{ikc(\gamma_v t'+\frac{\gamma_v v x_1'}{c^2})}$

$+[\gamma_v\overline{B}_{1,n,\epsilon}(\overline{k})-\gamma_v v B_{n,\epsilon}(\overline{k})]e^{-ikc(\gamma_v t'+\frac{\gamma_v v x_1'}{c^2})})e^{ik_1(\gamma_v x_1'+\gamma_v v t')}e^{ik_2x_2'}e^{ik_3x_3'}d\overline{k},$

$\int_{\mathcal{R}^3}(\overline{A}_{2,n,\epsilon}(\overline{k})e^{ikc(\gamma_v t'+\frac{\gamma_v v x_1'}{c^2})}+\overline{B}_{2,n,\epsilon}(\overline{k})e^{-ikc(\gamma_v t'+\frac{\gamma_v v x_1'}{c^2})})e^{ik_1(\gamma_v x_1'+\gamma_v v t')}e^{ik_2x_2'}e^{ik_3x_3'}d\overline{k},$

$\int_{\mathcal{R}^3}(\overline{A}_{3,n,\epsilon}(\overline{k})e^{ikc(\gamma_v t'+\frac{\gamma_v v x_1'}{c^2})}+\overline{B}_{3,n,\epsilon}(\overline{k})e^{-ikc(\gamma_v t'+\frac{\gamma_v v x_1'}{c^2})})e^{ik_1(\gamma_v x_1'+\gamma_v v t')}e^{ik_2x_2'}e^{ik_3x_3'}d\overline{k})$

$=\frac{1}{(2\pi)^3}(\int_{\mathcal{R}^3}([\gamma_v\overline{A}_{1,n,\epsilon}(\overline{k})-\gamma_v v A_{n,\epsilon}(\overline{k})]e^{i(kc\gamma_v+k_1\gamma_v v)t'}e^{i(k_1\gamma_v+\frac{k\gamma_v v}{c})x_1'}e^{ik_2x_2'}e^{ik_3x_3'}d\overline{k}$

$+[\gamma_v\overline{B}_{1,n,\epsilon}(\overline{k})-\gamma_v v B_{n,\epsilon}(\overline{k})]e^{-i(kc\gamma_v-k_1\gamma_v v)t'}e^{i(k_1\gamma_v-\frac{k\gamma_v v}{c})x_1'}e^{ik_2x_2'}e^{ik_3x_3'}d\overline{k},$

$\int_{\mathcal{R}^3}\overline{A}_{2,n,\epsilon}(\overline{k})e^{i(kc\gamma_v+k_1\gamma_v v)t'}e^{i(k_1\gamma_v+\frac{k\gamma_v v}{c})x_1'}e^{ik_2x_2'}e^{ik_3x_3'}d\overline{k}$

$+\overline{B}_{2,n,\epsilon}(\overline{k})e^{-i(kc\gamma_v-k_1\gamma_v v)t'}e^{i(k_1\gamma_v-\frac{k\gamma_v v}{c})x_1'}e^{ik_2x_2'}e^{ik_3x_3'}d\overline{k},$

$\int_{\mathcal{R}^3}\overline{A}_{3,n,\epsilon}(\overline{k})e^{i(kc\gamma_v+k_1\gamma_v v)t'}e^{i(k_1\gamma_v+\frac{k\gamma_v v}{c})x_1'}e^{ik_2x_2'}e^{ik_3x_3'}d\overline{k}$

$+\overline{B}_{3,n,\epsilon}(\overline{k})e^{-i(kc\gamma_v-k_1\gamma_v v)t'}e^{i(k_1\gamma_v-\frac{k\gamma_v v}{c})x_1'}e^{ik_2x_2'}e^{ik_3x_3'}d\overline{k})$

(A)

We make the change of variables, in (A) for the $\{\overline{A},A\}$ terms;

$k'c=kc\gamma_v+k_1\gamma_v v$

$k'=k\gamma_v+\frac{k_1\gamma_v v}{c}$

$k_1'=k_1\gamma_v+\frac{k\gamma_v v}{c}$

$k'_2 = k_2$

$k'_3 = k_3$

and, in (A) for the $\{\overline{B}, B\}$ terms;

$k'c = kc\gamma_v - k_1\gamma_v v$

$k' = k\gamma_v - \frac{k_1\gamma_v v}{c}$

$k'_1 = k_1\gamma_v - \frac{k\gamma_v v}{c}$

$k'_2 = k_2$

$k'_3 = k_3$

which are the Doppler shift transformations given in [14], for the velocity vector $(v, 0, 0)$. Calculating the Jacobian, and using the chain rule, we have that, for the $\{\overline{A}, A\}$ and $\{\overline{B}, B\}$ terms of (A) respectively;

$dk_1dk_2dk_3 = (\gamma_v - \frac{\gamma_v k'_1 v}{k'c})dk'_1dk'_2dk'_3$

$dk_1dk_2dk_3 = (\gamma_v + \frac{\gamma_v k'_1 v}{k'c})dk'_1dk'_2dk'_3$

so that, substituting in the equation (A);

$\overline{J}'_{n,\epsilon}(\overline{x}', t') = \frac{1}{(2\pi)^3}(\int_{\mathcal{R}^3}([\gamma_v\overline{A}_{1,n,\epsilon}(\overline{k}') - \gamma_v v A_{n,\epsilon}(\overline{k}')]e^{ik'ct'}e^{ik'_1x'_1}e^{ik'_2x'_2}e^{ik'_3x'_3}$

$(\gamma_v - \frac{\gamma_v k'_1 v}{k'c})d\overline{k}'$

$+[\gamma_v\overline{B}_{1,n,\epsilon}(\overline{k}') - \gamma_v v B_{n,\epsilon}(\overline{k}')]e^{-ik'ct'}e^{ik'_1x'_1}e^{ik_2x'_2}e^{ik_3x'_3}(\gamma_v + \frac{\gamma_v k'_1 v}{k'c})d\overline{k}',$

$\int_{\mathcal{R}^3}\overline{A}_{2,n,\epsilon}(\overline{k}')e^{ik'ct'}e^{ik'_1x'_1}e^{ik'_2x'_2}e^{ik'_3x'_3}(\gamma_v - \frac{\gamma_v k'_1 v}{k'c})d\overline{k}'$

$+\overline{B}_{2,n,\epsilon}(\overline{k}')e^{-ik'ct'}e^{ik'_1x'_1}e^{ik_2x'_2}e^{ik_3x'_3}(\gamma_v + \frac{\gamma_v k'_1 v}{k'c})d\overline{k}',$

$\int_{\mathcal{R}^3}\overline{A}_{3,n,\epsilon}(\overline{k}')e^{ik'ct'}e^{ik'_1x'_1}e^{ik'_2x'_2}e^{ik'_3x'_3}(\gamma_v - \frac{\gamma_v k'_1 v}{k'c})d\overline{k}'$

$+\overline{B}_{3,n,\epsilon}(\overline{k}')e^{-ik'ct'}e^{ik'_1x'_1}e^{ik_2x'_2}e^{ik_3x'_3}(\gamma_v + \frac{\gamma_v k'_1 v}{k'c})d\overline{k}')$

$= \frac{1}{(2\pi)^3}(\int_{\mathcal{R}^3}([\gamma_v\overline{A}_{1,n,\epsilon}(\overline{k}') - \gamma_v v A_{n,\epsilon}(\overline{k}')]e^{ik'ct'}e^{i\overline{k}'\centerdot\overline{x}'}(\gamma_v - \frac{\gamma_v k_1' v}{k'c})d\overline{k}'$

$+[\gamma_v\overline{B}_{1,n,\epsilon}(\overline{k}') - \gamma_v v B_{n,\epsilon}(\overline{k}')]e^{-ik'ct'}e^{i\overline{k}'\centerdot\overline{x}'}(\gamma_v + \frac{\gamma_v k_1' v}{k'c})d\overline{k}',$

$\int_{\mathcal{R}^3}\overline{A}_{2,n,\epsilon}(\overline{k}')e^{ik'ct'}e^{i\overline{k}'\centerdot\overline{x}'}(\gamma_v - \frac{\gamma_v k_1' v}{k'c})d\overline{k}' + \overline{B}_{2,n,\epsilon}(\overline{k}')e^{-ik'ct'}e^{i\overline{k}'\centerdot\overline{x}'}(\gamma_v + \frac{\gamma_v k_1' v}{k'c})d\overline{k}',$

$\int_{\mathcal{R}^3}\overline{A}_{3,n,\epsilon}(\overline{k}')e^{ik'ct'}e^{i\overline{k}'\centerdot\overline{x}'}(\gamma_v - \frac{\gamma_v k_1' v}{k'c})d\overline{k}' + \overline{B}_{3,n,\epsilon}(\overline{k}')e^{-ik'ct'}e^{i\overline{k}'\centerdot\overline{x}'}(\gamma_v + \frac{\gamma_v k_1' v}{k'c})d\overline{k}')$

so that equating coefficients, and using the wave representation for $\overline{J}'_{n,\epsilon}(\overline{x}', t')$, we have that;

$\overline{A}'_{1,n,\epsilon}(\overline{k}') = (\gamma_v - \frac{\gamma_v k_1' v}{k'c})[\gamma_v\overline{A}_{1,n,\epsilon}(\overline{k}') - \gamma_v v A_{n,\epsilon}(\overline{k}')]$

$\overline{A}'_{2,n,\epsilon}(\overline{k}') = (\gamma_v - \frac{\gamma_v k_1' v}{k'c})[\overline{A}_{2,n,\epsilon}(\overline{k}')]$

$\overline{A}'_{3,n,\epsilon}(\overline{k}') = (\gamma_v - \frac{\gamma_v k_1' v}{k'c})[\overline{A}_{3,n,\epsilon}(\overline{k}')]$

$\overline{B}'_{1,n,\epsilon}(\overline{k}') = (\gamma_v + \frac{\gamma_v k_1' v}{k'c})[\gamma_v\overline{B}_{1,n,\epsilon}(\overline{k}') - \gamma_v v B_{n,\epsilon}(\overline{k}')]$

$\overline{B}'_{2,n,\epsilon}(\overline{k}') = (\gamma_v + \frac{\gamma_v k_1' v}{k'c})[\overline{B}_{2,n,\epsilon}(\overline{k}')]$

$\overline{B}'_{3,n,\epsilon}(\overline{k}') = (\gamma_v + \frac{\gamma_v k_1' v}{k'c})[\overline{B}_{3,n,\epsilon}(\overline{k}')]$ (E)

and, using (D), with the inverse relations for the first line;

$k = k'\gamma_v - \frac{k_1'\gamma_v v}{c}$

$k_1 = k_1'\gamma_v - \frac{k'\gamma_v v}{c}$

$k_2' = k_2$

$k_3' = k_3$

$lim_{\epsilon\rightarrow 0}A_{n,\epsilon}(\overline{k}') = \frac{q}{2} + \frac{qwk_1}{2kc}$

$= \frac{q}{2} + \frac{qw(k_1'\gamma_v - \frac{k'\gamma_v v}{c})}{2(k'\gamma_v - \frac{k_1'\gamma_v v}{c})c}$

$lim_{\epsilon\rightarrow 0}\overline{A}_{1,n,\epsilon}(\overline{k}') = [-\frac{cq}{2k} - \frac{qwk_1}{2k^2}]k_1$

$= -\frac{cq(k_1'\gamma_v - \frac{k'\gamma_v v}{c})}{2(k'\gamma_v - \frac{k_1'\gamma_v v}{c})} - \frac{qw(k_1'\gamma_v - \frac{k'\gamma_v v}{c})^2}{2(k'\gamma_v - \frac{k_1'\gamma_v v}{c})^2}$

$lim_{\epsilon\rightarrow 0}\overline{A}_{2,n,\epsilon}(\overline{k}')=[-\frac{cq}{2k}-\frac{qwk_1}{2k^2}]k_2$

$=-\frac{cqk_2'}{2(k'\gamma_v-\frac{k_1'\gamma_v v}{c})}-\frac{qw(k_1'\gamma_v-\frac{k'\gamma_v v}{c})k_2'}{2(k'\gamma_v-\frac{k_1'\gamma_v v}{c})^2}$

$lim_{\epsilon\rightarrow 0}\overline{A}_{3,n,\epsilon}(\overline{k}')=[-\frac{cq}{2k}-\frac{qwk_1}{2k^2}]k_3$

$=-\frac{cqk_3'}{2(k'\gamma_v-\frac{k_1'\gamma_v v}{c})}-\frac{qw(k_1'\gamma_v-\frac{k'\gamma_v v}{c})k_3'}{2(k'\gamma_v-\frac{k_1'\gamma_v v}{c})^2}$

and, substituting into the first three lines of (E);

$\lim_{\epsilon\rightarrow 0}\overline{A}'_{1,n,\epsilon}(\overline{k}')=(\gamma_v-\frac{\gamma_v k_1' v}{k'c})[-v\gamma_v(\frac{q}{2}+\frac{qw(k_1'\gamma_v-\frac{k'\gamma_v v}{c})}{2(k'\gamma_v-\frac{k_1'\gamma_v v}{c})c})+\gamma_v(-\frac{cq(k_1'\gamma_v-\frac{k'\gamma_v v}{c})}{2(k'\gamma_v-\frac{k_1'\gamma_v v}{c})}$

$-\frac{qw(k_1'\gamma_v-\frac{k'\gamma_v v}{c})^2}{2(k'\gamma_v-\frac{k_1'\gamma_v v}{c})^2})]$

$\lim_{\epsilon\rightarrow 0}\overline{A}'_{2,n,\epsilon}(\overline{k}')=(\gamma_v-\frac{\gamma_v k_1' v}{k'c})[-\frac{cqk_2'}{2(k'\gamma_v-\frac{k_1'\gamma_v v}{c})}-\frac{qw(k_1'\gamma_v-\frac{k'\gamma_v v}{c})k_2'}{2(k'\gamma_v-\frac{k_1'\gamma_v v}{c})^2}]$

$\lim_{\epsilon\rightarrow 0}\overline{A}'_{3,n,\epsilon}(\overline{k}')=(\gamma_v-\frac{\gamma_v k_1' v}{k'c})[-\frac{cqk_3'}{2(k'\gamma_v-\frac{k_1'\gamma_v v}{c})}-\frac{qw(k_1'\gamma_v-\frac{k'\gamma_v v}{c})k_3'}{2(k'\gamma_v-\frac{k_1'\gamma_v v}{c})^2}]\ (F)$

Again, using (D), with the inverse relations for the second line;

$k=k'\gamma_v+\frac{k_1'\gamma_v v}{c}$

$k_1=k_1'\gamma_v+\frac{k'\gamma_v v}{c}$

$k_2'=k_2$

$k_3'=k_3$

$lim_{\epsilon\rightarrow 0}B_{n,\epsilon}(\overline{k}')=\frac{q}{2}-\frac{qwk_1}{2kc}$

$=\frac{q}{2}-\frac{qw(k_1'\gamma_v+\frac{k'\gamma_v v}{c})}{2(k'\gamma_v+\frac{k_1'\gamma_v v}{c})c}$

$lim_{\epsilon\rightarrow 0}\overline{B}_{1,n,\epsilon}(\overline{k}')=[\frac{cq}{2k}-\frac{qwk_1}{2k^2}]k_1$

$=\frac{cq(k_1'\gamma_v+\frac{k'\gamma_v v}{c})}{2(k'\gamma_v+\frac{k_1'\gamma_v v}{c})}-\frac{qw(k_1'\gamma_v+\frac{k'\gamma_v v}{c})^2}{2(k'\gamma_v+\frac{k_1'\gamma_v v}{c})^2}$

$lim_{\epsilon\rightarrow 0}\overline{B}_{2,n,\epsilon}(\overline{k}')=[\frac{cq}{2k}-\frac{qwk_1}{2k^2}]k_2$

$= \frac{cqk_2'}{2(k'\gamma_v+\frac{k_1'\gamma_v v}{c})} - \frac{qw(k_1'\gamma_v+\frac{k'\gamma_v v}{c})k_2'}{2(k'\gamma_v+\frac{k_1'\gamma_v v}{c})^2}$

$lim_{\epsilon\to 0}\overline{B}_{3,n,\epsilon}(\overline{k}') = [\frac{cq}{2k} - \frac{qwk_1}{2k^2}]k_3$

$= \frac{cqk_3'}{2(k'\gamma_v+\frac{k_1'\gamma_v v}{c})} - \frac{qw(k_1'\gamma_v+\frac{k'\gamma_v v}{c})k_3'}{2(k'\gamma_v+\frac{k_1'\gamma_v v}{c})^2}$

and, substituting into the last three lines of (E);

$\lim_{\epsilon\to 0}\overline{B}'_{1,n,\epsilon}(\overline{k}') = (\gamma_v+\frac{\gamma_v k_1' v}{k'c})[-v\gamma_v(\frac{q}{2}-\frac{qw(k_1'\gamma_v+\frac{k'\gamma_v v}{c})}{2(k'\gamma_v+\frac{k_1'\gamma_v v}{c})c})+\gamma_v(\frac{cq(k_1'\gamma_v+\frac{k'\gamma_v v}{c})}{2(k'\gamma_v+\frac{k_1'\gamma_v v}{c})}$
$-\frac{qw(k_1'\gamma_v+\frac{k'\gamma_v v}{c})^2}{2(k'\gamma_v+\frac{k_1'\gamma_v v}{c})^2})]$

$\lim_{\epsilon\to 0}\overline{B}'_{2,n,\epsilon}(\overline{k}') = (\gamma_v + \frac{\gamma_v k_1' v}{k'c})[\frac{cqk_2'}{2(k'\gamma_v+\frac{k_1'\gamma_v v}{c})} - \frac{qw(k_1'\gamma_v+\frac{k'\gamma_v v}{c})k_2'}{2(k'\gamma_v+\frac{k_1'\gamma_v v}{c})^2}]$

$\lim_{\epsilon\to 0}\overline{B}'_{3,n,\epsilon}(\overline{k}') = (\gamma_v + \frac{\gamma_v k_1' v}{k'c})[\frac{cqk_3'}{2(k'\gamma_v+\frac{k_1'\gamma_v v}{c})} - \frac{qw(k_1'\gamma_v+\frac{k'\gamma_v v}{c})k_3'}{2(k'\gamma_v+\frac{k_1'\gamma_v v}{c})^2}]$ (G)

We rearrange the last two terms of (F) and (G) first;

$\lim_{\epsilon\to 0}\overline{A}'_{2,n,\epsilon}(\overline{k}') = (1 - \frac{k_1' v}{k'c})[-\frac{cqk_2'}{2k'(1-\frac{k_1' v}{c})} - \frac{qw(k_1'-\frac{k' v}{c})k_2'}{2k'^2(1-\frac{k_1' v}{k'c})^2}]$

$= \frac{q}{2}[-\frac{ck_2'}{k'} - \frac{w(k_1'-\frac{k' v}{c})k_2'}{k'^2(1-\frac{k_1 v}{k'c})}]$

$= -\frac{cqk_2'}{2k'}[1 + \frac{wk'(k_1'-\frac{k' v}{c})k_2'}{ck_2'k'^2(1-\frac{k_1 v}{k'c})}]$

$= -\frac{cqk_2'}{2k'}[1 + \frac{w(k_1'-\frac{k' v}{c})}{ck'(1-\frac{k_1 v}{k'c})}]$

$= -\frac{cqk_2'}{2k'}[\frac{ck'-k_1' v+wk_1'-\frac{k' vw}{c}}{ck'(1-\frac{k_1 v}{k'c})}]$

$= -\frac{cqk_2'}{2k'}[\frac{ck'(1-\frac{vw}{c^2})+k_1'(w-v)}{ck'(1-\frac{k_1 v}{k'c})}]$

$= -\frac{cqk_2'}{2k'}(1 - \frac{vw}{c^2})[\frac{1+\frac{k_1'}{ck'}\frac{w-v}{1-\frac{vw}{c^2}}}{(1-\frac{k_1 v}{k'c})}]$

$= -\frac{cqk_2'}{2k'}(1 - \frac{vw}{c^2})[\frac{1+\frac{k_1'}{ck'}w'}{(1-\frac{k_1 v}{k'c})}]$

$= -\frac{cqk_2'}{2k'}\frac{\gamma_{-\overline{v}*\overline{w}}}{\gamma_w\gamma_v}[\frac{1+\frac{k_1'}{ck'}w'}{(1-\frac{k_1 v}{k'c})}]$

$= \frac{[-\frac{cq'k_2'}{2k'}-\frac{q'w'k_1'k_2'}{2k'^2}]}{\gamma_v(1-\frac{k_1v}{k'c})}$

$= lim_{\epsilon\rightarrow 0}\overline{A}''_{2,n,\epsilon}(\overline{k}')(q')\frac{k'}{k}$

where $\frac{k'}{k}$ is the Doppler shift, $lim_{\epsilon\rightarrow 0}\overline{A}''_{2,n,\epsilon}$ is the factor in the wave equation for an electron travelling at the transformed velocity $-\overline{v}*\overline{w}$ in S' and q' is the charge on the electron in the frame S' due to the change in mass $m' = m\frac{\gamma_{-\overline{v}*\overline{w}}}{\gamma_w}$. This follows as m is the mass of the electron as measured in S, so that $m_0 = \frac{m}{\gamma_w}$ is the rest mass, and the electron mass in S' is $m_0\gamma_{-\overline{v}*\overline{w}}$ in S' as the electron travels with velocity $-\overline{v}*\overline{w}$ in S'.

By a similar argument, replacing k_2' with k_3', we can show that;

$\lim_{\epsilon\rightarrow 0}\overline{A}'_{3,n,\epsilon}(\overline{k}') = lim_{\epsilon\rightarrow 0}\overline{A}''_{3,n,\epsilon}(q')\frac{k'}{k}$

and, by the symmetry, we also have;

$\lim_{\epsilon\rightarrow 0}\overline{B}'_{2,n,\epsilon}(\overline{k}') = lim_{\epsilon\rightarrow 0}\overline{B}''_{2,n,\epsilon}(q')\frac{k'}{k}$

$\lim_{\epsilon\rightarrow 0}\overline{B}'_{3,n,\epsilon}(\overline{k}') = lim_{\epsilon\rightarrow 0}\overline{B}''_{3,n,\epsilon}(q')\frac{k'}{k}$

For the first terms in (F) and (G), we have that;

$\lim_{\epsilon\rightarrow 0}\overline{A}'_{1,n,\epsilon}(\overline{k}') = -\frac{q\gamma_v^2}{2}(1-\frac{k_1'v}{k'c})[v+\frac{vw(k_1'-\frac{k'v}{c})}{k'(1-\frac{k_1'v}{k'c})c}+\frac{c(k_1'-\frac{k'v}{c})}{k'(1-\frac{k_1'v}{k'c})}$

$+\frac{w(k_1'-\frac{k'v}{c})^2}{k'^2(1-\frac{k_1'v}{k'c})^2}]$

$= -\frac{q\gamma_v^2}{2}[v(1-\frac{k_1'v}{k'c})+\frac{vw(k_1'-\frac{k'v}{c})}{k'c}+\frac{c(k_1'-\frac{k'v}{c})}{k'}+\frac{w(k_1'-\frac{k'v}{c})^2}{k'^2(1-\frac{k_1'v}{k'c})}]$

$= -\frac{q\gamma_v^2}{2}[v(1-\frac{k_1'v}{k'c})+\frac{c(k_1'-\frac{k'v}{c})}{k'}][1+\frac{w(k_1'-\frac{k'v}{c})}{ck'(1-\frac{k_1'v}{k'c})}]$

$= -\frac{q\gamma_v^2}{2}[v-\frac{k_1'v^2}{k'c}+\frac{ck_1'}{k'}-v][1+\frac{w(k_1'-\frac{k'v}{c})}{ck'(1-\frac{k_1'v}{k'c})}]$

$= -\frac{q\gamma_v^2}{2}[\frac{ck_1'}{k'}(1-\frac{v^2}{c^2})][1+\frac{w(k_1'-\frac{k'v}{c})}{ck'(1-\frac{k_1'v}{k'c})}]$

$= -\frac{q}{2}\frac{ck_1'}{k'}[1+\frac{w(k_1'-\frac{k'v}{c})}{ck'(1-\frac{k_1'v}{k'c})}]$

$= -\frac{q}{2}\frac{ck'_1}{k'}[\frac{ck'(1-\frac{k'_1 v}{k'c})+w(k'_1-\frac{k'v}{c})}{ck'(1-\frac{k'_1 v}{k'c})}]$

$= -\frac{q}{2}\frac{ck'_1}{k'}[\frac{ck'-vk'_1+wk'_1-\frac{k'vw}{c}}{ck'(1-\frac{k'_1 v}{k'c})}]$

$= \frac{q}{2}\frac{ck'_1}{k'}[\frac{vk'_1-wk'_1-ck'(1-\frac{vw}{c^2})}{ck'(1-\frac{k'_1 v}{k'c})}]$

$= \frac{q}{2}\frac{ck'_1}{k'}(1-\frac{vw}{c^2})[\frac{-w'k'_1-ck'}{ck'(1-\frac{k'_1 v}{k'c})}]$

$= \frac{q}{2}\frac{\gamma_{-\overline{v}*\overline{w}}}{\gamma_v\gamma_w}k'\frac{[-\frac{ck'_1}{k'}-\frac{w'k'^2_1}{2k'^2}]}{(k'-\frac{k'_1 v}{c})}$

$= \frac{q'}{2\gamma_v}k'\frac{[-\frac{ck'_1}{k'}-\frac{w'k'^2_1}{2k'^2}]}{(k'-\frac{k'_1 v}{c})}$

$= \frac{k'}{k}lim_{\epsilon\rightarrow 0}\overline{A}''_{1,n,\epsilon}(q')$

By the symmetry, we have that;

$\lim_{\epsilon\rightarrow 0}\overline{B}'_{1,n,\epsilon}(\overline{k}') = \frac{k'}{k}lim_{\epsilon\rightarrow 0}\overline{B}''_{1,n,\epsilon}(q')$

□

Lemma 0.17. *Let an electron move at velocity $\overline{w}$ in the base frame S, and let S' be a rotation of S by $g \in O(3)$, represented by the matrix $(g_{ij})_{1\leq i,j\leq 3}$, then there is no Doppler shift in the amplitudes of the spectrum of the current of the electron in S'.*

Proof. We consider a rotated frame S' relative to the base frame S. For $n \geq 6$,if the electron travels with velocity vector $\overline{w}$ in the base frame S, then by Lemma 0.15, we have that;

$lim_{\epsilon\rightarrow 0}\overline{A}_{n,\epsilon}(\overline{k}) = [-\frac{cq}{2k} - \frac{q\overline{w}.\overline{k}}{2k^2}]\overline{k}$

$lim_{\epsilon\rightarrow 0}\overline{B}_{n,\epsilon}(\overline{k}) = [\frac{cq}{2k} - \frac{q\overline{w}.\overline{k}}{2k^2}]\overline{k}$ (D)

We have that $\overline{J}_{n,\epsilon} = (j_{1,n,\epsilon}, j_{2,n,\epsilon}, j_{3,n,\epsilon})$ transforms to S' as;

$\overline{J}'_{n,\epsilon}(\overline{x}', t') = (j'_{1,n,\epsilon}, j'_{2,n,\epsilon}, j'_{3,n,\epsilon})$

where;

$j'_{i,n,\epsilon}(\overline{x}', t) = \sum_{j=1}^{3} g_{ij}j_{j,n,\epsilon}(\overline{x}, t)$, for $1 \leq i \leq 3$, $(\dagger)$

and $\overline{x}'=\overline{x}^{g}$

see [10], so that, using (†) and the wave representation for $j_{i,n,\epsilon}$, $1\leq i\leq 3$;

$j'_{i,n,\epsilon}(\overline{x}',t)=\frac{1}{(2\pi)^{3}}\sum_{j=1}^{3}g_{ij}\int_{\mathcal{R}^{3}}(\overline{A}_{n,j,\epsilon}(\overline{k})e^{ikct}+\overline{B}_{n,j,\epsilon}(\overline{k})e^{-ikct})e^{i\overline{k}\centerdot\overline{x}}d\overline{k}$

(††)

We have that, using the inverse rotation, $\overline{x}=\overline{x}'^{g^{-1}}$ and substituting into (††), using the fact that $g\in O(3)$;

$j'_{i,n,\epsilon}(\overline{x}',t')=\frac{1}{(2\pi)^{3}}\int_{\mathcal{R}^{3}}\int_{\mathcal{R}^{3}}((\sum_{j=1}^{3}g_{ij}\overline{A}_{n,j,\epsilon}(\overline{k}))e^{ikct}+(\sum_{j=1}^{3}g_{ij}\overline{B}_{n,j,\epsilon}(\overline{k}))e^{-ikct})e^{i\overline{k}\centerdot\overline{x}'^{g^{-1}}}d\overline{k}$

$=\frac{1}{(2\pi)^{3}}\int_{\mathcal{R}^{3}}\int_{\mathcal{R}^{3}}((\sum_{j=1}^{3}g_{ij}\overline{A}_{n,j,\epsilon}(\overline{k})e^{ikct}+(\sum_{j=1}^{3}g_{ij}\overline{B}_{n,j,\epsilon}(\overline{k}))e^{-ikct})e^{i\overline{k}^{g}\centerdot\overline{x}'}d\overline{k}$

(A)

We make the change of variables $\overline{k}'=\overline{k}^{g}$. Calculating the Jacobian, and using the fact that $g\in O(3)$, we have that;

$d\overline{k}=d\overline{k}'$

so that, substituting in (A), for $1\leq i\leq 3$;

$j'_{i,n,\epsilon}(\overline{x}',t)=\frac{1}{(2\pi)^{3}}\int_{\mathcal{R}^{3}}\int_{\mathcal{R}^{3}}(\sum_{j=1}^{3}g_{ij}\overline{A}_{n,j,\epsilon}(\overline{k}'^{g^{-1}}))e^{ik'ct}+(\sum_{j=1}^{3}g_{ij}\overline{B}_{n,j,\epsilon}(\overline{k}'^{g^{-1}}))e^{-ik'ct})e^{i\overline{k}'\centerdot\overline{x}'}d\overline{k}'$

and equating coefficients, using the wave representation for $j'_{i,n,\epsilon}(\overline{x}',t)$, we have that;

$\overline{A}'_{n,i,\epsilon}(\overline{k}')=\sum_{j=1}^{3}g_{ij}\overline{A}_{n,j,\epsilon}(\overline{k}'^{g^{-1}})$
$\overline{B}'_{n,i,\epsilon}(\overline{k}')=\sum_{j=1}^{3}g_{ij}\overline{B}_{n,j,\epsilon}(\overline{k}'^{g^{-1}})$ (E)

so that, using (D) and the fact that $g\in O(3)$, $k=k'$;

$\lim_{\epsilon\rightarrow 0}\overline{A}'_{n,i,\epsilon}(\overline{k}')=\sum_{j=1}^{3}g_{ij}[-\frac{cq}{2k'}-\frac{q\overline{w}\centerdot\overline{k}'^{g^{-1}}}{2k'^{2}}]k_{j}$

$=[-\frac{cq}{2k'}-\frac{q\overline{w}^{g}\centerdot\overline{k}'}{2k'^{2}}]\sum_{j=1}^{3}g_{ij}k_{j}$

$=[-\frac{cq}{2k'}-\frac{q\overline{w}^{g}\centerdot\overline{k}'}{2k'^{2}}]k'_{i}$

$\lim_{\epsilon\rightarrow 0}\overline{B}'_{n,i,\epsilon}(\overline{k}')=\sum_{j=1}^{3}g_{ij}[\frac{cq}{2k'}-\frac{q\overline{w}.\overline{k}'^{g^{-1}}}{2k'^2}]k_j$

$=[\frac{cq}{2k'}-\frac{q\overline{w}^g.\overline{k}'}{2k'^2}]\sum_{j=1}^{3}g_{ij}k_j$

$=[\frac{cq}{2k'}-\frac{q\overline{w}^g.\overline{k}'}{2k'^2}]k'_i$

which represent the current of an electron with charge q and velocity vector $\overline{w}^g$ moving in S'. There is no mass change and no Doppler effect as $k'=k$.

□

Definition 0.18. *Given a smooth trajectory $\overline{w}:[0,t_0]\rightarrow\mathcal{R}^3$, with velocity $\overline{v}:[0,t_0]\rightarrow\mathcal{R}^3$, $\overline{v}(s)=\overline{w}'(s)$, $|\overline{v}(s)|<c$, $s\in[0,t_0]$, we define;*

$D_{\overline{w}(s)}(\overline{x})=D(\overline{x}-\overline{w}(s))$

$D_{\overline{w}(s),\overline{v}(s)}(\overline{x})=\frac{d}{dt}|_{t=0}D_{\overline{w}(s)}(\overline{x}+t\overline{v}(s))$

$=v_1(s)D_{\overline{w}(s),x}+v_2(s)D_{\overline{w}(s),y}+v_3(s)D_{\overline{w}(s),z}$

in the sense of distributions, and the approximations $D_{n,\epsilon,\overline{w}(s)}$, for fixed $s\in[0,t_0]$, and $n\geq 6$, by;

$D_{n,\epsilon,\overline{w}(s)}(\overline{x})=D_{n,\epsilon}(\overline{x}-\overline{w}(s))$

$D_{n,\epsilon,\overline{w}(s),\overline{v}(s)}(\overline{x})=\frac{d}{dt}|_{t=0}D_{n,\epsilon,\overline{w}(s)}(\overline{x}+t\overline{v}(s))$

$=v_1(s)D_{n,\epsilon,\overline{w}(s),x}+v_2(s)D_{n,\epsilon,\overline{w}(s),y}+v_3(s)D_{n,\epsilon,\overline{w}(s),z}$

the reverse time derivative of $D_{n,\epsilon,\overline{w}(s)}$.

We define $\rho_{n,\epsilon,\overline{w}(s),\overline{v}(s)}$ to be the unique wave with $\square^2(\rho_{n,\epsilon,\overline{w}(s),\overline{v}(s)})=0$, generated by the initial conditions $\{qD_{n,\epsilon,\overline{w}(s)},qD_{n,\epsilon,\overline{w}(s),\overline{v}(s)}\}$ at the time $t=s$, $\rho_{n,\epsilon,\overline{w}(s),\overline{v}(s)}$ has compact support and we let $\overline{J}_{n,\epsilon,\overline{w}(s),\overline{v}(s)}$ be the associated current wave, $\square^2(\overline{J}_{n,\epsilon,\overline{w}(s),\overline{v}(s)})=\overline{0}$, satisfying the standard conditions in [9], with compact support.

Lemma 0.19. *We have, for $n\geq 6$, the representations;*

$\rho_{n,\epsilon,\overline{w}(s),\overline{v}(s)}(\overline{x},t)=\int_{\overline{k}\in\mathcal{R}^3}[A_{n,\epsilon}(\overline{v}(s))(\overline{k})e^{ick(t-s)}+B_{n,\epsilon}(\overline{v}(s))(\overline{k})e^{-ick(t-s)}]e^{i\overline{k}.(\overline{x}-\overline{w}(s))}d\overline{k}$

$\overline{J}_{n,\epsilon,\overline{w}(s),\overline{v}(s)}(\overline{x},t)=\int_{\overline{k}\in\mathcal{R}^3}[\overline{A}_{n,\epsilon}(\overline{v}(s))(\overline{k})e^{ick(t-s)}+\overline{B}_{n,\epsilon}(\overline{v}(s))(\overline{k})e^{-ick(t-s)}]e^{i\overline{k}\centerdot(\overline{x}-\overline{w}(s))}d\overline{k}$

Proof. For the proof, we can go through the previous steps in the paper where we computed $A_{n,\epsilon}$ and $B_{n,\epsilon}$, noting that they are natural functions of the velocity vector $\overline{v}(s)$. We have that, using Definition 0.18;

$\mathcal{F}(qD_{n,\epsilon,\overline{w}(s)})=\int_{\overline{x}\in\mathcal{R}^3}qD_{n,\epsilon,\overline{w}(s)}(\overline{x})e^{-i\overline{k}\centerdot\overline{x}}d\overline{x}$

$=q\int_{\overline{x}\in\mathcal{R}^3}D_{n,\epsilon}(\overline{x}-\overline{w}(s))e^{-i\overline{k}\centerdot\overline{x}}d\overline{x}$

$=q\int_{\overline{x}\in\mathcal{R}^3}D_{n,\epsilon}(\overline{y})e^{-i\overline{k}\centerdot(\overline{y}+\overline{w}(s))}d\overline{y}$, $\overline{y}=\overline{x}-\overline{w}(s)$, $d\overline{y}=d\overline{x}$

$=qe^{-i\overline{k}\centerdot\overline{w}(s)}\int_{\overline{x}\in\mathcal{R}^3}D_{n,\epsilon}(\overline{y})e^{-i\overline{k}\centerdot\overline{y}}d\overline{y}$

$=e^{-i\overline{k}\centerdot\overline{w}(s)}\mathcal{F}(qD_{n,\epsilon})$

and, similarly;

$\mathcal{F}(qD_{n,\epsilon,\overline{w}(s),\overline{v}(s)})=\mathcal{F}(v_1(s)D_{n,\epsilon,\overline{w}(s),x}+v_2(s)D_{n,\epsilon,\overline{w}(s),y}+v_3(s)D_{n,\epsilon,\overline{w}(s),z})$

$=iv_1(s)k_1\mathcal{F}(qD_{n,\epsilon,\overline{w}(s)})+iv_2(s)k_2\mathcal{F}(qD_{n,\epsilon,\overline{w}(s)})+iv_3(s)k_3\mathcal{F}(qD_{n,\epsilon,\overline{w}(s)})$

$=[iv_1(s)k_1e^{-i\overline{k}\centerdot\overline{w}(s)}+iv_2(s)k_2e^{-i\overline{k}\centerdot\overline{w}(s)}+iv_3(s)k_3e^{-i\overline{k}\centerdot\overline{w}(s)}]\mathcal{F}(qD_{n,\epsilon,\overline{w}(s)})$

$=i\overline{v}(s)\centerdot\overline{k}e^{-i\overline{k}\centerdot\overline{w}(s)}\mathcal{F}(qD_{n,\epsilon})$

$=e^{-i\overline{k}\centerdot\overline{w}(s)}\mathcal{F}(qD'_{n,\epsilon,t})(\overline{v}(s))$

It is then clear that the wave equation terms $\{A_{n,\epsilon,\overline{w}(s),\overline{v}(s)},B_{n,\epsilon,\overline{w}(s),\overline{v}(s)}\}$ for $\rho_{n,\epsilon,\overline{w}(s),\overline{v}(s),0}$ factorise as;

$A_{n,\epsilon,\overline{w}(s),\overline{v}(s),0}(\overline{k})=e^{-i\overline{k}\centerdot\overline{w}(s)}A_{n,\epsilon}(\overline{v}(s))(\overline{k})$

$B_{n,\epsilon,\overline{w}(s),\overline{v}(s),0}(\overline{k})=e^{-i\overline{k}\centerdot\overline{w}(s)}B_{n,\epsilon}(\overline{v}(s))(\overline{k})$

where $\rho_{n,\epsilon,\overline{w}(s),\overline{v}(s),0}$ is generated by the initial conditions;

$\{qD_{n,\epsilon,\overline{w}(s)},qD_{n,\epsilon,\overline{w}(s),\overline{v}(s)}\}$

at the time $t=0$. We then have that, using the inversion theorem;

$\rho_{n,\epsilon,\overline{w}(s),\overline{v}(s),0}(\overline{x},t)=\int_{\overline{k}\in\mathcal{R}^3}[A_{n,\epsilon,\overline{w}(s),\overline{v}(s)}(\overline{k})e^{ickt}+B_{n,\epsilon,\overline{w}(s),\overline{v}(s)}(\overline{k})e^{-ickt}]e^{i\overline{k}.\overline{x}}d\overline{k}$

$=\int_{\overline{k}\in\mathcal{R}^3}[A_{n,\epsilon}(\overline{v}(s))(\overline{k})e^{ickt}+B_{n,\epsilon}(\overline{v}(s))(\overline{k})e^{-ickt}]e^{i\overline{k}.(\overline{x}-\overline{w}(s))}d\overline{k}$ $(*)$

and we obtain the first result by taking;

$\rho_{n,\epsilon,\overline{w}(s),\overline{v}(s)}(\overline{x},t)=\rho_{n,\epsilon,\overline{w}(s),\overline{v}(s),0}(\overline{x},t-s)$

and substituting in $(*)$.

For the second result, just use the construction at the beginning of Lemma 0.12, together with the wave representation for $\rho_{n,\epsilon,\overline{w}(s),\overline{v}(s)}$.

□

Lemma 0.20. *Given a straight line path $\overline{w}:[0,t_0]\rightarrow\mathcal{R}^3$, with $\overline{w}'(s)=(v,0,0)$, $0\leq v<c$, $0\leq t\leq t_0$, there exists, for $\epsilon>0$, $G_\epsilon:\mathcal{R}^3\rightarrow\mathcal{R}$, defined on $V_\epsilon\subset\mathcal{R}^3$ open, with $\overline{V_\epsilon}$ compact, such that if $\rho_\epsilon:\mathcal{R}^3\times[0,t_0]$ is defined by;*

$\rho_\epsilon(\overline{x},t)=G_\epsilon(\overline{x}-\overline{w}(t))$

then, $\square^2(\rho_\epsilon)=0$ on $W_\epsilon=\{(\overline{x},t):t\in(0,t_0),\overline{x}\in\overline{w}(t)+V_\epsilon\}$ and $lim_{\epsilon\rightarrow 0}\rho_{\epsilon,t}=D_{\overline{w}(t)}$ in the sense of distributions, for $t\in(0,t_0)$.

Proof. By the chain rule, the condition that $\square^2(\rho_\epsilon)=0$ is given by;

$(G_{\epsilon,xx}+G_{\epsilon,yy}+G_{\epsilon,zz}-\frac{v^2}{c^2}G_{\epsilon,xx})|_{\overline{x}-\overline{w}(t)}$

$=((1-\frac{v^2}{c^2})G_{\epsilon,xx}+G_{\epsilon,yy}+G_{\epsilon,zz})|_{\overline{x}-\overline{w}(t)}$

$=(\frac{\partial}{\partial x},\frac{\partial}{\partial y},\frac{\partial}{\partial z})Diag(1-\frac{v^2}{c^2},1,1)(\frac{\partial}{\partial x},\frac{\partial}{\partial y},\frac{\partial}{\partial z})^tG_\epsilon$

$=0$ $(*)$

As $v<c$, $1-\frac{v^2}{c^2}>0$, so that, by Sylvester's theorem, there exists an invertible matrix P with;

$P^tDiag(1-\frac{v^2}{c^2},1,1)P=Diag(1,1,1)$

so that, the condition $(*)$ can be written as;

$(\frac{\partial}{\partial x},\frac{\partial}{\partial y},\frac{\partial}{\partial z})P^{-1,t}Diag(1,1,1)P^{-1}(\frac{\partial}{\partial x},\frac{\partial}{\partial y},\frac{\partial}{\partial z})^t G_\epsilon$

$=(\frac{\partial}{\partial x},\frac{\partial}{\partial y},\frac{\partial}{\partial z})P^{-1,t}Diag(1,1,1)((\frac{\partial}{\partial x},\frac{\partial}{\partial y},\frac{\partial}{\partial z})P^{-1,t})^t G_\epsilon$

$=(\frac{\partial}{\partial x'},\frac{\partial}{\partial y'},\frac{\partial}{\partial z'})Diag(1,1,1)(\frac{\partial}{\partial x'},\frac{\partial}{\partial y'},\frac{\partial}{\partial z'})^t H_\epsilon$

$=0$ $(**)$

where $(\frac{\partial}{\partial x'},\frac{\partial}{\partial y'},\frac{\partial}{\partial z'})=(\frac{\partial}{\partial x},\frac{\partial}{\partial y},\frac{\partial}{\partial z})P^{-1,t}$

Letting $(x',y',z')=(x,y,z)P$, we require that $\bigtriangledown^2(H_\epsilon)=0$, where $H_\epsilon(x',y',z')=G((x',y',z)P^{-1})$. Define H on $B(\overline{0},1)\setminus 0$ by;

$H(\overline{x}')=\frac{1}{|\overline{x}'|}-1$

Then;

$H_{x'x'}=-\frac{1}{|\overline{x}|^3}+3\frac{x'^2}{|\overline{x}'|^5}$

$H_{x'x'}+H_{y'y'}+H_{z'z'}=-\frac{3}{|\overline{x}|^3}+3\frac{|\overline{x}'|^2}{|\overline{x}'|^5}$

$=0$

so that $\bigtriangledown^2(H)=0$. Let H_ϵ be defined on the closed ball $B(\overline{0},\epsilon)$ by;

$H_\epsilon(\overline{x})=\frac{1}{\epsilon}-1$

and on the open annulus $Ann(\epsilon,2\epsilon)$, by;

$H_\epsilon(\overline{x}')=H(\overline{y}')$

where $|\overline{y}'|=\epsilon+\frac{(1-\epsilon)}{\epsilon}(|\overline{x}'|-\epsilon)$

Otherwise we define H_ϵ to be zero. By linearity, we still have that $\bigtriangledown^2 H_\epsilon=0$ on $Ann(\epsilon,2\epsilon)$ and trivially the same holds on the open ball $B^\circ(\overline{0},\epsilon)$ and on $\mathcal{R}^3\setminus B^\circ(\overline{0},2\epsilon)$. By construction, H_ϵ is continuous on $\mathcal{R}^3$ with compact support. Let $V_\epsilon=B^\circ(0,2\epsilon)P^{-1}$, then by continuity, V_ϵ is open and the closure $\overline{V}_\epsilon$ is compact. Define G_ϵ on V_ϵ by $G_\epsilon(\overline{x})=H_\epsilon(\overline{x}P)$ and zero elsewhere. By construction, G_ϵ is continuous on $\mathcal{R}^3$, $(*)$ is satisfied and $\Box^2(\rho_\epsilon)=0$ a.e, for the corresponding ρ_ϵ.

We have that, using polar coordinates;

$\int_{\mathcal{R}^3} H_\epsilon d\overline{x}' = \frac{4\pi\epsilon^3(\frac{1}{\epsilon}-1)}{3} + \int_{Ann(\epsilon,2\epsilon)} H_\epsilon d\overline{x}'$

$= \frac{4\pi\epsilon^3(\frac{1}{\epsilon}-1)}{3} + 4\pi\int_\epsilon^{2\epsilon}(\frac{1}{\epsilon+\frac{(1-\epsilon)}{\epsilon}(R-\epsilon)} - 1)R^2 dR$

$= \frac{4\pi\epsilon^3(\frac{1}{\epsilon}-1)}{3} - \frac{28\pi\epsilon^3}{3} + 4\pi\int_\epsilon^{2\epsilon}\frac{R^2}{\alpha+\beta R}dR$

where;

$\alpha = \epsilon - \frac{\epsilon(1-\epsilon)}{\epsilon} = 2\epsilon - 1$

$\beta = \frac{1-\epsilon}{\epsilon}$

We have, completing the square;

$\int_\epsilon^{2\epsilon}\frac{R^2}{\alpha+\beta R}dR$

$= \int_\epsilon^{2\epsilon}[\frac{\alpha+\beta R}{\beta^2} - \frac{2\alpha}{\beta^2} + \frac{\alpha^2}{\beta^2(\alpha+\beta R)}]dR$

$= [\frac{\alpha R}{\beta^2} + \frac{\beta R^2}{2\beta^2} - \frac{2\alpha R}{\beta^2} + \frac{\alpha^2 ln(\alpha+\beta R)}{\beta^3}]_\epsilon^{2\epsilon}$

$= \gamma(\epsilon)$

so that;

$\int_{\mathcal{R}^3}\frac{H_\epsilon}{\gamma_\epsilon}d\overline{x}' = 1$

By the change of variables formula;

$\int_{\mathcal{R}^3} G_\epsilon|det(P)|d\overline{x} = \int_{\mathcal{R}^3} H_\epsilon d\overline{x}'$

so that;

$\int_{\mathcal{R}^3}\frac{G_\epsilon|det(P)|}{\gamma(\epsilon)}d\overline{x} = 1$

as well.

By Lemma 0.3, $lim_{\epsilon\rightarrow 0}\rho_{\epsilon,\overline{w}(t)} = D_{\overline{w}(t)}$ in the sense of distributions, where;

$\rho_\epsilon(\overline{x},t)=\frac{G_\epsilon|det(P)|}{\gamma(\epsilon)}(\overline{x}-\overline{w}(t))$

$\square$

Lemma 0.21. *Given a straight line path $\overline{w}:[0,t_0]\rightarrow\mathcal{R}^3$, with $\overline{w}'(s)=(v,0,0)$, $0\leq v<c$, $0\leq t'\leq t_0$, there exist, for $\epsilon>0$, open subsets $V_{\epsilon,t'}\subset\mathcal{R}^3$, centred at $\overline{w}(t')$, and $\rho'_{\epsilon,1}$ a charge distribution supported on the compact sets $\overline{V}_{\epsilon,t'}$, such that $\square^2(\rho'_{\epsilon,1})=0$ on $V_{\epsilon,t'}$ and $lim_{\epsilon\rightarrow 0}\rho_{\epsilon,1,t'}=D_{\overline{w}(t')}$ in the sense of distributions, for $t'\in(0,t_0)$.*

Proof. We use the example given in Lemma 0.18 of [11] for the base frame S. $(\rho_\epsilon,\overline{J}_\epsilon)$ are given in polar coordinates, by;

$\rho_\epsilon(\overline{r},t)=\frac{sin(\gamma(\epsilon)r)}{r}e^{i\gamma(\epsilon)ct}$

$\overline{J}_\epsilon(\overline{r},t)=\frac{ic}{\gamma(\epsilon)}(\frac{\gamma(\epsilon)rcos(\gamma(\epsilon)r)-sin(\gamma(\epsilon)r)}{r^2})e^{i\gamma(\epsilon)ct}\hat{\overline{r}}$

with $tan(\gamma(\epsilon)\epsilon)=\gamma(\epsilon)\epsilon$. As is shown in [11], $\{\rho_\epsilon,\overline{J}_\epsilon\}$ satisfy the usual relations including $\square^2(\rho_\epsilon)=0$ with $\overline{J}_\epsilon|_{S(\epsilon)}=\overline{0}$. We restrict the given charge and current distributions to $B(\overline{0},\epsilon)$ and define them to be zero outside $B(\overline{0},\epsilon)$, introducing a discontinuity at the boundary. The properties are local and the representation $(*)$ still holds inside $B(\overline{0},\epsilon)$. Now let S' travel with velocity $-v$ relative to S. Then the transformed charge ρ'_ϵ is given by;

$\rho'_\epsilon=\gamma_v(\rho_\epsilon+\frac{vj_{1,\epsilon}}{c^2})$

and, by results in [7] or [9], we still have that $\square^2(\rho'_\epsilon)=0$.

Using the standard Lorentz transformation;

$x=\gamma_v(x'-vt')$, $t=\gamma_v(t'-\frac{vx'}{c^2})$

we have at time t'_0 in the moving frame S';

$\gamma_v^2(x'-vt'_0)^2+y'^2+z'^2=\epsilon^2$

is the locus of the image of $\delta B(\overline{0},\epsilon)$, for varying times in S, in S' at t'_0. Calling the interior of this locus V_{ϵ,t'_0}, V_{ϵ,t'_0} is an ellipsoid with diameter at most 2ϵ and centred at vt'_0, where t'_0 is measured in S'.

Clearly ρ'_ϵ is supported on $\overline{V}_\epsilon$.

By the change of variables formula and the definition of ρ'_ϵ, we have that;

$$\int_{V_{\epsilon,t'_0}} \rho'_{\epsilon,t'_0} d\overline{x}'$$

$$= \int_{B(\overline{0},\epsilon)} (\gamma_v\rho_\epsilon + \tfrac{\gamma_v v j_{1,\epsilon}}{c^2})|_{(x,y,z,\gamma_v(t'_0-\frac{vx'}{c^2}))}\tfrac{1}{\gamma_v}d\overline{x}$$

$$= \int_{B(\overline{0},\epsilon)} (\gamma_v\rho_\epsilon + \tfrac{\gamma_v v j_{1,\epsilon}}{c^2})|_{(x,y,z,\gamma_v(t'_0-\frac{v}{c^2}(\frac{x}{\gamma_v}+vt'_0)))}\tfrac{1}{\gamma_v}d\overline{x}$$

$$= \int_{B(\overline{0},\epsilon)} (\gamma_v\rho_\epsilon + \tfrac{\gamma_v v j_{1,\epsilon}}{c^2})|_{(x,y,z,\gamma_v t'_0\gamma_v^{-2}-\frac{vx}{c^2})}\tfrac{1}{\gamma_v}d\overline{x}$$

$$= \int_{B(\overline{0},\epsilon)} (\gamma_v\rho_\epsilon + \tfrac{\gamma_v v j_{1,\epsilon}}{c^2})|_{(x,y,z,\frac{t'_0}{\gamma_v}-\frac{vx}{c^2})}\tfrac{1}{\gamma_v}d\overline{x}$$

where we have used the fact that at time t'_0, $x = \gamma_v(x' - vt'_0)$, so that rearranging, $x' = \frac{x}{\gamma_v} + vt'_0$, together with the formula $t = \gamma_v(t'_0 - \frac{vx'}{c^2})$. The Jacobian of the change of coordinates is $\frac{1}{\gamma_v}$, as $x' = \frac{x}{\gamma_v} + vt'_0$, $y' = y$,$z' = z$. It follows that;

$$\int_{V_{\epsilon,t'_0}} \rho'_{\epsilon,t'_0} d\overline{x}'$$

$$= \int_{B(\overline{0},\epsilon)} (\gamma_v \tfrac{sin(\gamma(\epsilon)r)}{r} e^{i\gamma(\epsilon)c(\frac{t'_0}{\gamma_v}-\frac{vx}{c^2})} + \tfrac{v\gamma_v}{c^2}\tfrac{ic}{\gamma(\epsilon)}(\tfrac{\gamma(\epsilon)rcos(\gamma(\epsilon)r)-sin(\gamma(\epsilon)r)}{r^2})e^{i\gamma(\epsilon)c(\frac{t'_0}{\gamma_v}-\frac{vx}{c^2})}\hat{\overline{x}}_1)\tfrac{1}{\gamma_v}d\overline{x}$$

$$= \int_{B(\overline{0},\epsilon)} (\tfrac{sin(\gamma(\epsilon)r)}{r} e^{i\gamma(\epsilon)c(\frac{t'_0}{\gamma_v}-\frac{vx}{c^2})} + \tfrac{v}{c^2}\tfrac{ic}{\gamma(\epsilon)}(\tfrac{\gamma(\epsilon)rcos(\gamma(\epsilon)r)-sin(\gamma(\epsilon)r)}{r^2})e^{i\gamma(\epsilon)c(\frac{t'_0}{\gamma_v}-\frac{vx}{c^2})}\hat{\overline{x}}_1)d\overline{x}$$

We have that;

$$\int_{B(\overline{0},\epsilon)} (\tfrac{sin(\gamma(\epsilon)r)}{r} e^{i\gamma(\epsilon)c(\frac{t'_0}{\gamma_v}-\frac{vx}{c^2})}d\overline{x}$$

$$= e^{i\gamma(\epsilon)\frac{ct'_0}{\gamma_v}} \int_{B(\overline{0},\epsilon)} \tfrac{sin(\gamma(\epsilon)r)}{r}d\overline{x} + \int_{B(\overline{0},\epsilon)} \tfrac{sin(\gamma(\epsilon)r)}{r}e^{-i\gamma(\epsilon)\frac{vx}{c}}d\overline{x}$$

$$= \int_{B(\overline{0},\epsilon)} \tfrac{sin(\gamma(\epsilon)r)}{r}e^{-i\gamma(\epsilon)\frac{vx}{c}}d\overline{x}$$

by the calculation in [11]. Using polar coordinates and the first part of Lemma 0.22;

$$\int_{B(\overline{0},\epsilon)} \tfrac{sin(\gamma(\epsilon)r)}{r}e^{-i\gamma(\epsilon)\frac{vx}{c}}d\overline{x}$$

$= \int_0^\epsilon \frac{sin(\gamma(\epsilon)r)}{r}(\int_{S(\bar{0},r)} e^{-i\gamma(\epsilon)\frac{vx}{c}} dS(r))dr$

$= \int_0^\epsilon \frac{sin(\gamma(\epsilon)r)}{r}(\int_{S(\bar{0},r)} e^{i\bar{x}\centerdot(-\frac{\gamma(\epsilon)v}{c},0,0)} dS(r)))dr$

$= \int_0^\epsilon \frac{sin(\gamma(\epsilon)r)}{r}\frac{4\pi r sin(\frac{\gamma(\epsilon)vr}{c})}{\frac{\gamma(\epsilon)v}{c}}dr$

$= \frac{4\pi c}{\gamma(\epsilon)v}\int_0^\epsilon sin(\gamma(\epsilon)r)sin(\frac{\gamma(\epsilon)vr}{c})dr$, $(*)$

We have that;

$\int_{B(\bar{0},\epsilon)} \frac{v}{c^2}\frac{ic}{\gamma(\epsilon)}(\frac{\gamma(\epsilon)rcos(\gamma(\epsilon)r)-sin(\gamma(\epsilon)r)}{r^2})e^{i\gamma(\epsilon)c(\frac{t_0'}{\gamma_v}-\frac{vx}{c^2})}\hat{\bar{x}}_1)d\bar{x}$

$= e^{i\gamma(\epsilon)c(\frac{t_0'}{\gamma_v})}\int_{B(\bar{0},\epsilon)} \frac{v}{c^2}\frac{ic}{\gamma(\epsilon)}(\frac{\gamma(\epsilon)rcos(\gamma(\epsilon)r)-sin(\gamma(\epsilon)r)}{r^2})\hat{\bar{x}}_1)d\bar{x}$

$+ \int_{B(\bar{0},\epsilon)} \frac{v}{c^2}\frac{ic}{\gamma(\epsilon)}(\frac{\gamma(\epsilon)rcos(\gamma(\epsilon)r)-sin(\gamma(\epsilon)r)}{r^2})e^{-i\gamma(\epsilon)\frac{vx}{c}}\hat{\bar{x}}_1 d\bar{x}$

$= \frac{iv}{c\gamma(\epsilon)}\int_{B(\bar{0},\epsilon)}(\frac{\gamma(\epsilon)rcos(\gamma(\epsilon)r)-sin(\gamma(\epsilon)r)}{r^2})e^{-i\gamma(\epsilon)\frac{vx}{c}}\hat{\bar{x}}_1 d\bar{x}$

as, converting to polars, using the fact that $\int_{-\pi}^{\pi} cos(\phi)d\phi = 0$;

$e^{i\gamma(\epsilon)c(\frac{t_0'}{\gamma_v})}\int_{B(\bar{0},\epsilon)} \frac{v}{c^2}\frac{ic}{\gamma(\epsilon)}(\frac{\gamma(\epsilon)rcos(\gamma(\epsilon)r)-sin(\gamma(\epsilon)r)}{r^2})\hat{\bar{x}}_1)d\bar{x}$

$= e^{i\gamma(\epsilon)c(\frac{t_0'}{\gamma_v})}\int_0^\epsilon\int_0^\pi\int_{-pi}^{\pi} \frac{v}{c^2}\frac{ic}{\gamma(\epsilon)}(\frac{\gamma(\epsilon)rcos(\gamma(\epsilon)r)-sin(\gamma(\epsilon)r)}{r^2})r^2 sin^2(\theta)cos(\phi)d\phi d\theta dr$

$= 0$

We have that, converting to polars again, using the second part of Lemma 0.22, integrating by parts and using the definition of $\gamma(\epsilon)$;

$\frac{iv}{c\gamma(\epsilon)}\int_{B(\bar{0},\epsilon)}(\frac{\gamma(\epsilon)rcos(\gamma(\epsilon)r)-sin(\gamma(\epsilon)r)}{r^2})e^{-i\gamma(\epsilon)\frac{vx}{c}}\hat{\bar{x}}_1 d\bar{x}$

$= \frac{iv}{c\gamma(\epsilon)}\int_0^\epsilon(\frac{\gamma(\epsilon)rcos(\gamma(\epsilon)r)-sin(\gamma(\epsilon)r)}{r^2})(\int_{S(\bar{0},r)} e^{-i\gamma(\epsilon)\frac{vx}{c}}\hat{\bar{x}}_1)dS(r)dr$

$= \frac{iv}{c\gamma(\epsilon)}\int_0^\epsilon(\frac{\gamma(\epsilon)rcos(\gamma(\epsilon)r)-sin(\gamma(\epsilon)r)}{r^2})(\int_{S(\bar{0},r)} \frac{\bar{r}}{r}e^{i\bar{r}\centerdot(-\frac{\gamma(\epsilon)v}{c},0,0)}dS(r))_1 dr$

$= \frac{iv}{c\gamma(\epsilon)}\int_0^\epsilon(\frac{\gamma(\epsilon)rcos(\gamma(\epsilon)r)-sin(\gamma(\epsilon)r)}{r^2})\frac{1}{r}4\pi ir((-\widehat{\frac{\gamma(\epsilon)v}{c}},0,0))_1\frac{1}{\frac{\gamma(\epsilon)^2v^2}{c^2}}(sin(\frac{\gamma(\epsilon)vr}{c})$

$-\frac{\gamma(\epsilon)vr}{c}cos(\frac{\gamma(\epsilon)vr}{c}))dr$

$= \frac{iv}{c\gamma(\epsilon)}\frac{-4\pi ic^2}{\gamma(\epsilon)^2v^2}\int_0^\epsilon(\frac{\gamma(\epsilon)rcos(\gamma(\epsilon)r)-sin(\gamma(\epsilon)r)}{r^2})(sin(\frac{\gamma(\epsilon)vr}{c})-\frac{\gamma(\epsilon)vr}{c}cos(\frac{\gamma(\epsilon)vr}{c}))dr$

$= \frac{4\pi c}{\gamma(\epsilon)^3v}\int_0^\epsilon(\frac{\gamma(\epsilon)rcos(\gamma(\epsilon)r)-sin(\gamma(\epsilon)r)}{r^2})(sin(\frac{\gamma(\epsilon)vr}{c})-\frac{\gamma(\epsilon)vr}{c}cos(\frac{\gamma(\epsilon)vr}{c}))dr$

$= \frac{4\pi c}{\gamma(\epsilon)^3v}([\frac{sin(\gamma(\epsilon)r)sin(\frac{\gamma(\epsilon)vr}{c})}{r}]_0^\epsilon-\int_0^\epsilon\frac{\gamma(\epsilon)cos(\gamma(\epsilon)r)sin(\frac{\gamma(\epsilon)vr}{c})+\frac{\gamma(\epsilon)v}{c}cos(\frac{\gamma(\epsilon)vc}{r})sin(\gamma(\epsilon)r)}{r}dr$

$+\int_0^\epsilon\frac{\gamma(\epsilon)rcos(\gamma(\epsilon)r)sin(\frac{\gamma(\epsilon)vr}{c})+\frac{\gamma(\epsilon)vr}{c}cos(\frac{\gamma(\epsilon)vc}{r})sin(\gamma(\epsilon)r)}{r^2}dr-\int_0^\epsilon\frac{\frac{\gamma(\epsilon)^2r^2v}{c}cos(\gamma(\epsilon)r)cos(\frac{\gamma(\epsilon)vr}{c})}{r^2}dr)$

$= \frac{4\pi c}{\gamma(\epsilon)^3v}(\frac{sin(\gamma(\epsilon)\epsilon)sin(\frac{\gamma(\epsilon)v\epsilon}{c})}{\epsilon}-\frac{\gamma(\epsilon)^2v}{c}\int_0^\epsilon cos(\gamma(\epsilon)r)cos(\frac{\gamma(\epsilon)vr}{c}))dr$

$= \frac{4\pi c}{\gamma(\epsilon)^3v}(\gamma(\epsilon)cos(\gamma(\epsilon)\epsilon)sin(\frac{\gamma(\epsilon)v\epsilon}{c}))-\frac{4\pi}{\gamma(\epsilon)}\int_0^\epsilon cos(\gamma(\epsilon)r)cos(\frac{\gamma(\epsilon)vr}{c})dr$

$= \frac{4\pi c}{\gamma(\epsilon)^2v}(cos(\gamma(\epsilon)\epsilon)sin(\frac{\gamma(\epsilon)v\epsilon}{c}))-\frac{4\pi}{\gamma(\epsilon)}(\frac{[sin(\frac{\gamma(\epsilon)vr}{c})cos(\gamma(\epsilon)r)]_0^\epsilon}{\frac{v\gamma(\epsilon)}{c}}-\frac{1}{\frac{\gamma(\epsilon)v}{c}}\int_0^\epsilon sin(\frac{\gamma(\epsilon)vr}{c})sin(\gamma(\epsilon)r)\gamma(\epsilon)dr)$

$= -\frac{4\pi c}{v\gamma(\epsilon)}\int_0^\epsilon sin(\frac{\gamma(\epsilon)vr}{c})sin(\gamma(\epsilon)r)dr$

Cancelling with $(*)$, we obtain the result that;

$\int_{V_{\epsilon,t'_0}}\rho'_{\epsilon,t'_0}d\overline{x}'=0$

Now define $\rho_{\epsilon,1}=\rho_\epsilon+c(\epsilon)$, where $c(\epsilon)$ is a constant to be determined, and keep the current the same. Clearly, all the standard relations of [11] are still satisfied and we restrict to $B(\overline{0},\epsilon)$. As before we have that the transformed charge $\rho'_{\epsilon,1}$ satisfies $\square^2(\rho'_{\epsilon,1})=0$ and is supported on $V_{\epsilon,t'}$ as t' varies in S'. This time we have that;

$\int_{V_{\epsilon,t'_0}}\rho'_{\epsilon,1,t'_0}d\overline{x}'$

$=\int_{V_{\epsilon,t'_0}}\rho'_{\epsilon,t'_0}d\overline{x}'+\int_{B(\overline{0},\epsilon)}\gamma_v c(\epsilon)\frac{1}{\gamma_v}d\overline{x}$

$=\int_{B(\overline{0},\epsilon)}c(\epsilon)d\overline{x}$

$=\frac{4c(\epsilon)\pi\epsilon^3}{3}$

so with the choice of $c(\epsilon)$ to be $\frac{3}{4\pi\epsilon^3}$, we have that;

$\int_{V_{\epsilon,t'_0}}\rho'_{\epsilon,1,t'_0}d\overline{x}'=1$

Letting $\epsilon\rightarrow 0$ and using Lemma 0.3, we have that;

$lim_{\epsilon\rightarrow 0}\rho_{\epsilon,1,t'}=D_{\overline{w}(t')}$

in the sense of distributions, for $t'\in(0,t_0)$.

□

Lemma 0.22. *We have that, for $\overline{x}\in\mathcal{R}^3$, $\overline{x}\neq\overline{0}$;*

$\int_{S(\overline{0},k)}e^{i\overline{k}\centerdot\overline{x}}d\overline{k}=4\pi k\frac{sin(k|\overline{x}|)}{|\overline{x}|}$

so that, for $\epsilon>0$, $\overline{x}\in B(\overline{0},\epsilon)$;

$\int_{S(\overline{0},\gamma(\epsilon))}\alpha'(\epsilon)e^{i\overline{k}\centerdot\overline{x}}d\overline{k}=\rho_\epsilon(\overline{x},0)$

where $\alpha'(\epsilon)=\frac{1}{4\pi\gamma(\epsilon)}$

We have that, for $\overline{x}\in\mathcal{R}^3$, $\overline{x}\neq\overline{0}$;

$\int_{S(\overline{0},k)}\overline{k}e^{i\overline{k}\centerdot\overline{x}}d\overline{k}=4\pi ik\hat{\overline{x}}(\frac{sin(k|\overline{x}|)-cos(k|\overline{x}|)k|\overline{x}|}{|\overline{x}|^2})$

so that, for $\epsilon>0$, $\overline{x}\in B(\overline{0},\epsilon)$;

$\int_{S(\overline{0},\gamma(\epsilon))}\alpha(\epsilon)\overline{k}e^{i\overline{k}\centerdot\overline{x}}d\overline{k}=\overline{J}_\epsilon(\overline{x},0)$

where $\alpha(\epsilon)=-\frac{c}{4\pi\gamma(\epsilon)^2}$

We have that $\alpha(\epsilon)=-\frac{c\alpha'(\epsilon)}{\gamma(\epsilon)}$, as claimed in the calculations of previous lemmas, in particular Lemma 0.12.

Proof. For the first claim, choose $g\in SO(3)$ with $g(0,0,s)=\overline{x}$, then;

$\int_{S(\overline{0},k)}e^{i\overline{k}\centerdot\overline{x}}d\overline{k}$

$=\int_{S(\overline{0},k)}e^{i\overline{k}\centerdot g(0,0,s)}d\overline{k}$

$=\int_{S(\overline{0},k)}e^{ig^{-1}(\overline{k})\centerdot(0,0,s)}d\overline{k}$

so that making the change of variables $\overline{k}'=g^{-1}(\overline{k})$, and, as $g\in SO(3)$, $d\overline{k}=d\overline{k}'$, we have that;

$\int_{S(\overline{0},k)}e^{i\overline{k}\centerdot\overline{x}}d\overline{k}$

$= \int_{S(\bar{0},k)} e^{i\bar{k}'\cdot(0,0,s)} d\bar{k}'$

$= \int_{S(\bar{0},k)} e^{ikscos(\theta)} d\bar{k}'$

$= \int_0^{\pi} \int_{-\pi}^{\pi} e^{ikscos(\theta)} k^2 sin(\theta) d\theta d\phi$

$= 2\pi k^2 \int_0^{\pi} e^{ikscos(\theta)} sin(\theta) d\theta$

so that making the change of variables $u = cos(\theta)$, $du = -sin(\theta)d\theta$;

$\int_{S(\bar{0},k)} e^{i\bar{k}\cdot\bar{x}} d\bar{k}$

$= 2\pi k^2 \int_1^{-1} -e^{iksu} du$

$= 2\pi k^2 \int_{-1}^{1} e^{iksu} du$

$= 2\pi k^2 [\frac{e^{iksu}}{iks}]_{-1}^{1}$

$= 2\pi k^2 \frac{2isin(ks)}{iks}$

$= 4\pi k \frac{sin(ks)}{s}$

The second claim is clear from the definition of ρ_ϵ in Lemma 0.21.

For the third claim, working in polars, let $\bar{x} = (ssin(\theta_0)cos(\phi_0), ssin(\theta_0)sin(\phi_0), scos(\theta_0))$, $s > 0$, $0 \leq \theta_0 \leq \pi$, $0 \leq \phi_0 < 2\pi$ and choose $g \in SO(3)$ with $g(0,0,s) = \bar{x}$, then;

$\int_{S(\bar{0},k)} \bar{k} e^{i\bar{k}\cdot\bar{x}} d\bar{k}$

$= \int_{S(\bar{0},k)} \bar{k} e^{i\bar{k}\cdot g(0,0,s)} d\bar{k}$

$= \int_{S(\bar{0},k)} \bar{k} e^{ig^{-1}(\bar{k})\cdot(0,0,s)} d\bar{k}$

so that making the change of variables $\bar{k}' = g^{-1}(\bar{k})$, $\bar{k} = g(\bar{k}')$, and, as $g \in SO(3)$, $d\bar{k} = d\bar{k}'$, we have that;

$\int_{S(\bar{0},k)} \bar{k} e^{i\bar{k}\cdot\bar{x}} d\bar{k}$

$= \int_{S(\bar{0},k)} g(\bar{k}') e^{i\bar{k}'\cdot(0,0,s)} d\bar{k}'$

$= \int_{S(\bar{0},k)}(g_{11}k'_1 + g_{12}k'_2 + g_{13}k'_3, g_{21}k'_1 + g_{22}k'_2 + g_{23}k'_3, g_{31}k'_1 + g_{32}k'_2$

$+g_{33}k'_3)e^{ikscos(\theta)}d\bar{k}'$

$= \int_0^\pi \int_{-\pi}^\pi (g_{11}k'_1 + g_{12}k'_2 + g_{13}k'_3, g_{21}k'_1 + g_{22}k'_2 + g_{23}k'_3, g_{31}k'_1 + g_{32}k'_2$

$+g_{33}k'_3)k^2 sin(\theta)e^{ikscos(\theta)}d\theta d\phi$

so that, using the fact that $\int_{-\pi}^\pi cos(\phi)d\phi = 0$, and so;

$\int_0^\pi \int_{-\pi}^\pi k'_1 k^2 sin(\theta)e^{ikscos(\theta)}d\theta d\phi$

$= \int_0^\pi \int_{-\pi}^\pi k'_2 k^2 sin(\theta)e^{ikscos(\theta)}d\theta d\phi$

$= 0$

we have, with $k'_3 = kcos(\theta)$;

$\int_{S(\bar{0},k)} \bar{k}e^{i\bar{k}.\bar{x}}d\bar{k}$

$= \int_0^\pi \int_{-\pi}^\pi (g_{13}k'_3, g_{23}k'_3, g_{33}k'_3)k^2 sin(\theta)e^{ikscos(\theta)}d\theta d\phi$

$= \hat{\bar{x}} \int_0^\pi \int_{-\pi}^\pi k'_3 k^2 sin(\theta)e^{ikscos(\theta)}d\theta d\phi$

$= \hat{\bar{x}} \int_0^\pi \int_{-\pi}^\pi k^3 sin(\theta)cos(\theta)e^{ikscos(\theta)}d\theta d\phi$

$= 2\pi k^3 \hat{\bar{x}} \int_0^\pi sin(\theta)cos(\theta)e^{ikscos(\theta)}d\theta$

so that making the change of variables $u = cos(\theta)$, $du = -sin(\theta)d\theta$;

$\int_{S(\bar{0},k)} \bar{k}e^{i\bar{k}.\bar{x}}d\bar{k}$

$= 2\pi k^3 \hat{\bar{x}} \int_1^{-1} -ue^{iksu}du$

$= 2\pi k^3 \hat{\bar{x}} \int_{-1}^1 ue^{iksu}du$

$= 2\pi k^3 \hat{\bar{x}}([\frac{e^{iksu}u}{iks}]_{-1}^1 - \int_{-1}^1 \frac{e^{iksu}}{iks}du)$

$= 2\pi k^3 \hat{\bar{x}}(\frac{e^{iks}+e^{-iks}}{iks} - [\frac{e^{iksu}}{-k^2s^2}]_{-1}^1)$

$= 2\pi k^3 \hat{\bar{x}}(\frac{2cos(ks)}{iks} + \frac{2isin(ks)}{k^2s^2})$

$= 2\pi i k^3 \hat{\overline{x}}(-\frac{2cos(ks)}{ks} + \frac{2sin(ks)}{k^2s^2})$

$= 2\pi i k \hat{\overline{x}}(-\frac{2cos(ks)k}{s} + \frac{2sin(ks)}{s^2})$

$= 4\pi i k \hat{\overline{x}}(\frac{sin(ks)-cos(ks)ks}{s^2})$

The fourth claim is clear from the definition of $\overline{J}_\epsilon$ in Lemma 0.21. The final claim is clear.

□

Lemma 0.23. *Given a straight line path $\overline{w} : [0, t_0] \rightarrow \mathcal{R}^3$, with $\overline{w}'(s) = (v_1, v_2, v_3)$, $0 \leq |\overline{v}| < c$, $0 \leq t' \leq t_0$, there exist, for $\epsilon > 0$, open subsets $V''_{\epsilon,t'} \subset \mathcal{R}^3$, centred at $\overline{w}(t')$, and $\rho''_{\epsilon,1}$ a charge distribution supported on the compact sets $\overline{V}''_{\epsilon,t'}$, such that $\square^2(\rho''_{\epsilon,1}) = 0$ on $V''_{\epsilon,t'}$ and $lim_{\epsilon\rightarrow 0}\rho''_{\epsilon,1,t'} = D_{\overline{w}(t')}$ in the sense of distributions, for $t' \in (0, t_0)$.*

Proof. This is an easy generalisation of Lemma 0.21. Using the notation there, we define ρ''_ϵ to be the transformation of the original $(\rho_\epsilon, \overline{J}_\epsilon)$ on $B(\overline{0}, \epsilon)$ in S to S'', moving with velocity vector $-\overline{v}$ relative to S. Choose $g \in SO(3)$ with $g(\overline{v}) = (v, 0, 0)$. Then, by a result in [10], and using the fact that ρ_ϵ is invariant under the action of g, we have that;

$R_g B_{-\overline{v}}(\rho_\epsilon) = B_{g(-\overline{v})} R_g(\rho_\epsilon) = B_{-(v,0,0)}(\rho_\epsilon)$

so that;

$B_{-\overline{v}}(\rho_\epsilon) = R_{g^{-1}} B_{-(v,0,0)}(\rho_\epsilon) = Rg^{-1}(\rho'_\epsilon)$

with $g^{-1}(v, 0, 0) = \overline{v}$.

With this, the remaining results are straightforward to prove, the open sets $V''_{\epsilon,t'}$ are then just the rotations by g^{-1} of the original open ellipsoids in Lemma 0.21. In coordinates $\overline{x}'' = g^{-1}(\overline{x}')$, if the original ellipsoid is given by;

$\gamma_v^2(x' - vt'_0)^2 + y'^2 + z'^2 = \epsilon^2$

we have that;

$(x'-vt'_0,y',z')D(x'-vt'_0,y',z')^t=\epsilon^2$

iff

$(\overline{x}'-t'_0(v,0,0))D(\overline{x}'-t'_0(v,0,0))^t=\epsilon^2$

iff

$((\overline{x}'')g^t-(t'_0\overline{v})g^t)Dg((\overline{x}''-t'_0\overline{v})^t)=\epsilon^2$

iff

$((\overline{x}''-(t'_0\overline{v}))(g^{-1}Dg)((\overline{x}''-t'_0\overline{v})^t)=\epsilon^2$

which is an ellipsoid centred at $t'_0\overline{v}$ with coefficients determined by $g^{-1}Dg$. The Laplacian is rotation invariant so we have that $\square^2(\rho''_\epsilon)=0$. By the change of variables formula, we have that;

$\int_{W''_{\epsilon,t'_0}}\rho''_\epsilon dV=\int_{W'_{\epsilon,t'_0}}\rho'_\epsilon dV=0$

where $\{W''_{\epsilon,t'_0},W'_{\epsilon,t'_0}\}$ are the interiors of the two ellipsoids, so we have to renormalise by adding the same constant $\frac{3\gamma_v}{4\pi\epsilon^3}$ when defining $\rho''_{\epsilon,1}$ to obtain the distribution claim.

□

Definition 0.24. *We say that an electron moving with velocity $\overline{v}$ is represented by the corresponding $\rho_{\epsilon,\overline{v}}$ as described by Lemma 0.23.*

Lemma 0.25. *If an electron in S is represented by $\rho_{\epsilon,\overline{v}}$ in S, then if S' moves with velocity $\overline{w}$ relative to S, the velocity of the electron in S' given by $\overline{v}'$, where $B_{\overline{w}}(\rho_{\epsilon,\overline{v}})=\rho_{\epsilon,\overline{v}'}$ is the same as the velocity obtained from the velocity transformation rule. The same result applies for rotations.*

Proof. The result for rotations is easy. We have that;

$\rho_{\epsilon,\overline{v}}=B_{-\overline{v}}(\rho_\epsilon)$

so that;

$R_g(\rho_{\epsilon,\overline{v}}) = R_g B_{-\overline{v}}(\rho_\epsilon) = B_{-g(\overline{v})} R_g(\rho_\epsilon) = B_{-g(\overline{v})}(\rho_\epsilon) = \rho_{\epsilon,g(\overline{v})}$

and clearly $g(\overline{v})$ is the velocity of the electron in the rotated frame.

By results in [10], we have that, for some $g \in SO(3)$;

$B_{\overline{w}}(\rho_{\epsilon,\overline{v}}) = B_{\overline{w}} B_{-\overline{v}}(\rho_\epsilon) = B_{\overline{w}*(-\overline{v})} R_g(\rho_\epsilon) = B_{\overline{w}*(-\overline{v})}(\rho_\epsilon) = \rho_{\epsilon,-(\overline{w}*(-\overline{v}))}$

By the definition in [10], using the identity;

$\overline{w} \times (\overline{w} \times \overline{v}) = \overline{w}(\overline{w} \centerdot \overline{v}) - \overline{v}(\overline{w} \centerdot \overline{w})$

we have that;

$-(\overline{w} * (-\overline{v}))$

$= -[\frac{\overline{w}-\overline{v}}{1-\frac{\overline{w}\centerdot\overline{v}}{c^2}} + \frac{\gamma_w}{c^2(\gamma_w+1)} \frac{\overline{w}\times(\overline{w}\times(-\overline{v}))}{1-\frac{\overline{w}\centerdot\overline{v}}{c^2}}]$

$= \frac{1}{1-\frac{\overline{w}\centerdot\overline{v}}{c^2}} [\overline{v} - \overline{w} - \frac{\gamma_w}{c^2(\gamma_w+1)} (\overline{w} \times (\overline{w} \times (-\overline{v})))]$

$= \frac{1}{1-\frac{\overline{w}\centerdot\overline{v}}{c^2}} [\overline{v} - \overline{w} + \frac{\gamma_w}{c^2(\gamma_w+1)} \overline{w}(\overline{w} \centerdot \overline{v}) - \frac{\gamma_w}{c^2(\gamma_w+1)} \overline{v}(\overline{w} \centerdot \overline{w})]$

$= \frac{1}{1-\frac{\overline{w}\centerdot\overline{v}}{c^2}} [\frac{\overline{v}}{\gamma_w} - \overline{w} + \frac{\gamma_w}{c^2(\gamma_w+1)} \overline{w}(\overline{w} \centerdot \overline{v})]$

as;

$1 - \frac{w^2}{c^2} \frac{\gamma_w}{\gamma_w+1} = \frac{1}{\gamma_w}$

which is the formula for the transformation of $\overline{v}$ in the frame $S_{\overline{w}}$, given in [2].

□

Lemma 0.26. *In the notation of Lemma 0.21, the spectral realisation of ρ'_ϵ is given by;*

$\int_{Ell(k')} \alpha(k') \frac{k'}{k} (\overline{v}) e^{i\overline{k}' \centerdot \overline{x}'} e^{ik'ct'} dEll(k')$

where $Ell(k')$ is the boundary of the ellipsoid corresponding to the sphere under the change of variables given by the Doppler shift with velocity vector $\overline{v}$, where $\overline{v} = (v, 0, 0)$.

Proof. We have from Lemma 0.22, by a slight abuse of the notation in Lemma 0.21, that;

$\rho_\epsilon(\overline{x},t)=\int_{S(\overline{0},k)}\alpha'(k)e^{i\overline{k}.\overline{x}}e^{ikct}dS(k)$

$\overline{J}_\epsilon(\overline{x},t)=\int_{S(\overline{0},k)}\alpha(k)\overline{k}e^{i\overline{k}.\overline{x}}e^{ikct}dS(k)$

with $\alpha'(k)=\frac{1}{4\pi k}$, $\alpha(k)=-\frac{c\alpha'(k)}{k}$ and $k=\gamma(\epsilon)$.

It follows, with;

$t'=\gamma_v(t+\frac{vx}{c^2})$, $x'=\gamma_v(x+vt)$, $y'=y$, $z'=z$

$t=\gamma_v(t-\frac{vx'}{c^2})$, $x=\gamma_v(x'-vt)$

and;

$\rho'_\epsilon=\gamma_v(\rho_\epsilon+\frac{vj_{1,\epsilon}}{c^2})$

that;

$\rho'_\epsilon=\gamma_v\int_{S(\overline{0},k)}[\alpha'(k)+\frac{v\alpha(k)k_1}{c^2}]e^{i\overline{k}.\overline{x}}e^{ikct}dS(k)$

$=\gamma_v\int_{S(\overline{0},k)}[\alpha'(k)+\frac{v(-\frac{c\alpha'(k)}{k})k_1}{c^2}]e^{ik_1\gamma_v(x'-vt)}e^{ik_2y}e^{ik_3z}e^{ikc\gamma_v(t'-\frac{vx'}{c^2})}dS(k)$

$=\gamma_v\int_{S(\overline{0},k)}[1-\frac{vk_1}{ck}]\alpha'(k)e^{i(k_1\gamma_v-\frac{k\gamma_v v}{c})x'}e^{ik_2y'}e^{ik_3z'}e^{i(kc\gamma_v-k_1\gamma_v v)t'}dS(k)$

so that, making the substitutions;

$k'_1=k_1\gamma_v-\frac{k\gamma_v v}{c}$, $k'_2=k_2$, $k'_3=k_3$, $k'=k\gamma_v-\frac{k_1\gamma_v v}{c}$

$dS(k)=(\gamma_v+\frac{\gamma_v k'_1 v}{k'c})dEll(k')$

where $Ell(k')$ is the boundary of the ellipsoid corresponding to the sphere under the change of variables, we obtain

$\rho'_\epsilon=\int_{Ell(k')}\gamma_v(1-\frac{vk_1}{ck})\alpha'(k)e^{i\overline{k}'.\overline{x}'}e^{ik'ct'}(\gamma_v+\frac{\gamma_v k'_1 v}{k'c})dEll(k')$

$=\int_{Ell(k')}\gamma_v(1-\frac{v(k'_1\gamma_v+\frac{k'\gamma_v v}{c})}{c(k'\gamma_v+\frac{k'_1\gamma_v v}{c})})\alpha'(k'\gamma_v+\frac{k'_1\gamma_v v}{c})e^{i\overline{k}'.\overline{x}'}e^{ik'ct'}(\gamma_v+\frac{\gamma_v k'_1 v}{k'c})dEll(k')$

$= \int_{Ell(k')} \gamma_v (1 - \frac{v(k'_1\gamma_v + \frac{k'\gamma_v v}{c})}{c(k'\gamma_v + \frac{k'_1\gamma_v v}{c})}) e^{i\overline{k}' \cdot \overline{x}'} e^{ik'ct'} \frac{(\gamma_v + \frac{\gamma_v k'_1 v}{k'c})}{4\pi(k'\gamma_v + \frac{k'_1\gamma_v v}{c})} dEll(k')$

$= \int_{Ell(k')} \gamma_v^2 \frac{(ck'\gamma_v + k'_1\gamma_v v - k'_1\gamma_v v - \frac{k'\gamma_v v^2}{c})}{ck'\gamma_v + k'_1\gamma_v v} \frac{(1+\frac{k'_1 v}{k'c})}{4\pi(k'\gamma_v + \frac{k'_1\gamma_v v}{c})} e^{i\overline{k}' \cdot \overline{x}'} e^{ik'ct'} dEll(k')$

$= \int_{Ell(k')} \gamma_v^2 \frac{ck'(1-\frac{v^2}{c^2})}{ck' + k'_1 v} \frac{(1+\frac{k'_1 v}{k'c})}{4\pi(k'\gamma_v + \frac{k'_1\gamma_v v}{c})} e^{i\overline{k}' \cdot \overline{x}'} e^{ik'ct'} dEll(k')$

$= \int_{Ell(k')} \frac{ck'}{ck'(1+\frac{k'_1 v}{k'c})} \frac{(1+\frac{k'_1 v}{k'c})}{4\pi(k'\gamma_v + \frac{k'_1\gamma_v v}{c})} e^{i\overline{k}' \cdot \overline{x}'} e^{ik'ct'} dEll(k')$

$= \int_{Ell(k')} \frac{1}{(1+\frac{k'_1 v}{k'c})} \frac{(1+\frac{k'_1 v}{k'c})}{4\pi(k'\gamma_v + \frac{k'_1\gamma_v v}{c})} e^{i\overline{k}' \cdot \overline{x}'} e^{ik'ct'} dEll(k')$

$= \int_{Ell(k')} \frac{1}{4\pi(k'\gamma_v + \frac{k'_1\gamma_v v}{c})} e^{i\overline{k}' \cdot \overline{x}'} e^{ik'ct'} dEll(k')$

$= \int_{Ell(k')} \alpha(k') \frac{k'}{k}(\overline{v}) e^{i\overline{k}' \cdot \overline{x}'} e^{ik'ct'} dEll(k')$

□

Lemma 0.27. *For arbitrary $\overline{v}$, $v = |\overline{v}|$, the spectral realisation of $\rho_{\epsilon,\overline{v}}$ is given by;*

$\frac{3\gamma_v}{4\pi\epsilon^3} + \int_{Ell(k')} \alpha(k') \frac{k'}{k}(\overline{v}) e^{i\overline{k}' \cdot \overline{x}'} e^{ik'ct'} dEll(k')$

where $Ell(k')$ is the boundary of the ellipsoid corresponding to the sphere under the change of variables given by the Doppler shift with velocity vector $\overline{v}$.

Proof. Choose $g \in SO(3)$ with $g(\overline{v}) = (v, 0, 0)$, then;

$R_g B_{-\overline{v}} = B_{-(v,0,0)} R_g$

and;

$\rho_{\epsilon,\overline{v}} = B_{-\overline{v}}(\frac{3}{4\pi\epsilon^3} + \rho_\epsilon)$

$= R_{g^{-1}} B_{-(v,0,0)} R_g(\frac{3}{4\pi\epsilon^3} + \rho_\epsilon)$

$= R_{g^{-1}} B_{-(v,0,0)}(\frac{3}{4\pi\epsilon^3} + \rho_\epsilon)$

$= R_{g^{-1}}(\rho_{\epsilon,(v,0,0)})$

By Lemma 0.26, we have that;

$\rho_{\epsilon,(v,0,0)}(\overline{x}',t')=\frac{3\gamma_v}{4\pi\epsilon^3}+\int_{Ell(k')}\alpha(k')\frac{k'}{k}(v,0,0)e^{i\overline{k}'\centerdot\overline{x}'}e^{ik'ct'}dEll(k')$

where $Ell(k')$ is the boundary of the ellipsoid corresponding to the sphere under the change of variables given by the Doppler shift with velocity vector $(v,0,0)$.

so that with $\overline{x}''=g^{-1}(\overline{x}')$, $\overline{x}'=g(\overline{x}'')$

$\rho_{\epsilon,\overline{v}}(\overline{x}'',t'')=\frac{3\gamma_v}{4\pi\epsilon^3}+\int_{Ell(k')}\alpha(k')\frac{k'}{k}(v,0,0)e^{i\overline{k}'\centerdot\overline{x}'}e^{ik'ct'}dEll(k')$

$=\frac{3\gamma_v}{4\pi\epsilon^3}+\int_{Ell(k')}\alpha(k')\frac{k'}{k}(v,0,0)e^{i\overline{k}'\centerdot g(\overline{x}'')}e^{ik'ct''}dEll(k')$

$=\frac{3\gamma_v}{4\pi\epsilon^3}+\int_{Ell(k')}\alpha(k')\frac{k'}{k}(v,0,0)e^{ig^{-1}(\overline{k}')\centerdot\overline{x}''}e^{ik'ct''}dEll(k')$

so that making the change of variables $\overline{k}''=g^{-1}(\overline{k}')$, $k''=k'$, we have, with $R_{g^{-1}}(v,0,0)=\overline{v}$;

$\rho_{\epsilon,\overline{v}}(\overline{x}'',t'')=\frac{3\gamma_v}{4\pi\epsilon^3}+\int_{R_{g^{-1}}(Ell(k'))}\alpha(k'')\frac{k''}{k}(R_{g^{-1}}(v,0,0))e^{i\overline{k}''\centerdot\overline{x}''}e^{ik''ct''}dEll(k'')$

$=\frac{3\gamma_v}{4\pi\epsilon^3}+\int_{Ell(k'')}\alpha(k'')\frac{k''}{k}(\overline{v}R_g^{-1})e^{i\overline{k}''\centerdot\overline{x}''}e^{ik''ct''}dEll(k'')$

where $Ell(k'')$ is the boundary of the ellipsoid corresponding to the sphere under the change of variables given by the Doppler shift with velocity vector $\overline{v}$. To remove the factor R_g^{-1} in the Doppler shift, we apply the sequence of Doppler shifts $(B_{\overline{v}}R_gB_{-\overline{v}})$, so that;

$(B_{\overline{v}}R_gB_{-\overline{v}})B_{\overline{v}}R_{g^{-1}}=B_{\overline{v}}$

Reversing the directions, this corresponds to a change of variables, given by;

$(B_{-\overline{v}}R_{g^{-1}}B_{\overline{v}})$

We need to check this change fixes $\rho_{\epsilon,\overline{v}}$. We have that, as ρ_ϵ is invariant by rotations, that;

$(B_{-\overline{v}}R_{g^{-1}}B_{\overline{v}})\rho_{\epsilon,\overline{v}}$

$= (B_{-\overline{v}}R_{g^{-1}}B_{\overline{v}})B_{-\overline{v}}(\frac{3}{4\pi\epsilon^3} + \rho_\epsilon)$

$= B_{-\overline{v}}R_{g^{-1}}(\frac{3}{4\pi\epsilon^3} + \rho_\epsilon)$

$= B_{-\overline{v}}(\frac{3}{4\pi\epsilon^3} + \rho_\epsilon)$

$= \rho_{\epsilon,\overline{v}}$

so we obtain the result.

□

Lemma 0.28. *If an electron in S is represented by $\rho_{\epsilon,\overline{v}}$, then if S' moves with velocity $\overline{w}$ relative to S, the Doppler shift in the spectral realisation of the electron in S' is given by $-\overline{w}$ and the renormalisation factor between frames is given by $\frac{\gamma_{\overline{v}'}}{\gamma_{\overline{v}}}$ where $\overline{v}'$ is the velocity of the electron in S'.*

Proof. By Lemmas 0.25 and 0.27, the renormalisation factor of the electron in S' is given by $\frac{3\gamma_{\overline{v}'}}{4\pi\epsilon^3}$ and by $\frac{3\gamma_{\overline{v}}}{4\pi\epsilon^3}$ in S, so this involves a shift of $\frac{\gamma_{\overline{v}'}}{\gamma_{\overline{v}}}$ between frames. By Lemmas 0.25 and 0.27 again, we have a Doppler shift determined by $\overline{v}$ in the spectral realisation S and a Doppler shift determined by $\overline{v}'$ in the spectral realisation S'. The Doppler shift between frames is given by;

$B_{\overline{v}'}B_{\overline{v}}^{-1} = B_{\overline{v}'}B_{-\overline{v}} = B_{-\overline{w}*\overline{v}}B_{-\overline{v}} = B_{-\overline{w}}B_{\overline{v}}R_gB_{-\overline{v}}$

where $g \in SO(3)$ is a Thomas rotation. This time we apply the Doppler shift;

$(B_{-\overline{w}}B_{\overline{v}}R_{g^{-1}}B_{-\overline{v}}B_{\overline{w}})$

to the spectral realisation of $\rho_{\epsilon,\overline{v}'}$, so that;

$(B_{-\overline{w}}B_{\overline{v}}R_{g^{-1}}B_{-\overline{v}}B_{\overline{w}})B_{-\overline{w}}B_{\overline{v}}R_gB_{-\overline{v}} = B_{-\overline{w}}$

This corresponds to a change of variables;

$(B_{\overline{w}}B_{-\overline{v}}R_gB_{\overline{v}}B_{-\overline{w}})$

and we need to check this change fixes $\rho_{\epsilon,\overline{v}'}$. We have that, using the invariance of ρ_ϵ by rotations;

$(B_{\overline{w}}B_{-\overline{v}}R_gB_{\overline{v}}B_{-\overline{w}})(\rho_{\epsilon,\overline{v}'})$

$= (B_{\overline{w}}B_{-\overline{v}}R_gB_{\overline{v}}B_{-\overline{w}})(B_{-(-\overline{w}*\overline{v})})(\frac{3}{4\pi\epsilon^3}+\rho_\epsilon)$

$= (B_{\overline{w}}B_{-\overline{v}}R_gB_{\overline{v}}B_{-\overline{w}})(B_{\overline{w}*-\overline{v}})(\frac{3}{4\pi\epsilon^3}+\rho_\epsilon)$

$= (B_{\overline{w}}B_{-\overline{v}}R_gB_{\overline{v}}B_{-\overline{w}})(B_{\overline{w}}B_{-\overline{v}}R_h)(\frac{3}{4\pi\epsilon^3}+\rho_\epsilon)$

$= B_{\overline{w}}B_{-\overline{v}}(\frac{3}{4\pi\epsilon^3}+\rho_\epsilon)$

$= B_{-(-\overline{w}*\overline{v})}(\frac{3}{4\pi\epsilon^3}+\rho_\epsilon)$

$= \rho_{\epsilon,\overline{v}'}$

as required.

$\square$

Remarks 0.29. *This is the same result for the Doppler shift as in Lemma 0.15, with no charge defect and a matching velocity transformation given by Lemma 0.25. We used a backward derivative in Lemma 0.15, so if we redo the calculation with the forward derivative, we obtain for the velocity transformation of* $\overline{v}$ *in the frame* $S_{\overline{w}}$;

$$-(-\overline{v})' = -(-\overline{w}*-\overline{v}) = \overline{w}*\overline{v} = -(-\overline{w})*\overline{v}$$

which is the velocity transformation of $\overline{v}$ *in the frame* $S_{-\overline{w}}$.

In both representations, charge is conserved when moving to a new inertial frame, see Lemma 0.30. As the wave representation in Lemma 0.15 cannot move with the electron, while this representation can, there is a suggestion that the electron behaves like a martingale in a new inertial frame, with a splitting of velocities to $\{\overline{v}^{tr(\overline{w})}, \overline{v}^{tr(-\overline{w})}\}$ *and a splitting of charge* $q = q_1 + q_2$. *The free electron must approximate curved paths with straight line paths in the base frame* S, *to avoid losing energy in the Poynting vector. This leads to considerations of position and the behaviour of charge/momentum along paths in* S'. *We can use a martingale to define position;*

$\sum_i \frac{(\overline{v}_i^{tr(\overline{w})}+\overline{v}_i^{tr(-\overline{w})})}{2}+\omega_i \frac{(\overline{v}_i^{tr(\overline{w})}-\overline{v}_i^{tr(-\overline{w})})}{2}$

where ω_i is a random walk and the $\overline{v}_i$ are given.

If we use the splitting $q=\frac{q}{2}+\frac{q}{2}$ for charge, we obtain behaviour similar to the diffusion equation, see [6].

Lemma 0.30. *Let an electron with charge q be stationary in the base frame S, then if S' moves with velocity $-(v,0,0)$ relative to S, the same point charge q is observed in S' instantaneously at time $t'=0$, moving with velocity $-(v,0,0)$ for the representation at the beginning of this paper.*

Proof. For the velocity claim, we can use remark 0.29, to obtain;

$-(-\overline{0})'=-(--(v,0,0)*-\overline{0})=-(v,0,0)$

For the charge claim, we have that in S, for $n\geq 4$;

$\rho_{n,\epsilon}(\overline{x},t)=\frac{1}{(2\pi)^3}\int_{\overline{k}\in\mathcal{R}^3}[A_{n,\epsilon}e^{i\overline{k}.\overline{x}}e^{ikct}+B_{n,\epsilon}e^{i\overline{k}.\overline{x}}e^{-ikct}]d\overline{k}$

$\overline{J}_{n,\epsilon}(\overline{x},t)=\frac{1}{(2\pi)^3}\int_{\overline{k}\in\mathcal{R}^3}[\overline{A}_{n,\epsilon}e^{i\overline{k}.\overline{x}}e^{ikct}+\overline{B}_{n,\epsilon}e^{i\overline{k}.\overline{x}}e^{-ikct}]d\overline{k}$

where;

$lim_{\epsilon\rightarrow 0}A_{n,\epsilon}=\frac{q}{2}$

$lim_{\epsilon\rightarrow 0}B_{n,\epsilon}=\frac{q}{2}$

$lim_{\epsilon\rightarrow 0}\overline{A}_{n,\epsilon}=-\frac{cq}{2k}\overline{k}$

$lim_{\epsilon\rightarrow 0}\overline{B}_{n,\epsilon}=\frac{cq}{2k}\overline{k}$

Then $\rho'_{n,\epsilon}=\gamma_v[\rho_{n,\epsilon}+\frac{\gamma_v v j_{1,n,\epsilon}}{c^2}]$

and, using the construction of Lemma 0.21;

$\int_{V'_{n,\epsilon}}\rho'_{n,\epsilon}(\overline{x}',0)d\overline{x}'$

$=\int_{B(\overline{0},\epsilon)}[\gamma_v[\rho_{n,\epsilon}+\frac{v j_{1,n,\epsilon}}{c^2}]]_{t=-\frac{vx}{c^2}}\frac{d\overline{x}}{\gamma_v}$

$= \int_{B(\bar{0},\epsilon)}[\rho_{n,\epsilon} + \frac{vj_{1,n,\epsilon}}{c^2}]_{t=-\frac{vx}{c^2}} d\bar{x}$

$= \frac{1}{(2\pi)^3}\int_{B(\bar{0},\epsilon)}[(\int_{\bar{k}\in\mathcal{R}^3}[A_{n,\epsilon}e^{i\bar{k}.\bar{x}}e^{ikct} + B_{n,\epsilon}e^{i\bar{k}.\bar{x}}e^{-ikct}]d\bar{k})$

$+\frac{v}{c^2}(\int_{\bar{k}\in\mathcal{R}^3}[(\overline{A}_{n,\epsilon})_1e^{i\bar{k}.\bar{x}}e^{ikct} + (\overline{B}_{n,\epsilon})_1e^{i\bar{k}.\bar{x}}e^{-ikct}]d\bar{k})]_{t=-\frac{vx}{c^2}} d\bar{x}$

so that, making the substitutions;

$k_1' = k_1 - \frac{vk}{c}$, $d\bar{k}' = (1 - \frac{vk_1}{kc})d\bar{k}$

$k_1' = k_1 + \frac{vk}{c}$, $d\bar{k}' = (1 + \frac{vk_1}{kc})d\bar{k}$

for a test function g, with corresponding h, using the inversion theorem;

$lim_{\epsilon\rightarrow 0}\int_{V'_{n,\epsilon}}\rho'_{n,\epsilon}(\bar{x}',0)g(\bar{x}')d\bar{x}'$

$= \frac{1}{(2\pi)^3}\int_{\bar{x}\in\mathcal{R}^3}\int_{\bar{k}\in\mathcal{R}^3}[(\frac{q}{2}-\frac{v}{c^2}\frac{cqk_1}{2k})e^{i\bar{k}.\bar{x}}e^{ikc(-\frac{vx}{c^2})}+(\frac{q}{2}+\frac{v}{c^2}\frac{cqk_1}{2k})e^{i\bar{k}.\bar{x}}e^{-ikc(-\frac{vx}{c^2})}]d\bar{k}h(\bar{x})d\bar{x}$

$= \frac{1}{(2\pi)^3}\int_{\bar{x}\in\mathcal{R}^3}\int_{\bar{k}\in\mathcal{R}^3}[(\frac{q}{2}-\frac{vqk_1}{2kc})e^{i(k_1-\frac{vk}{c})x}e^{ik_2y}e^{ik_3z}+(\frac{q}{2}+\frac{vqk_1}{2kc})e^{i(k_1+\frac{vk}{c})x}e^{ik_2y}e^{ik_3z}]d\bar{k}h(\bar{x})d\bar{x}$

$= \frac{1}{(2\pi)^3}\int_{\bar{x}\in\mathcal{R}^3}[(\int_{\bar{k}'\in\mathcal{R}^3}[(\frac{q}{2}-\frac{vqk_1}{2kc})e^{ik_1'x}e^{ik_2'y}e^{ik_3'z}\frac{d\bar{k}'}{(1-\frac{vk_1}{kc})}+(\frac{q}{2}+\frac{vqk_1}{2kc})e^{ik_1'x}e^{ik_2'y}e^{ik_3'z}\frac{d\bar{k}'}{(1+\frac{vk_1}{kc})}]h(\bar{x})d\bar{x}$

$= \frac{1}{(2\pi)^3}\int_{\bar{x}\in\mathcal{R}^3}[(\int_{\bar{k}'\in\mathcal{R}^3}[qe^{ik_1'x}e^{ik_2'y}e^{ik_3'z}d\bar{k}'h(\bar{x})d\bar{x}$

$= \frac{1}{(2\pi)^3}\int_{\bar{k}'\in\mathcal{R}^3}q\mathcal{F}^{-1}(h)(\bar{k}')d\bar{k}'$

$= qh(\bar{0})$

$= qg(\bar{0})$

$\square$

Lemma 0.31. *The reverse velocity transformation* $(\bar{v})!$ *is self dual in the sense that* $(\bar{v})!! = \bar{v}$ *relative to the frame* $S_{\bar{w}}$. *The usual velocity transformation* $(\bar{v})'$ *is also self dual.*

Proof. For the first claim, we have, by Remark 0.29, that;

$(\bar{v})^{!(\bar{w})} = \bar{w} * \bar{v}$

for the reverse velocity transformation to the frame $S_{\overline{w}}$. Noting now that S travels relative to $S_{\overline{w}}$ with velocity $-\overline{w}$, we have that;

$$((\overline{v})^{!(\overline{w})})^{!(-\overline{w})}$$

$$= (\overline{w} * \overline{v})^{!(-\overline{w})}$$

$$= -(-(\overline{w} * \overline{v}))'^{(-\overline{w})}$$

$$= -(\overline{w} * (-\overline{w} * -\overline{v}))$$

$$= -\overline{w} * (\overline{w} * \overline{v})$$

$$= (-\overline{w} * \overline{w}) * Tom(-\overline{w}, \overline{w})\overline{v}$$

$$= \overline{0} * \overline{v}$$

$$= \overline{v}$$

where Tom is the Thomas rotation, see [13]. For the second claim;

$$((\overline{v})'^{(\overline{w})})'^{(-\overline{w})}$$

$$= (-\overline{w} * \overline{v})'^{(-\overline{w})}$$

$$= \overline{w} * (-\overline{w} * \overline{v})$$

$$= (\overline{w} * -\overline{w}) * Tom(\overline{w}, -\overline{w})\overline{v}$$

$$= \overline{0} * \overline{v}$$

$$= \overline{v}$$

□

Remarks 0.32. *The above lemma shows that we can keep track of the reverse particle paths by using the inverse boost matrix $B_{-\overline{w}}$, with the paths splitting at the origin $(0, 0, 0, 0)$. We can label each branch of the particle tree by a word in the boosts $B_{\overline{w}}$ and $B_{-\overline{w}}$, together with the velocities from the path in the base frame S.*

Definition 0.33. *Let $\overline{v}:[0,1]\rightarrow\mathcal{R}^3$ be a trajectory in the base frame S, $|\overline{v}|<c$, let S' travel with velocity $\overline{w}$ relative to S. Let $\overline{v}_1=(\overline{v})'^{(\overline{w})}$ and $\overline{v}_2=(\overline{v})^{!(\overline{w})}$ be the transformed velocities in S'. Then, motivated by Remark 0.29, we define the position of the electron in S' as;*

$$\overline{P}(t',\overline{\omega})=\int_0^{t'}\frac{\overline{v}_1(s')+\overline{v}_2(s')}{2}ds'+\int_0^{t'}\frac{\overline{v}_1(s')-\overline{v}_2(s')}{2}dW_{s'}$$

where $W_{s'}$ is Brownian motion.

We define the velocity $\overline{V}(t',\overline{\omega})$ of the electron in S' as $\frac{d\overline{P}}{dt'}$. By the FTC and the definition of white noise $Z_{t'}$, we have that;

$$\overline{V}(t',\omega)=\frac{\overline{v}_1(t')+\overline{v}_2(t')}{2}+\frac{\overline{v}_1(t')-\overline{v}_2(t')}{2}\frac{dW_{t'}}{dt'}$$

$$=\frac{\overline{v}_1(t')+\overline{v}_2(t')}{2}+\frac{\overline{v}_1(t')-\overline{v}_2(t')}{2}Z_{t'}$$

We define the generalised velocity $\overline{U}(t',\overline{\omega})$ of the electron in S' as;

$$=\frac{\overline{v}_1(t')+\overline{v}_2(t')}{2}+\frac{\overline{v}_1(t')-\overline{v}_2(t')}{4}Z_{t'}$$

so that the generalised velocity is the average of the velocity and the drift velocity;

$$\frac{\overline{v}_1(t')+\overline{v}_2(t')}{2}$$

We define the stochastic charge density $\rho'_{n,\epsilon}$ in S' by;

$$\rho'_{n,\epsilon}(\overline{x}',t',\overline{\omega})=D_{n,\epsilon}(\overline{x}'-\overline{P}(t',\overline{\omega}))$$

Lemma 0.34. *Letting E be expectation, and dropping the primed notation, we have that;*

$$E(\overline{P}_t)=\int_0^t\frac{\overline{v}_1(s)+\overline{v}_2(s)}{2}ds$$

$$E(\overline{V}_t)=E(\overline{U}_t)=\frac{\overline{v}_1(t)+\overline{v}_2(t)}{2}$$

For $1\leq i\leq 3$, letting $\{f_i(s),1\leq i\leq 3\}$ and $\{g_i(s):1\leq i\leq 3\}$ be the components of;

$\frac{\overline{v}_1(s)+\overline{v}_2(s)}{2}$ *and* $\frac{\overline{v}_1(s)-\overline{v}_2(s)}{2}$

respectively, we have that, for $\tau > 0$*;*

$$Cov(\overline{P}_{t,i}\overline{P}_{t+\tau,i}) = \int_0^t g_i^2 ds$$

$$Cov(\overline{V}_{t,i}\overline{V}_{t+\tau,i}) = Cov(\overline{U}_{t,i}\overline{U}_{t+\tau,i}) = 0$$

Proof. The first claim is clear as the second component of position is a martingale and $E(Z_t) = 0$. For the second claim, we compute;

$$Cov(\overline{P}_{t,i}\overline{P}_{t+\tau,i})$$

$$= E(\overline{P}_{t,i}\overline{P}_{t+\tau,i}) - E(\overline{P}_{t,i})E(\overline{P}_{t+\tau,i})$$

$$= \int_0^t f_i(s)ds \int_0^{t+\tau} f_i(s)ds + \int_0^t f_i(s)dsE(\int_0^{t+\tau} g_i(s)dW_s)$$

$$+ \int_0^{t+\tau} f_i(s)dsE(\int_0^t g_i(s)dW_s) + E(\int_0^t g_i(s)dW_s \int_0^{t+\tau} g_i(s)dW_s)$$

$$- \int_0^t f_i(s)ds \int_0^{t+\tau} f_i(s)(s)ds$$

$$= E(\int_0^t g_i(s)dW_s \int_0^{t+\tau} g_i(s)dW_s)$$

$$= E((\int_0^t g_i(s)dW_s)^2)$$

$$= \int_0^t g_i^2 ds$$

by standard properties of martingales and using Ito's integral formula. We then have;

$$Cov(\overline{V}_{t,i}\overline{V}_{t+\tau,i})$$

$$= E(\overline{V}_{t,i}\overline{V}_{t+\tau,i}) - E(\overline{V}_{t,i})E(\overline{V}_{t+\tau,i})$$

$$= E((f_i(t) + g_i(t)Z(t))(f_i(t+\tau) + g_i(t+\tau)Z(t+\tau))) - f_i(t)f_i(t+\tau)$$

$$= f_i(t)f_i(t+\tau) + f_i(t)g_i(t+\tau)E(Z(t+\tau)) + f_i(t+\tau)g_i(t)E(Z(t))$$

$$+ g_i(t)g_i(t+\tau)E(Z(t)Z(t+\tau)) - f_i(t)f_i(t+\tau)$$

$$= f_i(t)g_i(t+\tau)E(Z(t+\tau)) + f_i(t+\tau)g_i(t)E(Z(t))$$

$+g_i(t)g_i(t+\tau)E(Z(t)Z(t+\tau))$

$=0$

by standard properties of white noise, see [3]. The corresponding proof for $\overline{U}_{t,i}$ is similar.

∎

Lemma 0.35. *Stochastic representation of charge*

For a test function h, we have that, dropping the primed notation for S', and for trajectories starting at $\overline{0}$;

$lim_{\epsilon\rightarrow 0}\int_{\overline{x}\in\mathcal{R}^3}\rho_{n,\epsilon}(\overline{x},t,\overline{\omega})h(\overline{x})d\overline{x}$

$=h(\overline{0})+\int_0^t(\sum_{i=1}^3\frac{\partial h}{\partial x_i}|_{\overline{P}(s,\overline{\omega})}f_i(s)+\frac{1}{2}\sum_{j,k=1}^3\frac{\partial^2 h}{\partial x_j x_k}|_{\overline{P}(s,\overline{\omega})}g_jg_k(s))ds$

$+\int_0^t(\sum_{i=1}^3\frac{\partial h}{\partial x_i}|_{\overline{P}(s,\overline{\omega})}g_i(s))dW_s$

where $\{f_i(s),1\leq i\leq 3\}$ and $\{g_i(s):1\leq i\leq 3\}$ are the components of;

$\frac{\overline{v}_1(s)+\overline{v}_2(s)}{2}$ *and* $\frac{\overline{v}_1(s)-\overline{v}_2(s)}{2}$

respectively.

Proof. We have that, by the definition of position, using the box calculus and Ito's theorem, see [12];

$d\overline{P}(t,\overline{\omega})=\overline{f}(t)dt+\overline{g}(t)dW_t$

$d\overline{P}_j(t,\overline{\omega})d\overline{P}_k(t,\overline{\omega})=g_jg_k(t)dt$

$d\rho_{n,\epsilon}(\overline{x},t,\overline{\omega})$

$=D_{n,\epsilon}(\overline{x}-(\overline{P}(t,\overline{\omega})+d\overline{P}(t,\overline{\omega})))-D_{n,\epsilon}(\overline{x}-\overline{P}(t,\overline{\omega}))$

$=-\sum_{i=1}^3\frac{\partial D_{n,\epsilon}}{\partial x_i}|_{\overline{x}-\overline{P}(t,\overline{\omega})}d\overline{P}_i(t,\overline{\omega})+\frac{1}{2}\sum_{j,k=1}^3\frac{\partial^2 D_{n,\epsilon}}{\partial x_j\partial x_k}|_{\overline{x}-\overline{P}(t,\overline{\omega})}d\overline{P}_j(t,\overline{\omega})d\overline{P}_k(t,\overline{\omega})$

$=-\sum_{i=1}^3\frac{\partial D_{n,\epsilon}}{\partial x_i}|_{\overline{x}-\overline{P}(t,\overline{\omega})}(f_i(t)dt+g_i(t)dW_t)+\frac{1}{2}\sum_{j,k=1}^3\frac{\partial^2 D_{n,\epsilon}}{\partial x_j\partial x_k}|_{\overline{x}-\overline{P}(t,\overline{\omega})}g_jg_k(t)dt$

so that;

$\rho_{n,\epsilon}(\overline{x},t,\overline{\omega})-\rho_{n,\epsilon}(\overline{x},0,\overline{\omega})$

$=\int_0^t d\rho_{n,\epsilon}(\overline{x},t,\overline{\omega})$

$=\int_0^t[-\sum_{i=1}^3 \frac{\partial D_{n,\epsilon}}{\partial x_i}|_{\overline{x}-\overline{P}(t,\overline{\omega})}(f_i(t)dt+g_i(t)dW_t)+\frac{1}{2}\sum_{j,k=1}^3 \frac{\partial^2 D_{n,\epsilon}}{\partial x_j\partial x_k}|_{\overline{x}-\overline{P}(t,\overline{\omega})}g_jg_k(t)]dt$

$=\int_0^t[-\sum_{i=1}^3 \frac{\partial D_{n,\epsilon}}{\partial x_i}|_{\overline{x}-\overline{P}(t,\overline{\omega})}f_i(t)+\frac{1}{2}\sum_{j,k=1}^3 \frac{\partial^2 D_{n,\epsilon}}{\partial x_j\partial x_k}|_{\overline{x}-\overline{P}(t,\overline{\omega})}g_jg_k(t)]dt$

$-\int_0^t\sum_{i=1}^3 \frac{\partial D_{n,\epsilon}}{\partial x_i}|_{\overline{x}-\overline{P}(t,\overline{\omega})}g_i(t)dW_t$

and using the properties of $D_{n,\epsilon}$, for a test function h, interchanging the orders of integration;

$lim_{\epsilon\rightarrow 0}\int_{\overline{x}\in\mathcal{R}^3}\rho_{n,\epsilon}(\overline{x},t,\overline{\omega})h(\overline{x})d\overline{x}$

$=lim_{\epsilon\rightarrow 0}\int_{\overline{x}\in\mathcal{R}^3}[\int_0^t[-\sum_{i=1}^3 \frac{\partial D_{n,\epsilon}}{\partial x_i}|_{\overline{x}-\overline{P}(t,\overline{\omega})}f_i(t)+\frac{1}{2}\sum_{j,k=1}^3 \frac{\partial^2 D_{n,\epsilon}}{\partial x_j\partial x_k}|_{\overline{x}-\overline{P}(t,\overline{\omega})}g_jg_k(t)]dt$

$-\int_0^t\sum_{i=1}^3 \frac{\partial D_{n,\epsilon}}{\partial x_i}|_{\overline{x}-\overline{P}(t,\overline{\omega})}g_i(t)dW_t]h(\overline{x})d\overline{x}$

$=\int_0^t(\sum_{i=1}^3 \frac{\partial h}{\partial x_i}|_{\overline{P}(s,\overline{\omega})}f_i(s)+\frac{1}{2}\sum_{j,k=1}^3 \frac{\partial^2 h}{\partial x_ix_j}|_{\overline{P}(s,\overline{\omega})}g_jg_k(s))ds+\int_0^t(\sum_{i=1}^3 \frac{\partial h}{\partial x_i}|_{\overline{P}(s,\overline{\omega})}g_i(s))dW_s$

□

Definition 0.36. *We define* $\rho_n(\overline{x},\overline{\omega})=lim_{\epsilon\rightarrow 0}\rho_{n,\epsilon}$ *in the sense of generalised random variables.*

Lemma 0.37. *For a test function* h*, with the previous notation, we have that;*

$E(\rho_n)(h)=E(\int_0^t(\sum_{j=1}^3 \frac{\partial h}{\partial x_i}|_{\overline{P}(s,\overline{\omega})}f_i(s)+\frac{1}{2}\sum_{j,k=1}^3 \frac{\partial^2 h}{\partial x_ix_j}|_{\overline{P}(s,\overline{\omega})}g_jg_k(s))ds)$

Proof. Again, this is clear from Lemma 0.35 and the fact that the second term there is a martingale.

□

Lemma 0.38. *For a test function* h*, and trajectories starting at* $\overline{0}$*, we have that;*

$h(\overline{P}(t,\overline{\omega})-h(\overline{0})$

$= \int_0^t (\sum_{i=1}^3 \frac{\partial h}{\partial x_i}|_{\overline{P}(s,\overline{\omega})} f_i(s) + \frac{1}{2}\sum_{j,k=1}^3 \frac{\partial^2 h}{\partial x_j x_k}|_{\overline{P}(s,\overline{\omega})} g_j g_k(s))ds$

$+ \int_0^t (\sum_{i=1}^3 \frac{\partial h}{\partial x_i}|_{\overline{P}(s,\overline{\omega})} g_i(s))dW_s$

Proof. This is a consequence of Lemma 0.35 and the definition of charge;

$\rho_n(h(\overline{P}(t,\overline{\omega})))$

$= lim_{\epsilon\rightarrow 0}\int_{\overline{x}\in\mathcal{R}^3} D_{n,\epsilon}(\overline{x} - h(\overline{P}(t,\overline{\omega})))d\overline{x}$

$= h(\overline{P}(t,\overline{\omega}))$

□

Definition 0.39. *We define the associated current* $\overline{J}_n$ *to* ρ_n *by setting;*

$\overline{J}_{n,\epsilon}(\overline{x},t,\overline{\omega}) = \rho_{n,\epsilon}(\overline{x},t,\overline{\omega})\overline{U}(t,\omega)$

and letting $\epsilon \rightarrow 0$.

Lemma 0.40. *The pair* $(\rho_n, \overline{J}_n)$ *satisfies the continuity equation;*

$E(\frac{\partial \rho_n}{\partial t} + \bigtriangledown \centerdot \overline{J}_n) = 0$

Proof. We have that, by Lemma 0.38, for a test function h, using the FTC and the definition of white noise, with the constant term $h(\overline{0})$ vanishing;

$\frac{\partial \rho_n}{\partial t}(h) = \frac{\partial}{\partial t}[\int_0^t (\sum_{i=1}^3 \frac{\partial h}{\partial x_i}|_{\overline{P}(s,\overline{\omega})} f_i(s) + \frac{1}{2}\sum_{j,k=1}^3 \frac{\partial^2 h}{\partial x_j x_k}|_{\overline{P}(s,\overline{\omega})} g_j g_k(s))ds$

$+ \int_0^t (\sum_{i=1}^3 \frac{\partial h}{\partial x_i}|_{\overline{P}(s,\overline{\omega})} g_i(s))dW_s]$

$= \sum_{i=1}^3 \frac{\partial h}{\partial x_i}|_{\overline{P}(t,\overline{\omega})} f_i(t) + \frac{1}{2}\sum_{j,k=1}^3 \frac{\partial^2 h}{\partial x_j x_k}|_{\overline{P}(t,\overline{\omega})} g_j g_k(t) + \sum_{i=1}^3 \frac{\partial h}{\partial x_i}|_{\overline{P}(t,\overline{\omega})} g_i(t)Z(t)$

while, using the fact that $\overline{U}(t,\omega)$ is independent of position $\overline{x}$, integrating by parts;

$\bigtriangledown \centerdot \overline{J}_n(h) = \bigtriangledown(\rho_\epsilon) \centerdot \overline{U}(t,\omega)(h)$

$= -\sum_{l=1}^3 \frac{\partial h}{\partial x_l}(\overline{0})(f_l(t) + \frac{g_l(t)Z(t)}{2})$

$-\int_0^t[\sum_{i,l=1}^3 \frac{\partial^2 h}{\partial x_i \partial x_l}|_{\overline{P}(s,\overline{\omega})} f_i(s)ds(f_l(t)+\frac{g_l(t)Z(t)}{2})$

$-\frac{1}{2}\sum_{j,k,l=1}^3 \frac{\partial^3 h}{\partial x_j \partial x_k \partial x_l}|_{\overline{P}(s,\overline{\omega})} g_j g_k(s)ds(f_l(t)+\frac{g_l(t)Z(t)}{2})]$

$-\int_0^t[\sum_{i,l=1}^3 \frac{\partial^2 h}{\partial x_i \partial x_l}|_{\overline{P}(s,\overline{\omega})} g_i(s)dW_s(f_l(t)+\frac{g_l(t)Z(t)}{2})]$

As $Z(s)$ is uncorrelated with previous information;

$E(\sum_{i=1}^3 \frac{\partial h}{\partial x_i}|_{\overline{P}(t,\overline{\omega})} g_i(t)Z(t))=0$

and, therefore;

$E(\frac{\partial \rho_n}{\partial t}(h))=E(\sum_{i=1}^3 \frac{\partial h}{\partial x_i}|_{\overline{P}(t,\overline{\omega})} f_i(t))+E(\frac{1}{2}\sum_{j,k=1}^3 \frac{\partial^2 h}{\partial x_j x_k}|_{\overline{P}(t,\overline{\omega})} g_j g_k(t))$

Similarly;

$E(-\int_0^t[\sum_{i,l=1}^3 \frac{\partial^2 h}{\partial x_i \partial x_l}|_{\overline{P}(s,\overline{\omega})} f_i(s)ds(\frac{g_l(t)Z(t)}{2})])$

$=E(-\int_0^t[\frac{1}{2}\sum_{j,k,l=1}^3 \frac{\partial^3 h}{\partial x_j \partial x_k \partial x_l}|_{\overline{P}(s,\overline{\omega})} g_j g_k(s)ds(\frac{g_l(t)Z(t)}{2})])$

$=0$

and, using the fact that $dW_t Z(t)=1$, together with the martingale property ;

$E(-\int_0^t[\sum_{i,l=1}^3 \frac{\partial^2 h}{\partial x_i \partial x_l}|_{\overline{P}(s,\overline{\omega})} g_i(s)dW_s(f_l+\frac{g_l(t)Z(t)}{2})])$

$=E(-\frac{1}{2}\sum_{i,l=1}^3 \frac{\partial^2 h}{\partial x_i x_l}|_{\overline{P}(t,\overline{\omega})} g_i g_l(t))$

so that, as ;

$E(\sum_{l=1}^3 \frac{\partial h}{\partial x_l}(\overline{0})(\frac{h_l(t)Z(t)}{2}))=0$

we have;

$E((\frac{\partial \rho_n}{\partial t}+\bigtriangledown \centerdot \overline{J}_n)(h))$

$=E(\sum_{i=1}^3 \frac{\partial h}{\partial x_i}|_{\overline{P}(t,\overline{\omega})} f_i(t)-\int_0^t[\sum_{i,l=1}^3 \frac{\partial^2 h}{\partial x_i \partial x_l}|_{\overline{P}(s,\overline{\omega})} f_i(s)ds(f_l(t))$

$-\frac{1}{2}\sum_{j,k,l=1}^3 \frac{\partial^3 h}{\partial x_j \partial x_k \partial x_l}|_{\overline{P}(s,\overline{\omega})} g_j g_k(s)ds(f_l(t))])-E(\sum_{l=1}^3 \frac{\partial h}{\partial x_l}(\overline{0})(f_l(t)))$, $(*)$

Now using Lemma 0.38 and the martingale property again, we have that;

$E(\sum_{i=1}^{3}\frac{\partial h}{\partial x_i}|_{\overline{P}(t,\overline{\omega})}f_i(t))$

$=E(\sum_{l=1}^{3}\frac{\partial h}{\partial x_l}(\overline{0})(f_l(t)))$

$+E(\int_0^t[\sum_{i,l=1}^{3}\frac{\partial^2 h}{\partial x_i\partial x_l}|_{\overline{P}(s,\overline{\omega})}f_i(s)ds(f_l(t))+\frac{1}{2}\sum_{j,k,l=1}^{3}\frac{\partial^3 h}{\partial x_j\partial x_k\partial x_l}|_{\overline{P}(s,\overline{\omega})}g_jg_k(s)ds(f_l(t))])$

so the result follow from $(*)$.

□

Lemma 0.41. *Assume that the electron travels in a straight line in the base frame S, then for charge and current $(\rho_n,\overline{J}_n)$ in S', satisfying the conditions of Lemma 0.40, we have that, for $\tau>0$, $1\leq i,j\leq 3$, and with notation as in the proof;*

$Cov(\rho_{n,t}(h),\rho_{n,t+\tau}(h))$

$=\frac{1}{(2\pi)^3}det(\Sigma_t)^{-\frac{1}{2}}det(\Sigma_\tau)^{-\frac{1}{2}}(\int_{\overline{x},\overline{y}\in\mathcal{R}^6}h(\overline{x})h(\overline{x}+\overline{y})e^{-\frac{1}{2}(\overline{x}-\overline{\mu}_t)^T\Sigma_t^{-1}(\overline{x}-\overline{\mu}_t)}e^{-\frac{1}{2}(\overline{y}-\overline{\mu}_\tau)^T\Sigma_\tau^{-1}(\overline{y}-\overline{\mu}_\tau)})d\overline{x}d\overline{y}$

$-\frac{1}{(2\pi)^3}det(\Sigma_t)^{-\frac{1}{2}}det(\Sigma_{t+\tau})^{-\frac{1}{2}}(\int_{\overline{x}\in\mathcal{R}^3}h(\overline{x})e^{-\frac{1}{2}(\overline{x}-\overline{\mu}_t)^T\Sigma_t^{-1}(\overline{x}-\overline{\mu}_t)}d\overline{x})(\int_{\overline{y}\in\mathcal{R}^3}h(\overline{y})$

$e^{-\frac{1}{2}(\overline{y}-\overline{\mu}_{t+\tau})^T\Sigma_{t+\tau}^{-1}(\overline{y}-\overline{\mu}_{t+\tau})}d\overline{y})$

$Cov(j_{i,n,t}(h),j_{j,n,t+\tau}(h))$

$=\frac{f_if_j}{(2\pi)^3}det(\Sigma_t)^{-\frac{1}{2}}det(\Sigma_\tau)^{-\frac{1}{2}}(\int_{\overline{x},\overline{y}\in\mathcal{R}^6}h(\overline{x})h(\overline{x}+\overline{y})e^{-\frac{1}{2}(\overline{x}-\overline{\mu}_t)^T\Sigma_t^{-1}(\overline{x}-\overline{\mu}_t)}e^{-\frac{1}{2}(\overline{y}-\overline{\mu}_\tau)^T\Sigma_\tau^{-1}(\overline{y}-\overline{\mu}_\tau)})d\overline{x}d\overline{y}$

$+\frac{f_ig'_j}{(2\pi)^3}det(\Sigma_t)^{-\frac{1}{2}}det(\Sigma_\tau)^{-\frac{1}{2}}(\int_{\overline{x},\overline{y}\in\mathcal{R}^6}h(\overline{x})(\sum_{k=1}^{3}\frac{\partial h}{\partial x_k}(\overline{x}+\overline{y})g_k)e^{-\frac{1}{2}(\overline{x}-\overline{\mu}_t)^T\Sigma_t^{-1}(\overline{x}-\overline{\mu}_t)}$

$e^{-\frac{1}{2}(\overline{y}-\overline{\mu}_\tau)^T\Sigma_\tau^{-1}(\overline{y}-\overline{\mu}_\tau)})d\overline{x}d\overline{y}$

$+\frac{g'_if_j}{(2\pi)^3}det(\Sigma_t)^{-\frac{1}{2}}det(\Sigma_\tau)^{-\frac{1}{2}}(\int_{\overline{x},\overline{y}\in\mathcal{R}^6}(\sum_{k=1}^{3}\frac{\partial h}{\partial x_k}(\overline{x})g_k)h(\overline{x}+\overline{y})e^{-\frac{1}{2}(\overline{x}-\overline{\mu}_t)^T\Sigma_t^{-1}(\overline{x}-\overline{\mu}_t)}$

$e^{-\frac{1}{2}(\overline{y}-\overline{\mu}_\tau)^T\Sigma_\tau^{-1}(\overline{y}-\overline{\mu}_\tau)})d\overline{x}d\overline{y}$

$+\frac{g'_ig'_j}{(2\pi)^3}det(\Sigma_t)^{-\frac{1}{2}}det(\Sigma_\tau)^{-\frac{1}{2}}(\int_{\overline{x},\overline{y}\in\mathcal{R}^6}(\sum_{l=1}^{3}\frac{\partial h}{\partial x_l}(\overline{x})g_l)(\sum_{k=1}^{3}\frac{\partial h}{\partial x_k}(\overline{x}+\overline{y})g_k)e^{-\frac{1}{2}(\overline{x}-\overline{\mu}_t)^T\Sigma_t^{-1}(\overline{x}-\overline{\mu}_t)}$

$e^{-\frac{1}{2}(\overline{y}-\overline{\mu}_\tau)^T\Sigma_\tau^{-1}(\overline{y}-\overline{\mu}_\tau)})d\overline{x}d\overline{y}$

$-[\frac{f_i}{(2\pi)^{\frac{3}{2}}}det(\Sigma_t)^{-\frac{1}{2}}(\int_{\overline{x}\in\mathcal{R}^3} h(\overline{x})e^{-\frac{1}{2}(\overline{x}-\overline{\mu}_t)^T\Sigma_t^{-1}(\overline{x}-\overline{\mu}_t)}d\overline{x})$

$+\frac{g_i'}{(2\pi)^{\frac{3}{2}}}det(\Sigma_t)^{-\frac{1}{2}}(\int_{\overline{x}\in\mathcal{R}^3}[\sum_{k=1}^3 \frac{\partial h}{\partial x_k}(\overline{x})g_k]e^{-\frac{1}{2}(\overline{x}-\overline{\mu}_t)^T\Sigma_t^{-1}(\overline{x}-\overline{\mu}_t)}d\overline{x})]$

$[\frac{f_j}{(2\pi)^{\frac{3}{2}}}det(\Sigma_{t+\tau})^{-\frac{1}{2}}(\int_{\overline{x}\in\mathcal{R}^3} h(\overline{x})e^{-\frac{1}{2}(\overline{x}-\overline{\mu}_{t+\tau})^T\Sigma_t^{-1}(\overline{x}-\overline{\mu}_{t+\tau})}d\overline{x})$

$+\frac{g_j'}{(2\pi)^{\frac{3}{2}}}det(\Sigma_{t+\tau})^{-\frac{1}{2}}(\int_{\overline{x}\in\mathcal{R}^3}[\sum_{k=1}^3 \frac{\partial h}{\partial x_k}(\overline{x})g_k]e^{-\frac{1}{2}(\overline{x}-\overline{\mu}_{t+\tau})^T\Sigma_{t+\tau}^{-1}(\overline{x}-\overline{\mu}_{t+\tau})}d\overline{x})]$

In particularly, taking $h=1$ on a large but compact support, we have that, for $1\leq i,j\leq 3$, $\tau>0$;

$Cov(\rho_{n,t}(h),\rho_{n,t+\tau}(h))=Cov(j_{i,n,t}(h),j_{j,n,t+\tau}(h))=0$

and the variables are uncorrelated.

While, with $h=1$ on a large but compact support, $\tau=0$, we have that, for $1\leq i,j\leq 3$, $\tau=0$;

$Cov(\rho_{n,t}(h),\rho_{n,t+\tau}(h))=0$

$Cov(j_{i,n,t}(h),j_{j,n,t+\tau}(h))=g_i'g_j'\eta$

The autocorrelation function;

$r(\tau)=E(\rho_{n,t}(h)\rho_{n,t+\tau}(h))=1$

in the case of charge, and;

$r(\tau)=E(j_{i,n,t}(h)j_{i,n,t+\tau}(h))=f_i^2+g_i'^2\delta_0(\tau)$

in the case of the components of current. In particular, the spectral density of charge is δ_0 and the spectral density of the components of current is $f_i^2\delta_0+\frac{g_i'^2}{2\pi}$.

Proof. For the first claim, observe that the components $\{f_i(s):1\leq i\leq 3\}$ and $\{g_i(s):1\leq i\leq 3\}$ are constant with time. In particularly, the position random variable $\overline{P}(t,\overline{\omega})$ follows a multivariate normal distribution with mean vector $\overline{\mu}_t=(f_1t,f_2t,f_3t)$ and covariance matrix Σ_t, with;

$(\Sigma)_{t,ij} = Cov(\overline{P}_{t,i}, \overline{P}_{t,j})$

$= E(\overline{P}_{t,i}\overline{P}_{t,j}) - E(\overline{P}_{t,i})E(\overline{P}_{t,j})$

$= E((f_i t + g_i B_t)(f_j t + g_j B_t)) - f_i f_j t^2$

$= E(g_i B_t g_j B_t)$

$= g_i g_j E(B_t^2)$

$= g_i g_j t$

for $1 \leq i, j \leq 3$. It follows that, for a test function h;

$E(\rho_{n,t}(h)\rho_{n,t+\tau}(h)) = E(h(\overline{P}_t)h(\overline{P}_{t+\tau})) = E(h(\overline{P}_t)E(h(\overline{P}_{t+\tau}|\mathcal{F}_t))$

$= \frac{1}{(2\pi)^3} det(\Sigma_t)^{-\frac{1}{2}} det(\Sigma_\tau)^{-\frac{1}{2}} (\int_{\overline{x},\overline{y} \in \mathcal{R}^6} h(\overline{x})h(\overline{x}+\overline{y}) e^{-\frac{1}{2}(\overline{x}-\overline{\mu}_t)^T \Sigma_t^{-1}(\overline{x}-\overline{\mu}_t)} e^{-\frac{1}{2}(\overline{y}-\overline{\mu}_\tau)^T \Sigma_\tau^{-1}(\overline{y}-\overline{\mu}_\tau)}) d\overline{x} d\overline{y}$

whereas;

$E(\rho_{n,t}(h))E(\rho_{n,t+\tau}(h)) = E(h(\overline{P}_t))E(h(\overline{P}_{t+\tau}))$

$= \frac{1}{(2\pi)^3} det(\Sigma_t)^{-\frac{1}{2}} det(\Sigma_{t+\tau})^{-\frac{1}{2}} (\int_{\overline{x} \in \mathcal{R}^3} h(\overline{x}) e^{-\frac{1}{2}(\overline{x}-\overline{\mu}_t)^T \Sigma_t^{-1}(\overline{x}-\overline{\mu}_t)} d\overline{x})(\int_{\overline{y} \in \mathcal{R}^3} h(\overline{y})$

$e^{-\frac{1}{2}(\overline{y}-\overline{\mu}_{t+\tau})^T \Sigma_{t+\tau}^{-1}(\overline{y}-\overline{\mu}_{t+\tau})} d\overline{y})$

so the result follows. For the second claim, we have that, letting $g_i' = \frac{g_i}{2}$;

$Cov(j_{i,n,t}(h), j_{j,n,t+\tau}(h)) = E(j_{i,n,t}(h), j_{j,n,t+\tau}(h)) - E(j_{i,n,t}(h))E(j_{j,n,t+\tau}(h))$

$= E(\rho_{n,t}(h)\rho_{n,t+\tau}(h)(f_i + g_i' Z_t)(f_j + g_j' Z_{t+\tau})) - E(\rho_{n,t}(h)(f_i + g_i' Z_t))E(\rho_{n,t+\tau}(h)$

$(f_j + g_j' Z_{t+\tau}))$

We have, using Lemma 0.35, the first part of the lemma and the box calculus;

$E(\rho_{n,t}(h)(f_i + g_i' Z_t)) = f_i E(\rho_{n,t}(h)) + g_i' E(\rho_{n,t}(h) Z_t)$

$= f_i E(h(\overline{P}_t)) + g_i' E([\int_0^t (\sum_{k=1}^3 \frac{\partial h}{\partial x_k}|_{\overline{P}(s,\overline{\omega})} g_k) dW_s] Z_t)$

$= \frac{f_i}{(2\pi)^{\frac{3}{2}}} det(\Sigma_t)^{-\frac{1}{2}} (\int_{\overline{x}\in\mathcal{R}^3} h(\overline{x}) e^{-\frac{1}{2}(\overline{x}-\overline{\mu}_t)^T \Sigma_t^{-1}(\overline{x}-\overline{\mu}_t)} d\overline{x}) + g_i' E(\sum_{k=1}^3 \frac{\partial h}{\partial x_k}|_{\overline{P}(t,\overline{\omega})} g_k)$

$= \frac{f_i}{(2\pi)^{\frac{3}{2}}} det(\Sigma_t)^{-\frac{1}{2}} (\int_{\overline{x}\in\mathcal{R}^3} h(\overline{x}) e^{-\frac{1}{2}(\overline{x}-\overline{\mu}_t)^T \Sigma_t^{-1}(\overline{x}-\overline{\mu}_t)} d\overline{x})$

$+ \frac{g_i'}{(2\pi)^{\frac{3}{2}}} det(\Sigma_t)^{-\frac{1}{2}} (\int_{\overline{x}\in\mathcal{R}^3} [\sum_{k=1}^3 \frac{\partial h}{\partial x_k}(\overline{x}) g_k] e^{-\frac{1}{2}(\overline{x}-\overline{\mu}_t)^T \Sigma_t^{-1}(\overline{x}-\overline{\mu}_t)} d\overline{x})$

and, similarly;

$E(\rho_{n,t+\tau}(h)(f_j + g_j' Z_{t+\tau})) = \frac{f_j}{(2\pi)^{\frac{3}{2}}} det(\Sigma_{t+\tau})^{-\frac{1}{2}} (\int_{\overline{x}\in\mathcal{R}^3} h(\overline{x}) e^{-\frac{1}{2}(\overline{x}-\overline{\mu}_{t+\tau})^T \Sigma_t^{-1}(\overline{x}-\overline{\mu}_{t+\tau})} d\overline{x})$

$+ \frac{g_j'}{(2\pi)^{\frac{3}{2}}} det(\Sigma_{t+\tau})^{-\frac{1}{2}} (\int_{\overline{x}\in\mathcal{R}^3} [\sum_{k=1}^3 \frac{\partial h}{\partial x_k}(\overline{x}) g_k] e^{-\frac{1}{2}(\overline{x}-\overline{\mu}_{t+\tau})^T \Sigma_{t+\tau}^{-1}(\overline{x}-\overline{\mu}_{t+\tau})} d\overline{x})$

We have that, using the result above, lemma 0.35, the representation of Z_t as $\sqrt{\eta}\omega_{\frac{[\eta t]}{\eta}}$, see [5], and the box calculus;

$E(\rho_{n,t}(h)\rho_{n,t+\tau}(h)(f_i + g_i' Z_t)(f_j + g_j' Z_{t+\tau}))$

$= f_i f_j E(\rho_{n,t}(h)\rho_{n,t+\tau}(h)) + f_i g_j' E(\rho_{n,t}(h)\rho_{n,t+\tau}(h) Z_{t+\tau}) + g_i' f_j E(\rho_{n,t}(h) Z_t \rho_{n,t+\tau}(h))$

$+ g_i' g_j' E(\rho_{n,t}(h) Z_t \rho_{n,t+\tau}(h) Z_{t+\tau})$

$= \frac{f_i f_j}{(2\pi)^3} det(\Sigma_t)^{-\frac{1}{2}} det(\Sigma_\tau)^{-\frac{1}{2}} (\int_{\overline{x},\overline{y}\in\mathcal{R}^6} h(\overline{x}) h(\overline{x}+\overline{y}) e^{-\frac{1}{2}(\overline{x}-\overline{\mu}_t)^T \Sigma_t^{-1}(\overline{x}-\overline{\mu}_t)} e^{-\frac{1}{2}(\overline{y}-\overline{\mu}_\tau)^T \Sigma_\tau^{-1}(\overline{y}-\overline{\mu}_\tau)}) d\overline{x} d\overline{y}$

$+ f_i g_j' E(\rho_{n,t}(h)[\int_0^{t+\tau} (\sum_{k=1}^3 \frac{\partial h}{\partial x_k}|_{\overline{P}(s,\overline{\omega})} g_k) dW_s] Z_{t+\tau})$

$+ g_i' f_j (E(\rho_{n,t}(h) Z_t \rho_{n,t+\tau}(h) | Z_t = \sqrt{\eta}) P(Z_t = \sqrt{\eta}) + E(\rho_{n,t}(h) Z_t \rho_{n,t+\tau}(h) | Z_t =$

$-\sqrt{\eta}) P(Z_t = -\sqrt{\eta})$

$+ g_i' g_j' E(\rho_{n,t}(h) Z_t [\int_0^{t+\tau} (\sum_{k=1}^3 \frac{\partial h}{\partial x_k}|_{\overline{P}(s,\overline{\omega})} g_k) dW_s] Z_{t+\tau})$

$= \frac{f_i f_j}{(2\pi)^3} det(\Sigma_t)^{-\frac{1}{2}} det(\Sigma_\tau)^{-\frac{1}{2}} (\int_{\overline{x},\overline{y}\in\mathcal{R}^6} h(\overline{x}) h(\overline{x}+\overline{y}) e^{-\frac{1}{2}(\overline{x}-\overline{\mu}_t)^T \Sigma_t^{-1}(\overline{x}-\overline{\mu}_t)} e^{-\frac{1}{2}(\overline{y}-\overline{\mu}_\tau)^T \Sigma_\tau^{-1}(\overline{y}-\overline{\mu}_\tau)}) d\overline{x} d\overline{y}$

$+ f_i g_j' E(\rho_{n,t}(h)(\sum_{k=1}^3 \frac{\partial h}{\partial x_k}|_{\overline{P}(t+\tau,\overline{\omega})} g_k))$

$+ g_i' f_j (\frac{1}{2} E((\sum_{k=1}^3 \frac{\partial h}{\partial x_k}|_{\overline{P}(t,\overline{\omega})} g_k + \sqrt{\eta}\rho_{n,t-\frac{1}{\eta}}(h))\rho_{n,t+\tau}(h)) + \frac{1}{2} E((\sum_{k=1}^3 \frac{\partial h}{\partial x_k}|_{\overline{P}(t,\overline{\omega})} g_k$

$-\sqrt{\eta}\rho_{n,t-\frac{1}{\eta}}(h))\rho_{n,t+\tau}(h)))$

$+g_i'g_j'E(\rho_{n,t}(h)Z_t(\sum_{k=1}^3 \frac{\partial h}{\partial x_k}|_{\overline{P}(t+\tau,\overline{\omega})}g_k))$

$= \frac{f_i f_j}{(2\pi)^3}det(\Sigma_t)^{-\frac{1}{2}}det(\Sigma_\tau)^{-\frac{1}{2}}(\int_{\overline{x},\overline{y}\in\mathcal{R}^6} h(\overline{x})h(\overline{x}+\overline{y})e^{-\frac{1}{2}(\overline{x}-\overline{\mu}_t)^T\Sigma_t^{-1}(\overline{x}-\overline{\mu}_t)}e^{-\frac{1}{2}(\overline{y}-\overline{\mu}_\tau)^T\Sigma_\tau^{-1}(\overline{y}-\overline{\mu}_\tau)})d\overline{x}d\overline{y}$

$+\frac{f_i g_j'}{(2\pi)^3}det(\Sigma_t)^{-\frac{1}{2}}det(\Sigma_\tau)^{-\frac{1}{2}}(\int_{\overline{x},\overline{y}\in\mathcal{R}^6} h(\overline{x})(\sum_{k=1}^3 \frac{\partial h}{\partial x_k}(\overline{x}+\overline{y})g_k)e^{-\frac{1}{2}(\overline{x}-\overline{\mu}_t)^T\Sigma_t^{-1}(\overline{x}-\overline{\mu}_t)}$

$e^{-\frac{1}{2}(\overline{y}-\overline{\mu}_\tau)^T\Sigma_\tau^{-1}(\overline{y}-\overline{\mu}_\tau)})d\overline{x}d\overline{y}$

$+g_i'f_jE((\sum_{k=1}^3 \frac{\partial h}{\partial x_k}|_{\overline{P}(t,\overline{\omega})}g_k)\rho_{n,t+\tau}(h))$

$+g_i'g_j'E((\sum_{l=1}^3 \frac{\partial h}{\partial x_l}|_{\overline{P}(t,\overline{\omega})}g_l)(\sum_{k=1}^3 \frac{\partial h}{\partial x_k}|_{\overline{P}(t+\tau,\overline{\omega})}g_k))$

$= \frac{f_i f_j}{(2\pi)^3}det(\Sigma_t)^{-\frac{1}{2}}det(\Sigma_\tau)^{-\frac{1}{2}}(\int_{\overline{x},\overline{y}\in\mathcal{R}^6} h(\overline{x})h(\overline{x}+\overline{y})e^{-\frac{1}{2}(\overline{x}-\overline{\mu}_t)^T\Sigma_t^{-1}(\overline{x}-\overline{\mu}_t)}e^{-\frac{1}{2}(\overline{y}-\overline{\mu}_\tau)^T\Sigma_\tau^{-1}(\overline{y}-\overline{\mu}_\tau)})d\overline{x}d\overline{y}$

$+\frac{f_i g_j'}{(2\pi)^3}det(\Sigma_t)^{-\frac{1}{2}}det(\Sigma_\tau)^{-\frac{1}{2}}(\int_{\overline{x},\overline{y}\in\mathcal{R}^6} h(\overline{x})(\sum_{k=1}^3 \frac{\partial h}{\partial x_k}(\overline{x}+\overline{y})g_k)e^{-\frac{1}{2}(\overline{x}-\overline{\mu}_t)^T\Sigma_t^{-1}(\overline{x}-\overline{\mu}_t)}$

$e^{-\frac{1}{2}(\overline{y}-\overline{\mu}_\tau)^T\Sigma_\tau^{-1}(\overline{y}-\overline{\mu}_\tau)})d\overline{x}d\overline{y}$

$+\frac{g_i' f_j}{(2\pi)^3}det(\Sigma_t)^{-\frac{1}{2}}det(\Sigma_\tau)^{-\frac{1}{2}}(\int_{\overline{x},\overline{y}\in\mathcal{R}^6}(\sum_{k=1}^3 \frac{\partial h}{\partial x_k}(\overline{x})g_k)h(\overline{x}+\overline{y})e^{-\frac{1}{2}(\overline{x}-\overline{\mu}_t)^T\Sigma_t^{-1}(\overline{x}-\overline{\mu}_t)}$

$e^{-\frac{1}{2}(\overline{y}-\overline{\mu}_\tau)^T\Sigma_\tau^{-1}(\overline{y}-\overline{\mu}_\tau)})d\overline{x}d\overline{y}$

$+\frac{g_i' g_j'}{(2\pi)^3}det(\Sigma_t)^{-\frac{1}{2}}det(\Sigma_\tau)^{-\frac{1}{2}}(\int_{\overline{x},\overline{y}\in\mathcal{R}^6}(\sum_{l=1}^3 \frac{\partial h}{\partial x_l}(\overline{x})g_l)(\sum_{k=1}^3 \frac{\partial h}{\partial x_k}(\overline{x}+\overline{y})g_k)e^{-\frac{1}{2}(\overline{x}-\overline{\mu}_t)^T\Sigma_t^{-1}(\overline{x}-\overline{\mu}_t)}$

$e^{-\frac{1}{2}(\overline{y}-\overline{\mu}_\tau)^T\Sigma_\tau^{-1}(\overline{y}-\overline{\mu}_\tau)})d\overline{x}d\overline{y}$

For the final claim, use the fact that for $\tau > 0$, the multivariate distributions integrate to unity, and the partial derivatives of h vanish on a large support, so the variables are uncorrelated. When $\tau = 0$, a similar proof shows that;

$$Cov(\rho_{n,t}(h), \rho_{n,t}(h)) = Var(\rho_{n,t}(h)) = 0$$

while, using the main proof, and the nonstandard representation of white noise $Z_t = \sqrt{\eta}\omega_{\frac{[\eta t]}{\eta}}$, $Z_t^2 = \eta$;

$Cov(j_{i,n,t}(h), j_{j,n,t}(h)) = E(\rho_{n,t}(h)^2(f_i + g_i'Z_t)(f_j + g_j'Z_t))$

$-E(\rho_{n,t}(h)(f_i + g_i'Z_t))E(\rho_{n,t}(h)(f_j + g_j'Z_t))$

$= E(\rho_{n,t}(h)^2(f_i f_j + f_i g'_j Z_t + f_j g'_i Z_t + g'_i g'_j Z_t^2) - f_i f_j E(\rho_{n,t}(h))^2$

$= f_i f_j + g'_i g'_j \eta - f_i f_j$

$= g'_i g'_j \eta$

The autocorrelation claim is a simple computation using the main proof, in the case of charge, and in the case of the components of current. By the definition of the spectral density as the Fourier transform of the autocorrelation, and using the convention that $\mathcal{F}(1) = \delta_0$, we obtain the last result, that the current is a combination of white noise and a periodic signal.

□

Remarks 0.42. *We can summarise the results of this paper by saying that the wave equation for charge and current supports the idea that an electron can simultaneously exhibit wave and particle properties. The transformations between frames of charge and current obeying the wave equation is consistent with the transformation for particles in special relativity, up to a Doppler shift, provided we allow for reverse particle paths. The Doppler shift is well known in the literature and we use the reverse particle paths to develop a theory of the random motion of an electron, for observers travelling at a velocity relative to the source. This theory can explain noise received in radio signals and can even predict the velocity of the observer given knowledge of the components* g'_i *above. More specifically, if we have a stationary source* X*, then for an observer moving at velocity* $\overline{v}$ *relative to* X*;*

$(g'_1, g'_2, g'_3) = \frac{\overline{v}-(-\overline{v})}{4} = \frac{\overline{v}}{2}$ $(*)$

so that if we filter out the noisy part of the signal and estimate (g'_1, g'_2, g'_3) *from a computation of the autocorrelations of the time series provided by the noise, we can obtain the velocity from* $(*)$*. It's not clear that this can be carried out theoretically, but would require some experimental work, which the theory can only suggest. One obstacle is the presence of capacitor filters in radios which eliminate background noise; an examination of car or aircraft radio signals without filters might lead to interesting conclusions. Another theory is that noise is only generated in the direction of travel relative to a stationary source, in which case the use of antennae pointing in the direction of travel might help to eliminate the noise of a radio signal. A feature of this* *paper is the suggestion that the noise received in radio signals is white noise. If X_t is white noise and $Y_t = Acos(\omega t)$ is a deterministic signal, with $Z_t = X_t + Y_t$, then by the strong law of large number, if;*

$$Z_{t,n} = \frac{Z_t + Z_{t+\frac{2\pi}{\omega}} + \ldots + Z_{t+\frac{2\pi n}{\omega}}}{n}$$

$$Z_{t,n} \rightarrow_{(as)} Y_t$$

as $n \rightarrow \infty$. Averaging a signal with time lags can be achieved in certain types of transmitter, like the spherical cavity magnetron. A further feature of this paper is that noise is generated by free particle paths. Presumably, with regard to the spherical cavity magnetron example again, increasing the radius of the sphere, will produce more noise than with a smaller radius, as free particle paths inside the sphere are more prevalent. This controlled addition and removal of noise could be useful in heating as well as confining plasmas generated by electrolysis of steam.

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