SOME ARGUMENTS FOR THE WAVE EQUATION IN QUANTUM THEORY 10

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ABSTRACT. This paper continues the work in [6], in which it was proved that given $(\rho, \overline{J})$ satisfying the relations;

$(i)\ \square^2(\rho) = 0$

$(ii)\ \square^2(\overline{J}) = \overline{0}$

$(iii)\ \bigtriangledown(\rho) + \frac{1}{c^2}\frac{\partial \overline{J}}{\partial t} = \overline{0}$

with compact support, the fields $(\overline{E}, \overline{B})$ defined by Jefimenko's equations *exist*. We prove here that the fields $(\overline{E}, \overline{B})$ are quasi split normal, so that the methods of [2] apply.

Lemma 0.1. *For charge and current $(\rho, \overline{J})$, the relations;*

$(i)\ \square^2(\rho) = 0$

$(ii)\ \square^2(\overline{J}) = \overline{0}$

$(iii)\ \bigtriangledown(\rho) + \frac{1}{c^2}\frac{\partial \overline{J}}{\partial t} = \overline{0}$

$(iv)\ \bigtriangledown \times \overline{J} = \overline{0}$

are invariant under transformations of the base frame by a velocity vector $\overline{v}$, with $|\overline{v}| < c$.

Proof. The proof that $(i), (ii), (iv)$ hold for the transformed quantities $(\rho', \overline{J}')$ is done in [4]. We check that (iii) holds for the standard boost $v\overline{e}_1$, with $0 < v < c$. We have that;

$\rho' = \gamma_v(p - \frac{vj_1}{c^2})$

$\overline{J}' = \gamma_v(\overline{J}_{||} - \rho v \bar{e}_1) + \overline{J}_{\perp}$

$= (\gamma_v(j_1 - \rho v), j_2, j_3)$

$\frac{\partial}{\partial t'} = \gamma_v(\frac{\partial}{\partial t} + v\frac{\partial}{\partial x})$

$\bigtriangledown' = \gamma_v(\bigtriangledown_{||} + \frac{v}{c^2}\frac{\partial}{\partial t}) + \bigtriangledown_{\perp}$

$= (\gamma_v(\frac{\partial}{\partial x} + \frac{v}{c^2}\frac{\partial}{\partial t}), \frac{\partial}{\partial y}, \frac{\partial}{\partial z})$

so that;

$\bigtriangledown'(\rho') = (\gamma_v(\frac{\partial}{\partial x} + \frac{v}{c^2}\frac{\partial}{\partial t}), \frac{\partial}{\partial y}, \frac{\partial}{\partial z})(\gamma_v(p - \frac{vj_1}{c^2}))$

$= (\gamma_v^2\frac{\partial \rho}{\partial x} + \frac{\gamma_v^2 v}{c^2}\frac{\partial \rho}{\partial t} - \frac{\gamma_v^2 v}{c^2}\frac{\partial j_1}{\partial x} - \frac{\gamma_v^2 v^2}{c^4}\frac{\partial j_1}{\partial t}, \gamma_v\frac{\partial \rho}{\partial y} - \frac{\gamma_v v}{c^2}\frac{\partial j_1}{\partial y}, \gamma_v\frac{\partial \rho}{\partial z} - \frac{\gamma_v v}{c^2}\frac{\partial j_1}{\partial z})$ (A)

while using (iii)

$(\frac{\partial \rho}{\partial x}, \frac{\partial \rho}{\partial y}, \frac{\partial \rho}{\partial z}) = -\frac{1}{c^2}(\frac{\partial j_1}{\partial t}, \frac{\partial j_2}{\partial t}, \frac{\partial j_3}{\partial t})$

and using (iv)

$\frac{\partial j_3}{\partial y} = \frac{\partial j_2}{\partial z}, \frac{\partial j_3}{\partial x} = \frac{\partial j_1}{\partial z}, \frac{\partial j_2}{\partial x} = \frac{\partial j_1}{\partial y}$

so that substituting into (A), we obtain;

$\bigtriangledown'(\rho') = (-\frac{\gamma_v^2}{c^2}\frac{\partial j_1}{\partial t} + \frac{\gamma_v^2 v}{c^2}\frac{\partial \rho}{\partial t} - \frac{\gamma_v^2 v}{c^2}\frac{\partial j_1}{\partial x} + \frac{\gamma_v^2 v^2}{c^2}\frac{\partial \rho}{\partial x}, -\frac{\gamma_v}{c^2}\frac{\partial j_2}{\partial t} - \frac{\gamma_v v}{c^2}\frac{\partial j_2}{\partial x}, -\frac{\gamma_v}{c^2}\frac{\partial j_3}{\partial t}$

$-\frac{\gamma_v v}{c^2}\frac{\partial j_3}{\partial x})$ (B)

and;

$\frac{1}{c^2}\frac{\partial \overline{J}'}{\partial t'} = \frac{1}{c^2}\gamma_v(\frac{\partial}{\partial t} + v\frac{\partial}{\partial x})(\gamma_v(j_1 - \rho v), j_2, j_3)$

$= \frac{1}{c^2}(\gamma_v^2\frac{\partial j_1}{\partial t} + \gamma_v^2 v\frac{\partial j_1}{\partial x} - \gamma_v^2 v\frac{\partial \rho}{\partial t} - \gamma_v^2 v^2\frac{\partial \rho}{\partial x}, \gamma_v\frac{\partial j_2}{\partial t} + \gamma_v v\frac{\partial j_2}{\partial x}, \gamma_v\frac{\partial j_3}{\partial t} + \gamma_v v\frac{\partial j_3}{\partial x})$

(C)

so that;

$$\bigtriangledown'(\rho') + \frac{1}{c^2}\frac{\partial \overline{J}'}{\partial t'} = \overline{0}$$

as required.

□

Lemma 0.2. *Consistency*

For any initial conditions $\{\rho_0, \dot{\rho}_0\}$, smooth and having compact support in $\mathcal{R}^3$, there exist $(\rho, \overline{J}\}$ on $\mathcal{R}^4$ satisfying the conditions of Lemma 0.1 and the continuity equation, extending $\{\rho_0, \dot{\rho}_0\}$, with compact support as a process.

Proof. Most of the proof can be found in [5]. The initial conditions generate a unique process ρ on $\mathcal{R}^4$, satisfying (i) with compact support, such that $\square^2(\rho) = 0$, (i) of Lemma 0.1. Define $\overline{J}$ by;

$$\overline{J}(\overline{x}, t) = -c^2 \int_{-\infty}^{t} \bigtriangledown(\rho) ds$$

As shown in [5], $\overline{J}$ is well defined and has compact support as a process. By the FTC, (iii) of Lemma 0.1 holds. We have that, differentiating under the integral sign;

$$\bigtriangledown \times \overline{J} = -c^2 \int_{-\infty}^{t} \bigtriangledown \times (\bigtriangledown(\rho)) ds$$

$$= \overline{0}$$

so (iv) of Lemma 0.1 holds. For the continuity equation, we have that, using (i) of Lemma 0.1 and the FTC;

$$\bigtriangledown \cdot \overline{J} = -c^2 \int_{-\infty}^{t} \bigtriangledown \cdot \bigtriangledown(\rho) ds$$

$$= -c^2 \int_{-\infty}^{t} \bigtriangledown^2(\rho) ds$$

$$= -c^2 \int_{-\infty}^{t} \frac{\rho_{ss}}{c^2} ds$$

$$= -\int_{-\infty}^{t} \rho_{ss} ds$$

$$= -\rho_t$$

Finally, we have that, by (iii) and (iv) of Lemma 0.1;

$$\bigtriangledown^2(\overline{J}) = \bigtriangledown(\bigtriangledown \cdot \overline{J}) - \bigtriangledown \times (\bigtriangledown \times \overline{J})$$

$= \bigtriangledown(\bigtriangledown \cdot \overline{J})$

$= -\bigtriangledown(\rho_t)$

$= (-\bigtriangledown(\rho))_t$

$= \frac{\overline{J}_{tt}}{c^2}$

so that (ii) of Lemma 0.1 holds.

$\square$

Lemma 0.3. *Symmetry Lemma*

For charge and current $(\rho, \overline{J})$ on $\mathcal{R}^4$, satisfying the conditions of Lemma 0.1 and the continuity equation, if we define the processes;

$(\rho_1, \overline{J}_1)$ *and* $(\rho_2, \overline{J}_2)$ *by;*

$\rho_1(\overline{x,t}) = \rho(\overline{x}, t) + \rho(\overline{x}, -t)$, $(\overline{x}, t) \in \mathcal{R}^4$

$\overline{J}_1(\overline{x,t}) = \overline{J}(\overline{x}, t) - \overline{J}(\overline{x}, -t)$, $(\overline{x}, t) \in \mathcal{R}^4$

$\rho_2(\overline{x,t}) = \rho(\overline{x}, t) - \rho(\overline{x}, -t)$, $(\overline{x}, t) \in \mathcal{R}^4$

$\overline{J}_2(\overline{x,t}) = \overline{J}(\overline{x}, t) + \overline{J}(\overline{x}, -t)$, $(\overline{x}, t) \in \mathcal{R}^4$

Then the relations of Lemma 0.1 are still satisfied for the two processes, together with the continuity equation. Moreover, we have that the initial conditions have the property that;

$\dot{\rho}_{1,0} = 0$, $\overline{J}_{1,0} = \overline{0}$, $\rho_{2,0} = 0$, $\dot{\overline{J}}_{1,0} = \overline{0}$

and we can recover the original process as $\rho = \frac{\rho_1+\rho_2}{2}$, $\overline{J} = \frac{\overline{J}_1+\overline{J}_2}{2}$.

If we can prove the fields $(\overline{E}_1, \overline{B}_1)$ and $(\overline{E}_2, \overline{B}_2)$, obtained from Jefimenko's equations exist, and are smooth of very moderate decrease, then so are the fields $(\overline{E}, \overline{B})$ obtained from Jefimenko's equations.

Proof. The proof is straightforward, noting the sign reversal in the time derivative for $-t$.

$\square$

Lemma 0.4. *Addendum to Uniqueness of Representation of Arcs Lemma*

Given $\overline{a} \in B(\overline{0}, s)$, with $\overline{a} \neq \overline{0}$, there exists, up to a set Bl of measure zero in $B(\overline{0}, s)$, a unique $\overline{v} \in V = \bigcup_{d \in B(\overline{0}, s)} H_d$, such that $B(\overline{v}, |\overline{v} - \overline{r}|)$ passes through $\overline{a}$, with $B(\overline{0}, |\overline{a}|)$ and $B(\overline{v}, |\overline{v} - \overline{r}|)$ sharing a common tangent plane at $\overline{a}$. It follows that we can define a map $\gamma : B(\overline{0}, s) \setminus Bl \rightarrow V \setminus H_{\overline{0}}$ which is a homeomorphism onto its image.

Proof. The proof is straightforward, given a generic $\overline{a} \neq \overline{0}$, the line $l_{0,\overline{a}}$ intersects the hyperplane $H_{\overline{a}}$ in a unique point $\overline{v}$, unless $l_{0,\overline{a}}$ and $H_{\overline{a}}$ are parallel, in which case $\overline{a} \cdot (\overline{r} - \overline{a}) = 0$. Letting $\overline{r} = (r_1, r_2, r_3)$, this locus is defined by;

$a_1 r_1 + a_2 r_2 + a_3 r_3 - (a_1^2 + a_2^2 + a_3^2) = 0$

iff $a_1^2 - a_1 r_1 + a_2^2 - a_2 r_2 + a_3^2 - a_3 r_3 = 0$

iff $(a_1 - \frac{r_1}{2})^2 + (a_2 - \frac{r_2}{2})^2 + (a_3 - \frac{r_3}{2})^2 = \frac{r_1^2}{4} + \frac{r_2^2}{4} + \frac{r_3^2}{4}$

which is a sphere centred at $\frac{\overline{r}}{2}$, with radius $\frac{|\overline{r}|}{2}$. Clearly, the intersection of this sphere with $B(\overline{0}, s)$, Bl, is a set of measure zero in $B(\overline{0}, s)$.

For $\overline{y} \in B(\overline{0}, s)$, with $|\overline{y}| = w$, $0 < w \leq s$, we have that, for $\lambda \in \mathcal{R}$;

$|\lambda\overline{y} - \overline{y}| = |\lambda\overline{y} - \overline{r}|$

iff $w|\lambda - 1| = |\lambda\overline{y} - \overline{r}|$

iff $w^2(\lambda - 1)^2 = (\lambda y_1 - r_1)^2 + (\lambda y_2 - r_2)^2 + (\lambda y_3 - r_3)^2$

iff $\lambda^2 w^2 - 2\lambda w^2 + w^2 = \lambda^2 w^2 - 2\lambda\overline{y} \cdot \overline{r} + |\overline{r}|^2$

iff $\lambda(-2w^2 + 2\overline{y} \cdot \overline{r}) = |\overline{r}|^2 - w^2$

iff $\lambda = \frac{|\overline{r}|^2 - w^2}{-2w^2 + 2\overline{y} \cdot \overline{r}}$

The exceptional locus $Bl \cap \delta B(\overline{0}, w)$ corresponds to the locus;

$2\overline{y} \cdot \overline{r} - 2w^2 = 0$

iff $\overline{y}\centerdot\overline{r}=w^2$

which is a plane intersecting the sphere $\delta B(\overline{0},w)$ in a circle C_w. We define the map γ, for $\overline{y}\in B(\overline{0},s)\setminus Bl$ by;

$\gamma(\overline{y})=\frac{|\overline{r}|^2-|\overline{y}|^2}{-2|\overline{y}|^2+2\overline{y}\centerdot\overline{r}}\overline{y}$

The fact that γ is bijective and onto $V\setminus H_0$ follows from the original uniqueness of representation of arcs lemma in [6], noting that we excluded the case that an arc passed through the origin $\overline{0}\in B(\overline{0},s)$. The above argument allows us to define the map γ, which is continuous with a continuous inverse.

□

Lemma 0.5. *Let the potential V satisfy;*

$\square^2(V)=-\frac{\rho}{\epsilon_0}$ $(*)$

with initial conditions $V_{0,1}$ satisfying $\bigtriangledown^2(V_{0,1})=-\frac{\rho_0}{\epsilon_0}$ and $V_{0,2}=0$, be given as in [4]*. Then V is smooth on $\mathcal{R}^4$.*

Proof. By the construction of V, see [4], using the fact that $V_{0,1}$ is smooth, differentiating under the integral sign, V is smooth for $(\overline{x},t)$ with $t\neq 0$. To check smoothness at $t=0$, we follow the method of [5]. By the proof in [4], V_{01} and V_t are of very moderate decrease and quasi split normal, for $t\neq 0$. By the main result of [3], we can take Fourier transforms of $(*)$, for $k_1\neq 0$, $k_2\neq 0$, $k_3\neq 0$, to obtain;

$\mathcal{F}(V)_{tt}+c^2k^2\mathcal{F}(V)=\frac{c^2\mathcal{F}(\rho)}{\epsilon_0}$

and using the fact that $\square^2(\rho)=0$, we obtain;

$\mathcal{F}(V)_{tt}+c^2k^2\mathcal{F}(V)=\frac{c^2\mathcal{F}(\rho)}{\epsilon_0}$

$=\frac{c^2\rho_a}{\epsilon_0}(\overline{k})e^{ikct}+\frac{c^2\rho_b}{\epsilon_0}(\overline{k})e^{-ikct}$ (A)

We can solve this second order forced ODE using Lagrange's variation of parameters, see [1]. The particular solutions to the homogeneous equation are given by;

$y_1(t)=e^{ikct}$ and $y_2(t)=e^{-ikct}$

so that;

$W(y_1,y_2)=y_1y_2'-y_1'y_2$

$=-ikce^{ikct}e^{-ikct}-ikcte^{ikct}e^{-ikct}$

$=-2ikc$

The forcing term g is given by;

$\frac{c^2\rho_a}{\epsilon_0}(\overline{k})e^{ikct}+\frac{c^2\rho_b}{\epsilon_0}(\overline{k})e^{-ikct}$

so that a particular solution of (A), is given by;

$Y(t)=-y_1(t)\int\frac{y_2(t)g(t)}{W(y_1,y_2)(t)}dt+y_2(t)\int\frac{y_1(t)g(t)}{W(y_1,y_2)(t)}dt$

$=-c^2e^{ikct}\int\frac{e^{-ikct}[\frac{\rho_a}{\epsilon_0}(\overline{k})e^{ikct}+\frac{\rho_b}{\epsilon_0}(\overline{k})e^{-ikct}]}{-2ikc}dt+c^2e^{-ikct}\int\frac{e^{ikct}[\frac{\rho_a}{\epsilon_0}(\overline{k})e^{ikct}+\frac{\rho_b}{\epsilon_0}(\overline{k})e^{-ikct}]}{-2ikc}dt$

$=-c^2e^{ikct}\int\frac{[\frac{\rho_a}{\epsilon_0}(\overline{k})+\frac{\rho_b}{\epsilon_0}(\overline{k})e^{-2ikct}]}{-2ikc}dt+c^2e^{-ikct}\int\frac{[\frac{\rho_a}{\epsilon_0}(\overline{k})e^{2ikct}+\frac{\rho_b}{\epsilon_0}(\overline{k})]}{-2ikc}dt$

$=-c^2e^{ikct}(\frac{\frac{\rho_a}{\epsilon_0}(\overline{k})t}{-2ikc}+\frac{\frac{\rho_b}{\epsilon_0}(\overline{k})e^{-2ikct}}{-4k^2c^2})+c^2e^{-ikct}(\frac{\frac{\rho_a}{\epsilon_0}(\overline{k})e^{2ikct}}{4k^2c^2}+\frac{\frac{\rho_b}{\epsilon_0}(\overline{k})t}{-2ikc})$

$=-c^2e^{ikct}(\frac{-\frac{\rho_a}{\epsilon_0}(\overline{k})t}{2ikc}-\frac{\frac{\rho_b}{\epsilon_0}(\overline{k})e^{-2ikct}}{4k^2c^2})+c^2e^{-ikct}(\frac{\frac{\rho_a}{\epsilon_0}(\overline{k})e^{2ikct}}{4k^2c^2}-\frac{\frac{\rho_b}{\epsilon_0}(\overline{k})t}{2ikc})$

$=ce^{ikct}\frac{\frac{\rho_a}{\epsilon_0}(\overline{k})t}{2ik}-ce^{-ikct}\frac{\frac{\rho_b}{\epsilon_0}(\overline{k})t}{2ik}+\frac{e^{-ikct}}{4k^2}(\frac{\rho_a}{\epsilon_0}(\overline{k})+\frac{\rho_b}{\epsilon_0}(\overline{k}))$

and a general solution of (A) is given by;

$A(\overline{k})e^{ikct}+B(\overline{k})e^{-ikct}+ce^{ikct}\frac{\frac{\rho_a}{\epsilon_0}(\overline{k})t}{2ik}-ce^{-ikct}\frac{\frac{\rho_b}{\epsilon_0}(\overline{k})t}{2ik}+\frac{e^{-ikct}}{4k^2}(\frac{\rho_a}{\epsilon_0}(\overline{k})+\frac{\rho_b}{\epsilon_0}(\overline{k}))$

(FF)

where at $t=0$;

$A(\overline{k})+B(\overline{k})+\frac{1}{4k^2}(\frac{\rho_a}{\epsilon_0}(\overline{k})+\frac{\rho_b}{\epsilon_0}(\overline{k}))=\mathcal{F}(V_{01})(\overline{k})$ (CC)

and differentiating with respect to t at $t=0$;

$ikcA(\overline{k})-ikcB(\overline{k})+\frac{\frac{c\rho_a}{\epsilon_0}(\overline{k})}{2ik}-\frac{\frac{c\rho_b}{\epsilon_0}(\overline{k})}{2ik}-\frac{ikc}{4k^2}(\frac{\rho_a}{\epsilon_0}(\overline{k})+\frac{\rho_b}{\epsilon_0}(\overline{k}))$

$= ikcA(\overline{k}) - ikcB(\overline{k}) + \frac{\frac{c\rho_a}{\epsilon_0}(\overline{k})}{2ik} - \frac{\frac{c\rho_b}{\epsilon_0}(\overline{k})}{2ik} + \frac{c}{4ik}(\frac{\rho_a}{\epsilon_0}(\overline{k}) + \frac{\rho_b}{\epsilon_0}(\overline{k}))$

$= 0$ (DD)

As $\bigtriangledown^2(V_{01}) = -\frac{\rho_0}{\epsilon_0}$, we obtain;

$\mathcal{F}(\bigtriangledown^2(V_{01}))(\overline{k})$

$= -k^2\mathcal{F}(V_{01}) = -\mathcal{F}(\frac{\rho_0}{\epsilon_0})$

$= -\frac{\rho_a}{\epsilon_0}(\overline{k}) - \frac{\rho_b}{\epsilon_0}(\overline{k})$

so that;

$\mathcal{F}(V_{01})(\overline{k}) = \frac{1}{k^2}(\frac{\rho_a}{\epsilon_0}(\overline{k}) + \frac{\rho_b}{\epsilon_0}(\overline{k}))$

and, from (CC);

$A(\overline{k}) + B(\overline{k}) + \frac{1}{4k^2}(\frac{\rho_a}{\epsilon_0}(\overline{k}) + \frac{\rho_b}{\epsilon_0}(\overline{k})) = \frac{1}{k^2}(\frac{\rho_a}{\epsilon_0}(\overline{k}) + \frac{\rho_b}{\epsilon_0}(\overline{k}))$

$A(\overline{k}) + B(\overline{k}) = (-\frac{1}{4k^2} + \frac{1}{k^2})(\frac{\rho_a}{\epsilon_0}(\overline{k}) + \frac{\rho_b}{\epsilon_0}(\overline{k}))$

$= \frac{3}{4k^2\epsilon_0}\mathcal{F}(\rho_0)$ (EE)

and, from (DD);

$ikcA(\overline{k}) - ikcB(\overline{k}) = \frac{\frac{-c\rho_a}{\epsilon_0}(\overline{k})}{2ik} + \frac{\frac{c\rho_b}{\epsilon_0}(\overline{k})}{2ik} - \frac{c}{4ik}(\frac{\rho_a}{\epsilon_0}(\overline{k}) + \frac{\rho_b}{\epsilon_0}(\overline{k}))$

$= \frac{-c}{2ik}\frac{1}{ikc\epsilon_0}\mathcal{F}(\dot{\rho}_0) - \frac{c}{4ik\epsilon_0}\mathcal{F}(\rho_0)$

$= \frac{1}{2k^2\epsilon_0}\mathcal{F}(\dot{\rho}_0) - \frac{c}{4ik\epsilon_0}\mathcal{F}(\rho_0)$

$A(\overline{k}) - B(\overline{k}) = \frac{1}{2ik^3c\epsilon_0}\mathcal{F}(\dot{\rho}_0) + \frac{1}{4k^2\epsilon_0}\mathcal{F}(\rho_0)$ (FF)

Solving (EE) and (FF), we obtain;

$2A(\overline{k}) = \frac{3}{4k^2\epsilon_0}\mathcal{F}(\rho_0) + \frac{1}{2ik^3c\epsilon_0}\mathcal{F}(\dot{\rho}_0) + \frac{1}{4k^2\epsilon_0}\mathcal{F}(\rho_0)$

$= \frac{1}{k^2\epsilon_0}\mathcal{F}(\rho_0) + \frac{1}{2ik^3c\epsilon_0}\mathcal{F}(\dot{\rho}_0)$

$A(\bar{k})=\frac{1}{2k^2\epsilon_0}\mathcal{F}(\rho_0)+\frac{1}{4ik^3c\epsilon_0}\mathcal{F}(\dot{\rho}_0)$

$2B(\bar{k})=\frac{3}{4k^2\epsilon_0}\mathcal{F}(\rho_0)-\frac{1}{2ik^3c\epsilon_0}\mathcal{F}(\dot{\rho}_0)-\frac{1}{4k^2\epsilon_0}\mathcal{F}(\rho_0)$

$=\frac{1}{2k^2\epsilon_0}\mathcal{F}(\rho_0)-\frac{1}{2ik^3c\epsilon_0}\mathcal{F}(\dot{\rho}_0)$

$B(\bar{k})=\frac{1}{4k^2\epsilon_0}\mathcal{F}(\rho_0)-\frac{1}{4ik^3c\epsilon_0}\mathcal{F}(\dot{\rho}_0)$

It follow that, from (FF), the solution of (A), matching the initial conditions, for $t>0$, is given by;

$[\frac{1}{2k^2\epsilon_0}\mathcal{F}(\rho_0)+\frac{1}{4ik^3c\epsilon_0}\mathcal{F}(\dot{\rho}_0)]e^{ikct}+[\frac{1}{4k^2\epsilon_0}\mathcal{F}(\rho_0)-\frac{1}{4ik^3c\epsilon_0}\mathcal{F}(\dot{\rho}_0)]e^{-ikct}$

$+ce^{ikct}\frac{\frac{\rho_a}{\epsilon_0}(\bar{k})t}{2ik}-ce^{-ikct}\frac{\frac{\rho_b}{\epsilon_0}(\bar{k})t}{2ik}+\frac{e^{-ikct}}{4k^2}(\frac{\rho_a}{\epsilon_0}(\bar{k})+\frac{\rho_b}{\epsilon_0}(\bar{k}))$ (GG)

We want to differentiate (GG) with respect to t, so we require the formula for the n'th derivative of $e^{\alpha t}t$, where $\alpha\in\mathcal{R}$. Suppose inductively that;

$(e^{\alpha t}t)^{(n)}=n\alpha^{n-1}e^{\alpha t}+\alpha^n e^{\alpha t}t$ $(\dagger)$

Then;

$(e^{\alpha t}t)^{(n+1)}=[n\alpha^{n-1}e^{\alpha t}+\alpha^n e^{\alpha t}t]'$

$=n\alpha^n e^{\alpha t}+\alpha^n e^{\alpha t}+\alpha^{n+1}e^{\alpha t}t$

$=(n+1)\alpha^n e^{\alpha t}+\alpha^{n+1}e^{\alpha t}t$

so we can use the formula $(\dagger)$ applied to the terms te^{ikct} and te^{-ikct} in (GG). Then taking the n'th time derivative of the process $\mathcal{F}(V_t)$ and evaluating at $t=0$, we obtain that;

$lim_{t\rightarrow 0+}\mathcal{F}(V_t^{(n)})$

$=(ikc)^n[\frac{1}{2k^2\epsilon_0}\mathcal{F}(\rho_0)+\frac{1}{4ik^3c\epsilon_0}\mathcal{F}(\dot{\rho}_0)]+(-ikc)^n[\frac{1}{4k^2\epsilon_0}\mathcal{F}(\rho_0)-\frac{1}{4ik^3c\epsilon_0}\mathcal{F}(\dot{\rho}_0)]$

$+\frac{\frac{c\rho_a}{\epsilon_0}(\bar{k})}{2ik}n(ikc)^{n-1}-\frac{\frac{c\rho_b}{\epsilon_0}(\bar{k})}{2ik}n(-ikc)^{n-1}+\frac{(-ikc)^n}{4k^2}(\frac{\rho_a}{\epsilon_0}(\bar{k})+\frac{\rho_b}{\epsilon_0}(\bar{k}))$

$=(ikc)^n[\frac{1}{2k^2\epsilon_0}\mathcal{F}(\rho_0)+\frac{1}{4ik^3c\epsilon_0}\mathcal{F}(\dot{\rho}_0)]+(-ikc)^n[\frac{1}{4k^2\epsilon_0}\mathcal{F}(\rho_0)-\frac{1}{4ik^3c\epsilon_0}\mathcal{F}(\dot{\rho}_0)]$

$+\frac{nc}{2ik\epsilon_0}[\rho_a(\overline{k})(ikc)^{n-1}-\rho_b(\overline{k})(-ikc)^{n-1}]+\frac{(-ikc)^n}{4k^2\epsilon_0}\mathcal{F}(\rho_0)$

$=(ikc)^n[\frac{1}{2k^2\epsilon_0}\mathcal{F}(\rho_0)+\frac{1}{4ik^3c\epsilon_0}\mathcal{F}(\dot{\rho}_0)]+(-ikc)^n[\frac{1}{2k^2\epsilon_0}\mathcal{F}(\rho_0)-\frac{1}{4ik^3c\epsilon_0}\mathcal{F}(\dot{\rho}_0)]$

$+\frac{nc}{2ik\epsilon_0}[\rho_a(\overline{k})(ikc)^{n-1}-\rho_b(\overline{k})(-ikc)^{n-1}]$

It follows that for n even, $n\geq 2$, using the fact that ρ obeys the wave equation $\square^2(\rho)=0$ at $t=0$;

$lim_{t\rightarrow 0+}\mathcal{F}(V_t^{(n)})$

$=\frac{\mathcal{F}(\rho_0)}{\epsilon_0}[\frac{(ikc)^n}{k^2}+\frac{nc(ick)^{n-1}}{2ik}]$

$=\frac{\mathcal{F}(\rho_0)}{\epsilon_0}k^{n-2}c^n[i^n+\frac{ni^{n-2}}{2}]$

$=\frac{\mathcal{F}(\rho_0)}{\epsilon_0}\frac{k^{n-2}c^ni^{n-2}(n-2)}{2}$

$=\frac{\mathcal{F}(\rho_0)}{\epsilon_0}\frac{k^{n-2}c^n(-1)^{\frac{n-2}{2}}(n-2)}{2}$

$=\frac{c^2(n-2)}{2\epsilon_0}\mathcal{F}((c^2)^{\frac{n-2}{2}}(\bigtriangledown^2)^{\frac{n-2}{2}}\rho_0)$

$=\frac{c^2(n-2)}{2\epsilon_0}\mathcal{F}(\rho_0^{(2)\frac{n-2}{2}})$

$=\frac{c^2(n-2)}{2\epsilon_0}\mathcal{F}(\rho_0^{(n-2)})$

By the classical theory applied to Duhamel's principle or the inversion theorem for split normal functions, see [2];

$lim_{t\rightarrow 0+}V_t=V_{01}$

and using the DCT, together with the classical inversion theorem, applied to $V_t^{(n)}$, $n\geq 2$, which has compact support, see [4], we obtain that, for $n\geq 2$;

$lim_{t\rightarrow 0+}V_t^{(n)}=\frac{c^2(n-2)\rho_0^{(n-2)}}{2\epsilon_0}$

For n odd, $n\geq 3$, we obtain;

$lim_{t\rightarrow 0+}\mathcal{F}(V_t^{(n)})$

$$= \frac{\mathcal{F}(\dot{\rho}_0)}{\epsilon_0}[\frac{(ikc)^n}{2ik^3c} + \frac{nc(ick)^{n-1}}{2ik(ikc)}]$$

$$= \frac{\mathcal{F}(\dot{\rho}_0)}{\epsilon_0}k^{n-3}c^{n-1}[\frac{i^{n-1}}{2} + \frac{ni^{n-3}}{2}]$$

$$= \frac{\mathcal{F}(\dot{\rho}_0)}{\epsilon_0}\frac{k^{n-3}c^{n-1}i^{n-3}(n-1)}{2}$$

$$= \frac{\mathcal{F}(\dot{\rho}_0)}{\epsilon_0}\frac{k^{n-3}c^{n-1}(-1)^{\frac{n-3}{2}}(n-1)}{2}$$

$$= \frac{c^2(n-1)}{2\epsilon_0}\mathcal{F}((c^2)^{\frac{n-3}{2}}(\bigtriangledown^2)^{\frac{n-3}{2}}\dot{\rho}_0)$$

$$= \frac{c^2(n-1)}{2\epsilon_0}\mathcal{F}(\dot{\rho}_0^{(2)\frac{n-3}{2}})$$

$$= \frac{c^2(n-1)}{2\epsilon_0}\mathcal{F}(\dot{\rho}_0^{(n-3)})$$

$$= \frac{c^2(n-1)}{2\epsilon_0}\mathcal{F}(\rho_0^{(n-2)})$$

By the classical theory applied to Duhamel's principle or the inversion theorem for split normal functions again, see [2];

$$lim_{t\rightarrow 0+}\dot{V}_t = 0$$

and using the DCT, together with the classical inversion theorem, applied to $V_t^{(n)}$, $n \geq 3$, which has compact support, see [4], we obtain that, for $n \geq 3$;

$$lim_{t\rightarrow 0+}V_t^{(n)} = \frac{c^2(n-1)\rho_0^{(n-2)}}{2\epsilon_0}$$

It follows that all the time derivatives of V_t extend continuously to the boundary $t = 0$, and it is a straightforward argument to then prove that all the derivatives $\frac{\partial^{i+j+k+l}V}{\partial x^i \partial y^j \partial z^k \partial t^l}$ extend continuously to the boundary $t = 0$ as well, see the corresponding argument in [5]. In particular, by continuity, the inhomogeneous wave equation $\square^2(V) = -\frac{\rho}{\epsilon}$ holds at $t = 0$. We can use a similar argument for $t < 0$, noting that we have to make a sign reversal for $\dot{\rho}_0$ and also a sign reversal for $lim_{t\rightarrow 0-}\mathcal{F}(V_t^{(n)})$, when n is odd. The case for n even is unaffected, and the case for n odd is also unaffected as we obtain a double minus sign in the calculation. As the derivatives match at the boundary, this establishes the smoothness of V on $\mathcal{R}^4$, as required.

□

Definition 0.6. *Opposites*

Fixing $\overline{r}$*, we have, by Lemma 0.4, that up to a set of measure zero,* $V_{\overline{r}} = \bigcup_{d\in B(\overline{0},s)} H_d$*, where;*

$H_d = \{\overline{r}' : |\overline{r}' - d| = |\overline{r}' - \overline{r}|\}$
is parametrised by $d \in B(\overline{0}, s)$*. Given such a hyperplane* H_d*, with* $d \in B(\overline{0}, s)$*, we let* $\overline{p_d}$ *be the midpoint of* d *and* $\overline{r}$*, and note that* $\overline{r}'$ *is defined by the intersection of* $l_{d,\overline{r}'}$*, the line connecting* d *and* $\overline{r}'$ *with* H_d*. For* $\overline{r}' \in H_d$*, we let* $\overline{r}'_{opp}$ *be defined by;*

$\overline{r}'_{opp} = \overline{r}' - 2(\overline{r}' - \overline{p_d})$

$= 2\overline{p_d} - \overline{r}'$

We note first that, for $\overline{r}' \in H_d$*;*

$\overline{r}' + \overline{r}'_{opp}$

$= \overline{r}' + (2\overline{p_d} - \overline{r}')$

$= 2\overline{p_d}$

and secondly, that, using similar triangles;

$|\overline{r} - \overline{r}'| = |\overline{r} - \overline{r}'_{opp}|$

so that;

$(\overline{r} - \overline{r}')\hat{} + (\overline{r} - \overline{r}'_{opp})\hat{}$

$= \frac{\overline{r}-\overline{r}'}{|\overline{r}-\overline{r}'|} + \frac{\overline{r}-\overline{r}'_{opp}}{|\overline{r}-\overline{r}'_{opp}|}$

$= \frac{2\overline{r}-(\overline{r}'+\overline{r}'_{opp})}{|\overline{r}-\overline{r}'|}$

$= \frac{2\overline{r}-2\overline{p_d}}{|\overline{r}-\overline{r}'|}$ (DD)

...lemma; $\frac{|(\overline{r}'-\overline{r}'_{opp})_1|}{|\overline{r}-\overline{r}'|} = O(\frac{1}{r})$ *for* $\overline{r} = (r, y, z)$*,* y, z *fixed....*

For $v \in \mathcal{R}_{>0}$*, we let* $\overline{E_{0,v}}(\overline{r})$ *be defined by restricting the integral in Jefimenko's equations to the compact volume* $V \cap (|\overline{r} - \overline{r}'| \leq v)$*,* $\overline{E}_{0,v}$ *is* *the initial condition of the electric field defined by restricting the initial* $(\rho, \overline{J})$ *to* $(-v, \infty)$. *The basis of the hyperbolic method in* [6] *was to show that* $lim_{v\rightarrow\infty}\overline{E}_v = \overline{E}$. *Observe, that if* $\overline{r}' \in V \cap (|\overline{r} - \overline{r}'| \leq v)$ *then* $\overline{r}'_{opp} \in V \cap (|\overline{r} - \overline{r}'| \leq v)$.

...lemma $|\overline{r}'_{opp,opp} - \overline{r}'| = O(\frac{1}{|\overline{r}-\overline{r}'|})$

...lemma, wave equation ρ *with compact initial conditions,...*$\rho(\overline{r}', t) + \rho(\overline{r}'_{opp}, t)$ *has higher order decay on* V *than* $\rho(\overline{r}', t)$.

...lemma, higher decay rate from restricting angle to $O(\frac{1}{|\overline{r}|})$ *in blow up region (asymptotic cone).*

Lemma 0.7. $\frac{\overline{r}'_{opp,opp}}{|\overline{r}-\overline{r}'|} = \frac{\overline{r}'}{|\overline{r}-\overline{r}'|} + O(\frac{|\overline{r}|}{|\overline{r}'|})$

Proof. Fixing $\overline{r}$ and $\overline{r}'$, we have that;

$\overline{r}'_{opp} = \overline{r}' - 2(\overline{r}' - \overline{p_d})$

$= 2\overline{p_d} - \overline{r}'$

$= 2(\frac{\overline{d}+\overline{r}}{2}) - \overline{r}'$

$= \overline{d} + \overline{r} - \overline{r}'$ (A)

and;

$\overline{d} = \overline{r}' - \frac{|\overline{r}'-\overline{r}|}{|\overline{r}'|}\overline{r}'$

so that, substituting into (A);

$\overline{r}'_{opp} = \overline{r}' - \frac{|\overline{r}'-\overline{r}|}{|\overline{r}'|}\overline{r}' + \overline{r} - \overline{r}'$

$= \overline{r} - \frac{|\overline{r}'-\overline{r}|}{|\overline{r}'|}\overline{r}'$ (B)

By the same reasoning, we have that;

$\overline{r}'_{opp,opp} = \overline{r} - \frac{|\overline{r}'_{opp}-\overline{r}|}{|\overline{r}'_{opp}|}\overline{r}'_{opp}$

$\overline{r} - \frac{|\overline{r}'-\overline{r}|}{|\overline{r}'_{opp}|}\overline{r}'_{opp}$

so that, substituting from (B);

$$\overline{r}'_{opp,opp}=\overline{r}-\frac{|\overline{r}'-\overline{r}|}{|\overline{r}-\frac{|\overline{r}'-\overline{r}|}{|\overline{r}'|}\overline{r}'|}(\overline{r}-\frac{|\overline{r}'-\overline{r}|}{|\overline{r}'|}\overline{r}')\ (C)$$

Fixing $\overline{r}$, we have that, using Cauchy-Schwartz and Newton's theorem, that;

$$\frac{|\overline{r}'-\overline{r}|}{|\overline{r}'|}=[1-\frac{2<\overline{r}',\overline{r}>-|\overline{r}|^2}{|\overline{r}'|^2}]^{\frac{1}{2}}$$

$$=[1+O(\frac{|\overline{r}|}{|\overline{r}'|})]^{\frac{1}{2}}$$

$$=1+O(\frac{|\overline{r}|}{|\overline{r}'|})$$

and, similarly;

$$\frac{|\overline{r}'|}{|\overline{r}'-\overline{r}|}=[1-\frac{2<\overline{r}',\overline{r}>-|\overline{r}|^2}{|\overline{r}'|^2}]^{\frac{-1}{2}}$$

$$=[1+O(\frac{|\overline{r}|}{|\overline{r}'|})]^{\frac{-1}{2}}$$

$$=1+O(\frac{|\overline{r}|}{|\overline{r}'|}),\ (*)$$

so that, using Cauchy-Schwartz, $(*)$, and Newton's Theorem again;

$$\frac{|\overline{r}'-\overline{r}|}{|\overline{r}-\frac{|\overline{r}'-\overline{r}|}{|\overline{r}'|}\overline{r}'|}$$

$$=\frac{|\overline{r}-\overline{r}'|}{|\overline{r}-\overline{r}'+O(\frac{|\overline{r}|}{|\overline{r}'|})\overline{r}'|}$$

$$=[1-\frac{2<\overline{r}-\overline{r}',\overline{r}'O(\frac{|\overline{r}|}{|\overline{r}'|})>-|\overline{r}'|^2O(\frac{|\overline{r}|^2}{|\overline{r}'|^2})}{|\overline{r}-\overline{r}'|^2}]^{\frac{-1}{2}}$$

$$=[1-2O(\frac{|\overline{r}|}{|\overline{r}'|})(1+O(\frac{|\overline{r}|}{|\overline{r}'|}))+O(\frac{|\overline{r}|^2}{|\overline{r}'|^2})(1+O(\frac{|\overline{r}|}{|\overline{r}'|}))^2]^{\frac{-1}{2}}$$

$$=[1+O(\frac{|\overline{r}|}{|\overline{r}'|})]^{\frac{-1}{2}}$$

$$=1+O(\frac{|\overline{r}|}{|\overline{r}'|})\ (DD)$$

It follows, from $(C),(DD)$, that;

$$\overline{r}'_{opp,opp}=\overline{r}-(1+O(\frac{|\overline{r}|}{|\overline{r}'|}))(\overline{r}-(1+O(\frac{|\overline{r}|}{|\overline{r}'|}))\overline{r}')$$

$$=-O(\frac{|\overline{r}|}{|\overline{r}'|})\overline{r}+(1+O(\frac{|\overline{r}|}{|\overline{r}'|}))^2\overline{r}'$$

so that;

$\frac{\overline{r}'_{opp,opp}-\overline{r}'}{|\overline{r}-\overline{r}'|} = -O(\frac{|\overline{r}|}{|\overline{r}'|})\frac{\overline{r}}{|\overline{r}-\overline{r}'|} + O(\frac{|\overline{r}|}{|\overline{r}'|})\frac{\overline{r}'}{|\overline{r}-\overline{r}'|}$

$= -O(\frac{|\overline{r}|}{|\overline{r}'|})^2 + O(\frac{|\overline{r}|}{|\overline{r}'|})(1+O(\frac{|\overline{r}|}{|\overline{r}'|}))$

$= O(\frac{|\overline{r}|}{|\overline{r}'|})$

□

Lemma 0.8. *Let $\overline{r}$ vary as (r,y,z) with y,z fixed. Then for $\overline{r}' \in V_{\overline{r}}$;*

$\frac{|(\overline{r}'-\overline{r}'_{opp})_1|}{|\overline{r}-\overline{r}'|} = O(\frac{1}{r})$.

Proof. First observe that, by the definition of opposites and the fact that $\overline{r}-\overline{d}(\overline{r}')$ is perpendicular to H_d;

$(\overline{r}'-\overline{r}'_{opp}) \cdot \frac{\overline{r}-\overline{d}(\overline{r}')}{|\overline{r}-\overline{d}(\overline{r}')|} = 0$

where $\overline{d}(\overline{r}') \in B(\overline{0},s)$. It follows that, using Cauchy Schwartz and the observations on opposites above, that;

$|(\overline{r}'-\overline{r}'_{opp})_1| = |(\overline{r}'-\overline{r}'_{opp}) \cdot \overline{e}_1|$

$= |(\overline{r}'-\overline{r}'_{opp}) \cdot (\overline{e}_1 - \frac{\overline{r}-\overline{d}(\overline{r}')}{|\overline{r}-\overline{d}(\overline{r}')})|$

$\leq |\overline{r}'-\overline{r}'_{opp}||\overline{e}_1 - \frac{\overline{r}-\overline{d}(\overline{r}')}{|\overline{r}-\overline{d}(\overline{r}')}|$

$= |(\overline{r}'-\overline{r})-(\overline{r}'_{opp}-\overline{r})||\overline{e}_1 - \frac{\overline{r}-\overline{d}(\overline{r}')}{|\overline{r}-\overline{d}(\overline{r}')}|$

$\leq 2|\overline{r}'-\overline{r}||\overline{e}_1 - \frac{\overline{r}-\overline{d}(\overline{r}')}{|\overline{r}-\overline{d}(\overline{r}')}|$

so it is sufficient to show that;

$|\overline{e}_1 - \frac{\overline{r}-\overline{d}(\overline{r}')}{|\overline{r}-\overline{d}(\overline{r}')}| = O(\frac{1}{r})$

We have that;

$\overline{e}_1 - \frac{\overline{r}-\overline{d}(\overline{r}')}{|\overline{r}-\overline{d}(\overline{r}')}$

$= (1,0,0) - \frac{1}{[(r-d_1)^2+(y-d_2)^2+(z-d_3)^2]^{\frac{1}{2}}}(r-d_1, y-d_2, z-d_3)$

$= \frac{1}{[(r-d_1)^2+(y-d_2)^2+(z-d_3)^2]^{\frac{1}{2}}}([(r-d_1)^2+(y-d_2)^2+(z-d_3)^2]^{\frac{1}{2}}$

$-(r-d_1), y-d_2, z-d_3)$

so then, using the fact that $|\overline{d}| \leq s$, uniformly in $\overline{r}'$, and Newton's theorem, for sufficiently large r;

$|\overline{e}_1 - \frac{\overline{r}-\overline{d}(\overline{r}')}{|\overline{r}-\overline{d}(\overline{r}')}|$

$\leq \frac{1}{[(r-d_1)^2+(y-d_2)^2+(z-d_3)^2]^{\frac{1}{2}}}(|[(r-d_1)^2+(y-d_2)^2+(z-d_3)^2]^{\frac{1}{2}}$

$-(r-d_1)|+|y-d_2|+|z-d_3|)$

$\leq \frac{C_{y,z}}{[(r-d_1)^2+(y-d_2)^2+(z-d_3)^2]^{\frac{1}{2}}} + \frac{|[(r-d_1)^2+(y-d_2)^2+(z-d_3)^2]^{\frac{1}{2}}-(r-d_1)|}{[(r-d_1)^2+(y-d_2)^2+(z-d_3)^2]^{\frac{1}{2}}}$

$\leq \frac{C_{y,z}(1+O(\frac{1}{r}))^{\frac{-1}{2}}}{r} + \frac{|r(1+O(\frac{1}{r}))^{\frac{1}{2}}-r+d_1|}{r(1+O(\frac{1}{r}))^{\frac{1}{2}}}$

$= \frac{C_{y,z}(1+O(\frac{1}{r}))}{r} + \frac{|r(1+O(\frac{1}{r}))-r+d_1|(1+O(\frac{1}{r}))}{r}$

$\leq \frac{D_{y,z}}{r} + \frac{|O(1)+d_1|(1+O(\frac{1}{r}))}{r}$

$\leq \frac{E_{y,z}}{r}$

where $\{C_{y,z}, D_{y,z}, E_{y,z}\}$ are constants depending on $\{y,z\}$.

□

Lemma 0.9. *Reynolds Transport Theorem for Hyperplanes and Spheres*

Let $f:\mathcal{R}^4 \rightarrow \mathcal{R}$ be a smooth function such that uniformly in $t \in \mathcal{R}$, $f(\overline{x},t)$ is supported on $B(\overline{0},s)$, for some fixed $s>0$. Let H_t be a variation of the hyperplane H_0, obtained by translating H_0 by the vector $t\overline{v}$. Then we have that;

$\frac{d}{dt}|_{t=0}\int_{H_t} f(\overline{x},t)dA$

$= \int_{H_0}\frac{\partial f}{\partial t}(\overline{x},0)dA + \int_{H_0}\bigtriangledown(f)(\overline{x},0)\centerdot\overline{v}dA$

Let δB_t be a variation of the sphere $\delta B_0 = \delta B(\overline{x}_0, ct_0)$, obtained by altering the radius of δB_0 to ct and keeping the centre $\overline{x}_0$ fixed. Then we have that;

$\frac{d}{dt}|_{t=0}\int_{\delta B_t} f(\overline{x},t)dS(t)$

$= \int_{\delta B_0} \frac{\partial f}{\partial t}(\overline{x},0)dS(0) + \int_{\delta B_0} \bigtriangledown(f)(\overline{x},0) \centerdot c\hat{\overline{n}}dS(0) + \int_{\delta B_0} \frac{2f(\overline{x},0)}{t_0}dS(0)$

where $\hat{\overline{n}}$ *is the unit normal to the sphere* δB_0*.*

For a variation δB_{-ct} *of* δB_0*, with* $t < 0$*, we have that;*

$\frac{d}{dt}|_{t=0}\int_{\delta B_{-ct}} f(\overline{x},t)dS(t)$

$= \int_{\delta B_0} \frac{\partial f}{\partial t}(\overline{x},0)dS(0) - \int_{\delta B_0} \bigtriangledown(f)(\overline{x},0) \centerdot c\hat{\overline{n}}dS(0) - \int_{\delta B_0} \frac{2f(\overline{x},0)}{t_0}dS(0)$

Proof. We have that;

$\frac{d}{dt}|_{t=0}\int_{H_t} f(\overline{x},t)dA$

$= lim_{h\rightarrow 0}\frac{1}{h}(\int_{H_h} f(\overline{x},h)dA - \int_{H_0} f(\overline{x},0)dA)$

$= lim_{h\rightarrow 0}\frac{1}{h}(\int_{H_h}(f(\overline{x},h)-f(\overline{x},0))dA+\int_{H_h} f(\overline{x},0)dA-\int_{H_0} f(\overline{x},0)dA)$

Checking the conditions of the Moore-Osgood Theorem, we have that;

(i). For h' fixed, using the DCT;

$lim_{h\rightarrow 0}\frac{1}{h}\int_{H_{h'}}(f(\overline{x},h) - f(\overline{x},0))dA = \int_{H_{h'}} \frac{\partial f}{\partial t}(\overline{x},0)dA$

(ii). For h fixed, using continuity;

$lim_{h'\rightarrow 0}\frac{1}{h}\int_{H_{h'}}(f(\overline{x},h) - f(\overline{x},0))dA = \int_{H_0} \frac{(f(\overline{x},h)-f(\overline{x},0))}{h}dA$

(iii). The limit in (i) is uniform in h', as;

$lim_{h\rightarrow 0}\frac{1}{h}(f(\overline{x},h) - f(\overline{x},0)) = \frac{\partial f}{\partial t}(\overline{x},0)$ uniformly in $\overline{x} \in B(\overline{0},s)$

because f is smooth, so $\frac{\partial^2 f}{\partial t^2}(\overline{x},t)$ is bounded on $B(\overline{0},s)\times(-\epsilon,\epsilon)$ and we can use the MVT. It follows that;

$lim_{h\rightarrow 0}\frac{1}{h}(\int_{H_h}(f(\overline{x},h) - f(\overline{x},0))dA$

$= lim_{h,h'\rightarrow 0}\frac{1}{h}(\int_{H_{h'}}(f(\overline{x},h)-f(\overline{x},0))dA$

$= lim_{h'\rightarrow 0}\int_{H_{h'}}\frac{\partial f}{\partial t}(\overline{x},0)dA$

$= \int_{H_0}\frac{\partial f}{\partial t}(\overline{x},0)dA$ $(*)$

We have that, using the fact that the Jacobian of a translation is the identity and Taylor's theorem;

$lim_{h\rightarrow 0}\frac{1}{h}(\int_{H_h} f(\overline{x},0)dA - \int_{H_0} f(\overline{x},0)dA)$

$= lim_{h\rightarrow 0}\frac{1}{h}(\int_{H_0}(f(\overline{x}+h\overline{v},0)-f(\overline{x},0))dA)$

$= lim_{h\rightarrow 0}\frac{1}{h}(\int_{H_0} f(\overline{x},0)+h\bigtriangledown(f)(\overline{x},0)\centerdot\overline{v}+O(h^2)-f(\overline{x},0)dA)$

$= lim_{h\rightarrow 0}\frac{1}{h}(\int_{H_0} h\bigtriangledown(f)(\overline{x},0)\centerdot\overline{v}+O(h^2)dA)$

$= lim_{h\rightarrow 0}(\int_{H_0}(\bigtriangledown(f)(\overline{x},0)\centerdot\overline{v}+O(h)dA)$

$= \int_{H_0}\bigtriangledown(f)(\overline{x},0)\centerdot\overline{v}dA$

so that we obtain the result.

For the second part, we follow the proof of the first part up to $(*)$, replacing H_t with B_t. Then, we have that, using the fact that the change of measure $dS(h)=\frac{4\pi c^2(t_0+h)^2}{4\pi c^2 t_0^2}dS_0=\frac{(t_0+h)^2}{t_0^2}dS_0$, and Taylor's theorem;

$lim_{h\rightarrow 0}\frac{1}{h}(\int_{\delta B_h} f(\overline{x},0)dS(h)-\int_{\delta B_0} f(\overline{x},0)dS(0))$

$= lim_{h\rightarrow 0}\frac{1}{h}(\int_{\delta B_0} f(\overline{x}+ch\frac{\overline{x}-\overline{x}_0}{|\overline{x}-\overline{x}_0|},0)\frac{(t_0+h)^2}{t_0^2}dS_0-\int_{\delta B_0} f(\overline{x},0)dS(0))$

$= lim_{h\rightarrow 0}\frac{1}{h}(\int_{\delta B_0}(f(\overline{x},0)+ch\bigtriangledown(f)(\overline{x},0)\centerdot\hat{\overline{n}}+O(h^2))(\frac{(t_0+h)^2}{t_0^2})-f(\overline{x},0)dS(0))$

$= lim_{h\rightarrow 0}\frac{1}{h}(\int_{\delta B_0} ch\bigtriangledown(f)(\overline{x},0)\centerdot\hat{\overline{n}}+O(h^2)dS(0))$

$+lim_{h\rightarrow 0}\frac{1}{h}\int_{\delta B_0}(f(\overline{x},0)+ch\bigtriangledown(f)(\overline{x},0)\centerdot\hat{\overline{n}}+O(h^2))(\frac{2t_0h+h^2}{t_0^2})dS_0$

$= \int_{\delta B_0}\bigtriangledown(f)(\overline{x},0)\centerdot c\hat{\overline{n}}dS(0)+\int_{\delta B_0}\frac{2f(\overline{x},0)}{t_0}dS(0)$

as required.

For the last part, we have that, with $s = -t$, $ds = -dt$, and using the second part;

$\frac{d}{dt}|_{t=0} \int_{\delta B_{-c(t-t_0)}} f(\overline{x}, t) dS(t)$

$= -\frac{d}{ds}|_{s=0} \int_{\delta B_{c(s+t_0)}} f(\overline{x}, -s) dS(-s)$

$= -[\int_{\delta B_0} \frac{\partial f}{\partial s}(\overline{x}, 0) dS(0) + \int_{\delta B_0} \bigtriangledown(f)(\overline{x}, 0) \centerdot c\hat{\overline{n}} dS(0) + \int_{\delta B_0} \frac{2f(\overline{x},0)}{t_0} dS(0)]$

$= -[\int_{\delta B_0} -\frac{\partial f}{\partial t}(\overline{x}, 0) dS(0) + \int_{\delta B_0} \bigtriangledown(f)(\overline{x}, 0) \centerdot c\hat{\overline{n}} dS(0) + \int_{\delta B_0} \frac{2f(\overline{x},0)}{t_0} dS(0)]$

$= \int_{\delta B_0} \frac{\partial f}{\partial t}(\overline{x}, 0) dS(0) - \int_{\delta B_0} \bigtriangledown(f)(\overline{x}, 0) \centerdot c\hat{\overline{n}} dS(0) - \int_{\delta B_0} \frac{2f(\overline{x},0)}{t_0} dS(0)]$

as required.

□

Lemma 0.10. *Convergence of opposite centres, angles and improved version of Lemma 0.7*

We have that;

$< \frac{\overline{r}'}{|\overline{r}'|}, \frac{\overline{r}'_{opp}}{|\overline{r}'_{opp}|} >= -1 + O(\frac{|\overline{r}|}{|\overline{r}'|})$

and;

$\overline{d}(\overline{r}') = \overline{d}(\overline{r}'_{opp}) + O(\frac{|\overline{r}|}{|\overline{r}'|}^{\frac{1}{2}})$

and;

$\overline{r}' = \overline{r}'_{opp} + O(\frac{|\overline{r}|}{|\overline{r}'|}^{\frac{1}{2}})$

Proof. For the first claim, by the definition of opposites, fixing $\overline{r}$, we have that;

$\overline{r}'_{opp} = \overline{r}' - 2(\overline{r}' - \overline{p_d})$

$= -\overline{r}' + 2\overline{p_d}$

where $d \in B(\overline{0}, s)$. Then, using the definition of $\overline{p_d}$, the calculation above and Newton's theorem;

$< \frac{\overline{r}'}{|\overline{r}'|}, \frac{\overline{r}'_{op}}{|\overline{r}'_{opp}|} >$

$= \frac{1}{|\overline{r}'||\overline{r}'_{opp}|} < \overline{r}', \overline{r}'_{opp} >$

$= \frac{1}{|\overline{r}'||\overline{r}'_{opp}|} < \overline{r}', -\overline{r}' + 2\overline{p_d} >$

$= -\frac{|\overline{r}'|^2}{|\overline{r}'||\overline{r}'_{opp}|} + 2\frac{<\overline{r}', \overline{p_d}>}{|\overline{r}'||\overline{r}'_{opp}|}$

$= -\frac{|\overline{r}'|}{|\overline{r}'_{opp}|} + \frac{<\overline{r}', \overline{d}+\overline{r}>}{|\overline{r}'||\overline{r}'_{opp}|}$

$= -\frac{\frac{|\overline{r}'|}{|\overline{r}'-\overline{d}}}{\frac{|\overline{r}'_{opp}|}{|\overline{r}'_{opp}-\overline{d}}} + \epsilon$

$= -\frac{(1+O(\frac{|\overline{d}|}{|\overline{r}'|}))}{1+O(\frac{|\overline{d}|}{|\overline{r}'_{opp}|})} + \epsilon$

$= -1 + O(\frac{1}{|\overline{r}'|}) + \epsilon$

as $\overline{d} \in B(\overline{0}, s)$ with s fixed, and, by Cauchy-Schwartz;

$|\epsilon| \leq \frac{|\overline{r}'||\overline{d}+\overline{r}|}{|\overline{r}'||\overline{r}'_{opp}|}$

$= \frac{|\overline{d}+\overline{r}|}{|\overline{r}'_{opp}|}$

$= O(\frac{|\overline{r}|}{|\overline{r}'_{opp}|})$

$= O(\frac{|\overline{r}|}{|\overline{r}'|})$

as;

$|\overline{r}'| = |\overline{r}' - \overline{d} + \overline{d}|$

$\leq ||\overline{r}' - \overline{d}| + |\overline{d}|$

$= |\overline{r}'_{opp} - \overline{d}| + |\overline{d}|$

$\leq |\overline{r}'_{opp}| + 2|\overline{d}|$

and similarly;

$|\overline{r}'_{opp}| \leq |\overline{r}'| + 2|\overline{d}|$

which gives the result.

For the second claim, let $l_{O,\overline{r}'}$ intersect $\delta B(\overline{0},1)$ at p and $l_{O,\overline{r}'_{opp}}$ intersect $\delta B(\overline{0},1)$ at q. Let S be a great circle containing the points $\{p,-p,q\}$ and suppose, without loss of generality, that p is situated at $(-1,0)$, $-p$ is situated at $(1,0)$ and q is situated at $(cos(\theta), sin(\theta))$, in coordinates on S. By the first result, we have that;

$< p, q >= -cos(\theta)$

$= -1 + O(\frac{|\overline{r}|}{|\overline{r}'|})$

so that;

$1 - cos(\theta) = O(\frac{|\overline{r}|}{|\overline{r}'|})$

and;

$|q - (-p)| = (sin^2(\theta) + (cos(\theta) - 1)^2)^{\frac{1}{2}}$

$= (1 - 2cos(\theta) + 1)^{\frac{1}{2}}$

$= \sqrt{2}(1 - cos(\theta))^{\frac{1}{2}}$

$= \sqrt{2}O(\frac{|\overline{r}|}{|\overline{r}'|}^{\frac{1}{2}})$

$= O(\frac{|\overline{r}|}{|\overline{r}'|}^{\frac{1}{2}})$

Now $l_{O,\overline{r}'}$ intersects $\delta B(\overline{0},w)$ at $d(\overline{r}')$, for some w with $0 < w \leq s$, so rescaling;

$|l_{O,\overline{r}'_{opp}} \cap B(\overline{0},w) - (-d\overline{r}')| = wO(\frac{|\overline{r}|}{|\overline{r}'|}^{\frac{1}{2}})$

$= O(\frac{|\overline{r}|}{|\overline{r}'|}^{\frac{1}{2}})$

By similar triangles, we have that $span(\overline{r}'_{opp})$ intersects $\delta B(\overline{0},w)$ at a further point $c(\overline{r}'_{opp})$ with;

$|c(\overline{r}'_{opp})-d(\overline{r}')|=O(\frac{|\overline{r}|}{|\overline{r}'|}^{\frac{1}{2}})$

Let $e(\overline{r}'_{opp})$ be the closest point on $span(\overline{r}'_{opp})$ to $d(\overline{r}')$, so that $l_{\overline{r}'_{opp},e(\overline{r}'_{opp})}$ is perpendicular to $l_{e(\overline{r}'_{opp}),d(\overline{r}')}$. As $c(\overline{r}'_{opp})\in span(\overline{r}'_{opp})$ as well, we must have that;

$|e(\overline{r}'_{opp})-d(\overline{r}')|\leq|c(\overline{r}'_{opp})-d(\overline{r}')|=O(\frac{|\overline{r}|}{|\overline{r}'|}^{\frac{1}{2}})$, $(\dagger)$

and, by Pythagoras' theorem;

$|d(\overline{r}'_{opp})-d(\overline{r}')|=(|e(\overline{r}'_{opp})-d(\overline{r}')|^2+|e(\overline{r}'_{opp})-d(\overline{r}'_{opp})|^2)^{\frac{1}{2}}$, (Q)

Let θ be the angle between $l_{\overline{r}'_{opp},d(\overline{r}')}$ and $l_{\overline{r}'_{opp},d(\overline{r}'_{opp})}$, then;

$|e(\overline{r}'_{opp})-d(\overline{r}'_{opp})|=|\overline{r}'_{opp}-d(\overline{r}')|(1-cos(\theta))$

$=|\overline{r}'_{opp}-\overline{r}|(1-cos(\theta))$

$=|\overline{r}'-\overline{r}|(1-cos(\theta))$, (R)

whereas, by $(\dagger)$;

$sin(\theta)=\frac{|e(\overline{r}'_{opp})-d(\overline{r}')|}{|\overline{r}'_{opp}-d(\overline{r}')|}$

$=\frac{O(\frac{|\overline{r}|}{|\overline{r}'|}^{\frac{1}{2}})}{|\overline{r}'_{opp}-\overline{r}|}$

$=\frac{O(\frac{|\overline{r}|}{|\overline{r}'|}^{\frac{1}{2}})}{|\overline{r}'-\overline{r}|}$

$=O(\frac{|\overline{r}|^{\frac{1}{2}}}{|\overline{r}'|^{\frac{3}{2}}})$

so that, using Newton's theorem;

$cos(\theta)=\sqrt{1-sin^2(\theta)}$

$=1+O(\frac{|\overline{r}|}{|\overline{r}'|^3})$

$1 - cos(\theta) = O(\frac{|\overline{r}|}{|\overline{r}'|^3})$

and, from (R);

$|e(\overline{r}'_{opp}) - d(\overline{r}'_{opp})|$

$= |\overline{r}' - \overline{r}| O(\frac{|\overline{r}|}{|\overline{r}'|^3})$

$= O(\frac{|\overline{r}|}{|\overline{r}'|^2})$

so that, using (Q);

$|d(\overline{r}'_{opp}) - d(\overline{r}')|$

$= (O(\frac{|\overline{r}|}{|\overline{r}'|}^{\frac{1}{2}})^2 + O(\frac{|\overline{r}|}{|\overline{r}'|^2})^2)^{\frac{1}{2}}$

$= O(\frac{|\overline{r}|}{|\overline{r}'|}^{\frac{1}{2}})$

as required.

For the final claim, we have that;

$\overline{r}' - \overline{r}'_{opp,opp} = \overline{r}' - (\overline{r}'_{opp} - 2(\overline{r}'_{opp} - \overline{p}_{d(\overline{r}'_{opp})})$

$= \overline{r}' - \overline{r}'_{opp} + 2(\overline{r}'_{opp} - \overline{p}_{d(\overline{r}'_{opp})})$

$= \overline{r}' - \overline{r}'_{opp} + 2\overline{r}'_{opp} - 2(\frac{d(\overline{r}'_{opp}) + \overline{r}}{2})$

$= \overline{r}' + \overline{r}'_{opp} - d(\overline{r}'_{opp}) - \overline{r}$

so that, rearranging;

$d(\overline{r}'_{opp}) = \overline{r}' + \overline{r}'_{opp} - \overline{r} - (\overline{r}' - \overline{r}'_{opp,opp})$

whereas;

$d(\overline{r}') = \overline{r}' - \frac{|\overline{r}' - \overline{r}|}{|\overline{r}'|} \overline{r}'$

It follows that, using the definition of opposites;

$d(\overline{r}')-d(\overline{r}'_{opp})=(\overline{r}'-\frac{|\overline{r}'-\overline{r}|}{|\overline{r}'|}\overline{r}')-(\overline{r}'+\overline{r}'_{opp}-\overline{r}-(\overline{r}'-\overline{r}'_{opp,opp}))$

$=\overline{r}-\overline{r}'_{opp}-\frac{|\overline{r}'-\overline{r}|}{|\overline{r}'|}\overline{r}'+(\overline{r}'-\overline{r}'_{opp,opp})$

$=\overline{r}-(\overline{r}'-2(\overline{r}'-\overline{p}_{d(\overline{r}')}))-\frac{|\overline{r}'-\overline{r}|}{|\overline{r}'|}\overline{r}'+(\overline{r}'-\overline{r}'_{opp,opp})$

$=\overline{r}-(-\overline{r}'+2\overline{p}_{d(\overline{r}')})-\frac{|\overline{r}'-\overline{r}|}{|\overline{r}'|}\overline{r}'+(\overline{r}'-\overline{r}'_{opp,opp})$

$=\overline{r}+\overline{r}'-2(\frac{d(\overline{r}')+\overline{r}}{2})-\frac{|\overline{r}'-\overline{r}|}{|\overline{r}'|}\overline{r}'+(\overline{r}'-\overline{r}'_{opp,opp})$

$=\overline{r}'-d(\overline{r}')-\frac{|\overline{r}'-\overline{r}|}{|\overline{r}'|}\overline{r}'+(\overline{r}'-\overline{r}'_{opp,opp})$

$=\overline{r}'-d(\overline{r}')-\frac{|\overline{r}'-d(\overline{r}')|}{|\overline{r}'|}\overline{r}'+(\overline{r}'-\overline{r}'_{opp,opp})$

$=\overline{r}'-\overline{r}'_{opp,opp}$

as $\overline{r}', d(\overline{r}')$ and $\overline{0}$ are collinear, so that;

$\overline{r}'-d(\overline{r}')-\frac{|\overline{r}'-d(\overline{r}')|}{|\overline{r}'|}\overline{r}'=\overline{0}$

Now, use the second result.

□

Lemma 0.11. *Wave Equation and Opposites*

Let ρ satisfy the wave equation, $\square^2(\rho)=0$, on $\mathcal{R}^4$, with compactly supported on $B(\overline{0},s)$ initial conditions $\{\rho_0,\dot{\rho}_0\}$, let $\overline{r}$ be fixed, and, with the above notation, let $\{\overline{x},\overline{x}_{opp}\}$ be a pair of opposites, then;

$\dot{\rho}(\overline{x},t_r)=O(\frac{1}{|\overline{x}-\overline{r}|^3})$

$\dot{\rho}(\overline{x},t_r)+\dot{\rho}(\overline{x}_{opp},t_r)=O(\frac{1}{|\overline{x}-\overline{r}|^3})O(\frac{|\overline{r}|}{|\overline{x}})$

where $t_r=-\frac{|\overline{x}-\overline{r}|}{c}$.

Proof. By Kirchoff's formula, for $t<0$;

$\rho(\overline{x},t)=\frac{1}{4\pi c^2t^2}\int_{\delta B(\overline{x},-ct)}(\rho_0-ct\dot{\rho}_0+D\rho_0\centerdot(\overline{y}-\overline{x}))dS(\overline{y})$

so that, using the last part of Lemma 0.9 and the product rule;

$\dot{\rho}(\overline{x},t) = [\frac{-1}{2\pi c^2 t^3}\int_{\delta B(\overline{x},-ct)}(\rho_0 - ct\dot{\rho}_0 + D\rho_0 \cdot (\overline{y}-\overline{x}))dS(\overline{y})$

$+\frac{1}{4\pi c^2 t^2}\int_{\delta B(\overline{x},-ct)}(-c\dot{\rho}_0)dS(\overline{y})$

$-2\frac{1}{4\pi c^2 t^3}\int_{\delta B(\overline{x},-ct)}(\rho_0 - ct\dot{\rho}_0 + D\rho_0 \cdot (\overline{y}-\overline{x}))dS(\overline{y})$

$-\frac{1}{4\pi c^2 t^2}\int_{\delta B(\overline{x},-ct)}(\bigtriangledown(\rho_0)\cdot c\hat{\overline{n}}+\bigtriangledown(-ct\dot{\rho}_0)\cdot c\hat{\overline{n}}+\bigtriangledown(D\rho_0\cdot(\overline{y}-\overline{x}))\cdot c\hat{\overline{n}})dS(\overline{y})]$

where $\hat{\overline{n}}$ is the unit normal to the sphere $\delta B(\overline{x},-ct)$. Similarly, making the substitution for the opposite;

$\dot{\rho}(\overline{x}_{opp},t) = [\frac{-1}{2\pi c^2 t^3}\int_{\delta B(\overline{x}_{opp},-ct)}(\rho_0 - ct\dot{\rho}_0 + D\rho_0 \cdot (\overline{y}-\overline{x}_{opp}))dS(\overline{y})$

$+\frac{1}{4\pi c^2 t^2}\int_{\delta B(\overline{x}_{opp},-ct)}(-c\dot{\rho}_0)dS(\overline{y})$

$-2\frac{1}{4\pi c^2 t^3}\int_{\delta B(\overline{x}_{opp},-ct)}(\rho_0 - ct\dot{\rho}_0 + D\rho_0 \cdot (\overline{y}-\overline{x}_{opp}))dS(\overline{y})$

$-\frac{1}{4\pi c^2 t^2}\int_{\delta B(\overline{x}_{opp},-ct)}(\bigtriangledown(\rho_0)\cdot c\hat{\overline{n}}_{opp}+\bigtriangledown(-ct\dot{\rho}_0)\cdot c\hat{\overline{n}}_{opp}+\bigtriangledown(D\rho_0\cdot(\overline{y}$

$-\overline{x}_{opp}))\cdot c\hat{\overline{n}}_{opp})dS(\overline{y})], (A)$

Let pr be the projection from the sphere $\delta B(\overline{x},-ct_r)$ onto the hyperplane $H_{d(\overline{x})}$ and let pr_{opp} be the projection from the sphere $\delta B(\overline{x}_{opp},-ct_r)$ onto the hyperplane $H_{d(\overline{x}_{opp})}$, let $d\mu(\overline{y}$ be Lebesgue measure on the hyperplanes, then by the change of variables formula, we have that;

$\dot{\rho}(\overline{x},t_r) = [\frac{-1}{2\pi c^2 t_r^3}\int_{H_{d(\overline{x})}} pr^{-1,*}(\rho_0 - ct_r\dot{\rho}_0 + D\rho_0 \cdot (\overline{y}-\overline{x}))d\mu(\overline{y})$

$+\frac{1}{4\pi c^2 t_r^2}\int_{H_{d(\overline{x})}} pr^{-1,*}(-c\dot{\rho}_0)d\mu(\overline{y})$

$-2\frac{1}{4\pi c^2 t_r^3}\int_{H_{d(\overline{x})}} pr^{-1,*}(\rho_0 - ct_r\dot{\rho}_0 + D\rho_0 \cdot (\overline{y}-\overline{x}))d\mu(\overline{y})$

$-\frac{1}{4\pi c^2 t_r^2}\int_{H_{d(\overline{x})}} pr^{-1,*}(\bigtriangledown(\rho_0)\cdot c\hat{\overline{n}}+\bigtriangledown(-ct_r\dot{\rho}_0)\cdot c\hat{\overline{n}}+\bigtriangledown(D\rho_0\cdot(\overline{y}-\overline{x}))\cdot$

$c\hat{\overline{n}})d\mu(\overline{y})]+\epsilon$

and;

$\dot{\rho}(\overline{x}_{opp},t_r) = [\frac{-1}{2\pi c^2 t_r^3}\int_{H_{d(\overline{x}_{opp})}}(\rho_0 - ct_r\dot{\rho}_0 + D\rho_0 \cdot (\overline{y}-\overline{x}_{opp}))d\mu(\overline{y})$

$+\frac{1}{4\pi c^2 t_r^2}\int_{H_{d(\overline{x}_{opp})}}(-c\dot{\rho}_0)d\mu(\overline{y})$

$-2\frac{1}{4\pi c^2t^3}\int_{H_{d(\overline{x}_{opp})}}(\rho_0-ct_r\dot{\rho}_0+D\rho_0\centerdot(\overline{y}-\overline{x}_{opp}))d\mu(\overline{y})$

$-\frac{1}{4\pi c^2t_r^2}\int_{H_{d(\overline{x}_{opp})}}(\bigtriangledown(\rho_0)\centerdot c\hat{\overline{n}}_{opp}+\bigtriangledown(-ct_r\dot{\rho}_0)\centerdot c\hat{\overline{n}}_{opp}+\bigtriangledown(D\rho_0\centerdot(\overline{y}$

$-\overline{x}_{opp}))\centerdot c\hat{\overline{n}}_{opp})d\mu(\overline{y})]+\epsilon_{opp}$ (B)

where ϵ and ϵ_{opp} are the error terms;

$\epsilon=[\frac{-1}{2\pi c^2t_r^3}\int_{H_{d(\overline{x})}}pr^{-1,*}(\rho_0-ct_r\dot{\rho}_0+D\rho_0\centerdot(\overline{y}-\overline{x}))(pr^{-1,*}(dS(\overline{y}))-d\mu(\overline{y}))$

$+\frac{1}{4\pi c^2t_r^2}\int_{H_{d(\overline{x})}}pr^{-1,*}(-c\dot{\rho}_0)(pr^{-1,*}(dS(\overline{y}))-d\mu(\overline{y}))$

$-2\frac{1}{4\pi c^2t_r^3}\int_{H_{d(\overline{x})}}pr^{-1,*}(\rho_0-ct_r\dot{\rho}_0+D\rho_0\centerdot(\overline{y}-\overline{x}))(pr^{-1,*}(dS(\overline{y}))-d\mu(\overline{y}))$

$-\frac{1}{4\pi c^2t_r^2}\int_{H_{d(\overline{x})}}pr^{-1,*}(\bigtriangledown(\rho_0)\centerdot c\hat{\overline{n}}+\bigtriangledown(-ct_r\dot{\rho}_0)\centerdot c\hat{\overline{n}}+\bigtriangledown(D\rho_0\centerdot(\overline{y}-\overline{x}))$

$\centerdot c\hat{\overline{n}})(pr^{-1,*}(dS(\overline{y}))-d\mu(\overline{y}))]$

and;

$\epsilon_{opp}=[\frac{-1}{2\pi c^2t_r^3}\int_{H_{d(\overline{x}_{opp})}}(\rho_0-ct_r\dot{\rho}_0+D\rho_0\centerdot(\overline{y}-\overline{x}_{opp}))(pr_{opp}^{-1,*}(dS(\overline{y}))$

$-d\mu(\overline{y}))$

$+\frac{1}{4\pi c^2t_r^2}\int_{H_{d(\overline{x}_{opp})}}(-c\dot{\rho}_0)(pr_{opp}^{-1,*}(dS(\overline{y}))-d\mu(\overline{y}))$

$-2\frac{1}{4\pi c^2t^3}\int_{H_{d(\overline{x}_{opp})}}(\rho_0-ct_r\dot{\rho}_0+D\rho_0\centerdot(\overline{y}-\overline{x}_{opp}))(pr_{opp}^{-1,*}(dS(\overline{y}))-d\mu(\overline{y}))$

$-\frac{1}{4\pi c^2t_r^2}\int_{H_{d(\overline{x}_{opp})}}(\bigtriangledown(\rho_0)\centerdot c\hat{\overline{n}}_{opp}+\bigtriangledown(-ct_r\dot{\rho}_0)\centerdot c\hat{\overline{n}}_{opp}+\bigtriangledown(D\rho_0\centerdot(\overline{y}$

$-\overline{x}_{opp}))\centerdot c\hat{\overline{n}}_{opp})(pr_{opp}^{-1,*}(dS(\overline{y}))-d\mu(\overline{y}))]$ (C)

We simplify the error term $\epsilon+\epsilon_{opp}$ first, recalling from Lemma **??** that;

$(pr^{-1,*}(dS(\overline{y}))-d\mu(\overline{y}))$ and $(pr_{opp}^{-1,*}(dS(\overline{y}))-d\mu(\overline{y}))$

are order $O(\frac{1}{|\overline{x}-\overline{r}|^2})$. We translate Tr the hyperplane $H_{d(\overline{x}_{opp})}$ to $H_{d(\overline{x})}$ by the vector $O(\frac{|\overline{r}|}{|\overline{x}|}^{\frac{1}{2}})$, recalling from Lemma 0.10 that $d(\overline{x})-d(\overline{x}_{opp})=$

$O(\frac{|\overline{r}|}{|\overline{x}|}^{\frac{1}{2}})$, and noting there is no change of measure

so that;

$\epsilon = O(\frac{1}{|\overline{x}-\overline{r}|^3})O(\frac{1}{|\overline{x}-\overline{r}|^2}) + O(\frac{1}{|\overline{x}-\overline{r}|^2})O(\frac{1}{|\overline{x}-\overline{r}|^2}) + O(\frac{1}{|\overline{x}-\overline{r}|^3})O(\frac{|\overline{x}|}{|\overline{x}-\overline{r}|^2})$

$-\frac{1}{4\pi c^2 t_r^2}\int_{H_{d(\overline{x})}}[pr^{-1,*}(\bigtriangledown(-ct_r\dot{\rho}_0)\centerdot c\hat{\overline{n}}+\bigtriangledown(D\rho_0\centerdot(\overline{y}-\overline{x}))\centerdot c\hat{\overline{n}})(pr^{-1,*}(dS(\overline{y}))$

$-d\mu(\overline{y}))]$

$= O(\frac{1}{|\overline{x}-\overline{r}|^4}) + O(\frac{|\overline{x}|}{|\overline{x}-\overline{r}|^5})$

$-\frac{1}{4\pi c^2 t_r^2}\int_{H_{d(\overline{x})}}[pr^{-1,*}(\bigtriangledown(-ct_r\dot{\rho}_0)\centerdot c\hat{\overline{n}}+\bigtriangledown(D\rho_0\centerdot(\overline{y}-\overline{x}))\centerdot c\hat{\overline{n}})(pr^{-1,*}(dS(\overline{y}))$

$-d\mu(\overline{y}))]$

$\epsilon_{opp} = O(\frac{1}{|\overline{x}-\overline{r}|^4}) + O(\frac{|\overline{x}_{opp}|}{|\overline{x}-\overline{r}|^5})$

$-\frac{1}{4\pi c^2 t_r^2}\int_{H_{d(\overline{x}_{opp})}}[pr_{opp}^{-1,*}(\bigtriangledown(-ct_r\dot{\rho}_0)\centerdot c\hat{\overline{n}}_{opp}+\bigtriangledown(D\rho_0\centerdot(\overline{y}-\overline{x}_{opp}))\centerdot c\hat{\overline{n}}_{opp})(pr_{opp}^{-1,*}(dS(\overline{y}))$

$-d\mu(\overline{y}))]$

$\epsilon + \epsilon_{opp} = O(\frac{1}{|\overline{x}-\overline{r}|^4}) + O(\frac{|\overline{x}|}{|\overline{x}-\overline{r}|^5}) + O(\frac{|\overline{x}_{opp}|}{|\overline{x}-\overline{r}|^5})$

$-\frac{1}{4\pi c^2 t_r^2}\int_{H_{d(\overline{x})}}[pr^{-1,*}(\bigtriangledown(-ct_r\dot{\rho}_0)\centerdot c\hat{\overline{n}}+\bigtriangledown(D\rho_0\centerdot(\overline{y}-\overline{x}))\centerdot c\hat{\overline{n}})(pr^{-1,*}(dS(\overline{y}))$

$-d\mu(\overline{y}))]$

$-\frac{1}{4\pi c^2 t_r^2}\int_{H_{d(\overline{x})}}[Tr^* pr_{opp}^{-1,*}(\bigtriangledown(-ct_r\dot{\rho}_0)\centerdot c\hat{\overline{n}}_{opp}+\bigtriangledown(D\rho_0\centerdot(\overline{y}-\overline{x}_{opp}))\centerdot c\hat{\overline{n}}_{opp})(pr_{opp}^{-1,*}(dS(\overline{y}))$

$-d\mu(\overline{y}))]$

By the proof of Lemma 0.10, we have that on the respective spheres restricted to $B(\overline{0}, s)$, that;

......

..........Strategy; Reduce to hyperplanes and cancel $\hat{\overline{n}}$ and $\hat{\overline{n}}_{opp}$ in pairs at t_r, using Lemma 0.10, obtain $\bigtriangledown(D(\rho)\centerdot\overline{x}-\overline{x}_{opp}$ in one term, use the extra derivative on ρ and Lemma **??** with the higher decay in the z

variable.

□

Lemma 0.12. *Let the fields* $\{\overline{E},\overline{B}\}$ *be as in Lemma* **??**, *then* $\overline{E}_0(\overline{r})$ *and* $\overline{B}_0(\overline{r})$ *are of moderate decrease in the sense of* [3]*?*

Moreover, the fields $\{\frac{\partial^{i+j+k}(\overline{E})}{\partial x^i\partial y^j\partial z^k},\frac{\partial^{i+j+k}(\overline{B})}{\partial x^i\partial y^j\partial z^k}\}$ *are of moderate decrease* $2+i+j+k$ *in the sense of* [3]*?*

Proof. ....Using the hyperbolic method and further refinements of the hyperplane method, keeping track of the change of measure.

□

References

[1] Elementary Differential Equations and Boundary Value Problems, Seventh Edition, W. Boyce and R. DiPrima, Wiley, (2001).

[2] A Fourier Inversion Theorem for Normal Functions, Tristram de Piro, available at http://www.curvalinea.net (77) or Accademia, (2024).

[3] Functions Analytic at Infinity and Normality, Tristram de Piro, available at http://www.curvalinea.net (77) or Accademia, (2024).

[4] Some Arguments for the Wave Equation in Quantum Theory, Tristram de Piro, Open Journal of Mathematical Sciences, also available at http://www.curvaline.net (58) or Accademia, (2021).

[5] Some Arguments for the Wave Equation in Quantum Theory 6: Waves, Current and Charge, Tristram de Piro, available at http://www.curvaline.net (70) or Accademia, (2022).

[6] Some Arguments for the Wave Equation in Quantum Theory 7: The Hyperbolic Method, Tristram de Piro, available at http://www.curvaline.net (70) or Accademia, (2022).

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