SOME ARGUMENTS FOR THE WAVE EQUATION IN QUANTUM THEORY 10

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Abstract. This paper continues the work in [4], in which it was proved that given $(\rho, \overline{J})$ satisfying the relations;

(i) $\square^2(\rho) = 0$

(ii) $\square^2(\overline{J}) = \overline{0}$

(iii) $\bigtriangledown(\rho) + \frac{1}{c^2}\frac{\partial \overline{J}}{\partial t} = \overline{0}$

with compact support, the fields $(\overline{E}, \overline{B})$ defined by Jefimenko's equations *exist*. We prove here that the fields $(\overline{E}, \overline{B})$ are quasi split normal, so that the methods of [1] apply.

Lemma 0.1. *For charge and current $(\rho, \overline{J})$, the relations;*

(i) $\square^2(\rho) = 0$

(ii) $\square^2(\overline{J}) = \overline{0}$

(iii) $\bigtriangledown(\rho) + \frac{1}{c^2}\frac{\partial \overline{J}}{\partial t} = \overline{0}$

(iv) $\bigtriangledown \times \overline{J} = \overline{0}$

are invariant under transformations of the base frame by a velocity vector $\overline{v}$, with $|\overline{v}| < c$.

Proof. The proof that $(i), (ii), (iv)$ hold for the transformed quantities $(\rho', \overline{J}')$ is done in [3]. We check that (iii) holds for the standard boost $v\overline{e}_1$, with $0 < v < c$. We have that;

$\rho' = \gamma_v(p - \frac{vj_1}{c^2})$

$\overline{J}' = \gamma_v(\overline{J}_{||} - \rho v\bar{e}_1) + \overline{J}_\perp$

$= (\gamma_v(j_1 - \rho v), j_2, j_3)$

$\frac{\partial}{\partial t'} = \gamma_v(\frac{\partial}{\partial t} + v\frac{\partial}{\partial x})$

$\bigtriangledown' = \gamma_v(\bigtriangledown_{||} + \frac{v}{c^2}\frac{\partial}{\partial t}) + \bigtriangledown_\perp$

$= (\gamma_v(\frac{\partial}{\partial x} + \frac{v}{c^2}\frac{\partial}{\partial t}), \frac{\partial}{\partial y}, \frac{\partial}{\partial z})$

so that;

$\bigtriangledown'(\rho') = (\gamma_v(\frac{\partial}{\partial x} + \frac{v}{c^2}\frac{\partial}{\partial t}), \frac{\partial}{\partial y}, \frac{\partial}{\partial z})(\gamma_v(p - \frac{vj_1}{c^2}))$

$= (\gamma_v^2\frac{\partial \rho}{\partial x} + \frac{\gamma_v^2 v}{c^2}\frac{\partial \rho}{\partial t} - \frac{\gamma_v^2 v}{c^2}\frac{\partial j_1}{\partial x} - \frac{\gamma_v^2 v^2}{c^4}\frac{\partial j_1}{\partial t}, \gamma_v\frac{\partial \rho}{\partial y} - \frac{\gamma_v v}{c^2}\frac{\partial j_1}{\partial y}, \gamma_v\frac{\partial \rho}{\partial z} - \frac{\gamma_v v}{c^2}\frac{\partial j_1}{\partial z})\ (A)$

while using (iii)

$(\frac{\partial \rho}{\partial x}, \frac{\partial \rho}{\partial y}, \frac{\partial \rho}{\partial z}) = -\frac{1}{c^2}(\frac{\partial j_1}{\partial t}, \frac{\partial j_2}{\partial t}, \frac{\partial j_3}{\partial t})$

and using (iv)

$\frac{\partial j_3}{\partial y} = \frac{\partial j_2}{\partial z}, \frac{\partial j_3}{\partial x} = \frac{\partial j_1}{\partial z}, \frac{\partial j_2}{\partial x} = \frac{\partial j_1}{\partial y}$

so that substituting into (A), we obtain;

$\bigtriangledown'(\rho') = (-\frac{\gamma_v^2}{c^2}\frac{\partial j_1}{\partial t} + \frac{\gamma_v^2 v}{c^2}\frac{\partial \rho}{\partial t} - \frac{\gamma_v^2 v}{c^2}\frac{\partial j_1}{\partial x} + \frac{\gamma_v^2 v^2}{c^2}\frac{\partial \rho}{\partial x}, -\frac{\gamma_v}{c^2}\frac{\partial j_2}{\partial t} - \frac{\gamma_v v}{c^2}\frac{\partial j_2}{\partial x}, -\frac{\gamma_v}{c^2}\frac{\partial j_3}{\partial t}$

$-\frac{\gamma_v v}{c^2}\frac{\partial j_3}{\partial x})\ (B)$

and;

$\frac{1}{c^2}\frac{\partial \overline{J}'}{\partial t'} = \frac{1}{c^2}\gamma_v(\frac{\partial}{\partial t} + v\frac{\partial}{\partial x})(\gamma_v(j_1 - \rho v), j_2, j_3)$

$= \frac{1}{c^2}(\gamma_v^2\frac{\partial j_1}{\partial t} + \gamma_v^2 v\frac{\partial j_1}{\partial x} - \gamma_v^2 v\frac{\partial \rho}{\partial t} - \gamma_v^2 v^2\frac{\partial \rho}{\partial x}, \gamma_v\frac{\partial j_2}{\partial t} + \gamma_v v\frac{\partial j_2}{\partial x}, \gamma_v\frac{\partial j_3}{\partial t} + \gamma_v v\frac{\partial j_3}{\partial x})$

(C)

so that;

$\bigtriangledown'(\rho') + \frac{1}{c^2}\frac{\partial \overline{J}'}{\partial t'} = \overline{0}$

as required.

$\square$

Lemma 0.2. *Let $\{\overline{E}, \overline{B}\}$ be the electric and magnetic fields defined from the charge and current $(\rho, \overline{J})$ satisfying the relations;*

(i) $\square^2(\rho) = 0$

(ii) $\square^2(\overline{J}) = \overline{0}$

(iii) $\bigtriangledown(\rho) + \frac{1}{c^2}\frac{\partial \overline{J}}{\partial t} = \overline{0}$

Then the fields $\overline{E}$ and $\overline{B}$ are smooth and of very moderate decrease.

Proof. The existence and smoothness of the fields follows from the results of [4]. In particular, we obtain that $\square^2(\overline{E}) = \overline{0}$, from the property (iii) and a result in [3]. We claim that the initial conditions $\overline{E}_0$ and $\frac{\partial \overline{E}}{\partial t}|_0$ are of very moderate decrease, and that the 9 components of $D(\overline{E})_0$ are of very moderate decrease. By Jefimenko's equations, noting that the retarded time $t_r = -\frac{|\overline{r}-\overline{r}'|}{c}$, and using (iii), we have that;

$\overline{E}_0(\overline{r}) = \frac{1}{4\pi\epsilon_0}\int_V [\frac{\rho(\overline{r}',t_r)}{|\overline{r}-\overline{r}'|^2}(\overline{r}-\overline{r}')\hat{} + \frac{\frac{\partial \rho}{\partial t}(\overline{r}',t_r)}{c|\overline{r}-\overline{r}'|}(\overline{r}-\overline{r}')\hat{} - \frac{\frac{\partial \overline{J}}{\partial t}(\overline{r}',t_r)}{c^2|\overline{r}-\overline{r}'|}]d\tau'$

$= \frac{1}{4\pi\epsilon_0}\int_V [\frac{\rho(\overline{r}',t_r)}{|\overline{r}-\overline{r}'|^2}(\overline{r}-\overline{r}')\hat{} + \frac{\frac{\partial \rho}{\partial t}(\overline{r}',t_r)}{c|\overline{r}-\overline{r}'|}(\overline{r}-\overline{r}')\hat{} + \frac{\bigtriangledown(p)(\overline{r}',t_r)}{|\overline{r}-\overline{r}'|}]d\tau'$

while by Kirchoff's formula, with the support of $\{\rho_0, \frac{\partial \rho}{\partial t}|_0, \bigtriangledown(\rho_0), \frac{\partial \bigtriangledown(\rho)}{\partial t}|_0\}$ supported on a ball $B(\overline{0}, s)$, we have that;

$\rho(\overline{r}', t_r) = \frac{1}{4\pi c^2 t_r^2}\int_{\delta B(\overline{r}', -ct_r)} -ct_r\frac{\partial \rho}{\partial t}|_0(\overline{y}) + \rho_0(y) + D\rho_0(\overline{y}) \centerdot (\overline{y}-\overline{r}')dS(\overline{y})$

$= \frac{1}{4\pi|\overline{r}-\overline{r}'|^2}\int_{\delta B(\overline{r}', |\overline{r}-\overline{r}'|)} |\overline{r}-\overline{r}'|\frac{\partial \rho}{\partial t}|_0(\overline{y}) + \rho_0(\overline{y}) + D\rho_0(\overline{y}) \centerdot (\overline{y}-\overline{r}')dS(\overline{y})$

$\frac{\partial \rho}{\partial t}(\overline{r}', t_r) = \frac{1}{4\pi|\overline{r}-\overline{r}'|^2}\int_{\delta B(\overline{r}', |\overline{r}-\overline{r}'|)} |\overline{r}-\overline{r}'|\frac{\partial^2 \rho}{\partial t^2}|_0(\overline{y}) + \frac{\partial \rho}{\partial t}|_0(\overline{y}) + D(\frac{\partial \rho}{\partial t}|_0)(\overline{y})$

$\centerdot (\overline{y}-\overline{r}')dS(\overline{y})$

$\bigtriangledown(p)(\overline{r}', t_r) = \frac{1}{4\pi|\overline{r}-\overline{r}'|^2}\int_{\delta B(\overline{r}', |\overline{r}-\overline{r}'|)} |\overline{r}-\overline{r}'|\frac{\partial \bigtriangledown(\rho)}{\partial t}|_0(\overline{y}) + \bigtriangledown(\rho_0)(\overline{y})$

$+D(\bigtriangledown(\rho_0))(\overline{y})\centerdot(\overline{y}-\overline{r}')dS(\overline{y})$

so that;

$\overline{E}_0(\overline{r})=\frac{1}{4\pi\epsilon_0}\int_V[\frac{(\frac{1}{4\pi|\overline{r}-\overline{r}'|^2}\int_{\delta B(\overline{r}',|\overline{r}-\overline{r}'|)}|\overline{r}-\overline{r}'|\frac{\partial\rho}{\partial t}|_0(\overline{y})+\rho_0(\overline{y})+D\rho_0(\overline{y})\centerdot(\overline{y}-\overline{r}')dS(\overline{y}))}{|\overline{r}-\overline{r}'|^2}(\overline{r}-\overline{r}')\hat{}$

$+\frac{(\frac{1}{4\pi|\overline{r}-\overline{r}'|^2}\int_{\delta B(\overline{r}',|\overline{r}-\overline{r}'|)}|\overline{r}-\overline{r}'|\frac{\partial^2\rho}{\partial t^2}|_0(\overline{y})+\frac{\partial\rho}{\partial t}|_0(\overline{y})+D(\frac{\partial\rho}{\partial t}|_0)(\overline{y})\centerdot(\overline{y}-\overline{r}')dS(\overline{y}))}{c|\overline{r}-\overline{r}'|}(\overline{r}-\overline{r}')\hat{}$

$+\frac{(\frac{1}{4\pi|\overline{r}-\overline{r}'|^2}\int_{\delta B(\overline{r}',|\overline{r}-\overline{r}'|)}|\overline{r}-\overline{r}'|\frac{\partial\bigtriangledown(\rho)}{\partial t}|_0(\overline{y})+\bigtriangledown(\rho_0)(\overline{y})+D(\bigtriangledown(\rho_0))(\overline{y})\centerdot(\overline{y}-\overline{r}')dS(\overline{y}))}{|\overline{r}-\overline{r}'|}]d\tau'$

$=\frac{1}{4\pi\epsilon_0}\int_V[(\frac{1}{4\pi|\overline{r}-\overline{r}'|^4}\int_{\delta B(\overline{r}',|\overline{r}-\overline{r}'|)}|\overline{r}-\overline{r}'|\frac{\partial\rho}{\partial t}|_0(\overline{y})+\rho_0(\overline{y})+D\rho_0(\overline{y})$

$\centerdot(\overline{y}-\overline{r}')dS(\overline{y}))(\overline{r}-\overline{r}')\hat{}$

$+(\frac{1}{4\pi c|\overline{r}-\overline{r}'|^3}\int_{\delta B(\overline{r}',|\overline{r}-\overline{r}'|)}|\overline{r}-\overline{r}'|\frac{\partial^2\rho}{\partial t^2}|_0(\overline{y})+\frac{\partial\rho}{\partial t}|_0(\overline{y})+D(\frac{\partial\rho}{\partial t})|_0(\overline{y})$

$\centerdot(\overline{y}-\overline{r}')dS(\overline{y}))(\overline{r}-\overline{r}')\hat{}$

$+(\frac{1}{4\pi|\overline{r}-\overline{r}'|^3}\int_{\delta B(\overline{r}',|\overline{r}-\overline{r}'|)}|\overline{r}-\overline{r}'|\frac{\partial\bigtriangledown(\rho)}{\partial t}|_0(\overline{y})+\bigtriangledown(\rho_0)(\overline{y})+D(\bigtriangledown(\rho_0))(\overline{y})$

$\centerdot(\overline{y}-\overline{r}')dS(\overline{y}))]d\tau'$

$=\overline{E}_{0,1}(\overline{r})+\overline{E}_{0,2}(\overline{r})+\overline{E}_{0,3}(\overline{r})$

where $V=\cup_{d\in B(\overline{0},s)}H_d$ and H_d is the hyperplane defined by;

$\{\overline{r}':|d-\overline{r}'|=|\overline{r}-\overline{r}'|\}$

We have that for $\overline{r}$ sufficiently large and;

$max_{B(\overline{0},s)}(|\frac{\partial\rho}{\partial t}|_0|,|\rho_0|,|D\rho_0|)\leq M$;

$|\overline{E}_{0,1}(\overline{r})|\leq|\frac{1}{4\pi\epsilon_0}\int_V[(\frac{1}{4\pi|\overline{r}-\overline{r}'|^4}\int_{\delta B(\overline{r}',|\overline{r}-\overline{r}'|)\cap B(\overline{0},s)}|\overline{r}-\overline{r}'|\frac{\partial\rho}{\partial t}|_0(\overline{y})+\rho_0(\overline{y})$

$+D\rho_0(\overline{y})\centerdot(\overline{y}-\overline{r}')dS(\overline{y}))(\overline{r}-\overline{r}')\hat{}]d\tau'|$

$\leq\frac{1}{4\pi\epsilon_0}\int_V[(\frac{4\pi Ms^2}{4\pi|\overline{r}-\overline{r}'|^3}+\frac{4\pi Ms^2}{4\pi|\overline{r}-\overline{r}'|^4}+\frac{4\pi Ms^2}{4\pi|\overline{r}-\overline{r}'|^3})]d\tau'$

$\leq\frac{1}{4\pi\epsilon_0}\frac{4\pi Ms^2}{4\pi}\int_V\frac{3}{|\overline{r}-\overline{r}'|^3}d\tau'$

$= \frac{3Ms^2}{4\pi}\int_V \frac{3}{|\overline{r}-\overline{r}'|^3}d\tau'$

$\leq \frac{3Ms^2}{4\pi}Vol(B(\overline{0},s))max_{d\in B(\overline{0},s)}\int_{H_d}\frac{d\tau'}{|\overline{r}-\overline{r}'|^3}$

$= \frac{3Ms^2}{4\pi}\frac{4\pi s^3}{3}max_{d\in B(\overline{0},s)}\int_{\mathcal{R}^2}\frac{dxdy}{(x^2+y^2+r_d^2)^{\frac{3}{2}}}$

$= \frac{12M\pi s^5}{12\pi}max_{d\in B(\overline{0},s)}2\pi\int_0^\infty \frac{RdR}{(R^2+r_d^2)^{\frac{3}{2}}}$

$= 2\pi Ms^5 max_{d\in B(\overline{0},s)}[-\frac{1}{(R^2+r_d^2)^{\frac{1}{2}}}]_0^\infty$

$= 2\pi Ms^5 max_{d\in B(\overline{0},s)}\frac{1}{r_d}$

where r_d is the shortest distance between $\overline{r}$ and H_d.

so that;

$|\overline{E}_{0,1}(\overline{r})| \leq \frac{2\pi Ms^5}{\frac{|\overline{r}|-s}{2}}$

$= \frac{4\pi Ms^5}{|\overline{r}|-s}$

$\leq 4\pi Ms^5\frac{2}{|\overline{r}|}\ (|\overline{r}| > 2s)$

$= \frac{8\pi Ms^5}{|\overline{r}|}$

so that $\overline{E}_{0,1}(\overline{r})$ is of very moderate decrease.

We have that;

$\overline{E}_{0,2}(\overline{r}) = \frac{1}{4\pi\epsilon_0}\int_V (C(\overline{r},\overline{r}'))(\overline{r}-\overline{r}')\hat{}d\tau'$

where, by the wave equation for ρ;

$C(\overline{r},\overline{r}') = \frac{1}{4\pi c|\overline{r}-\overline{r}'|^3}\int_{\delta B(\overline{r}',|\overline{r}-\overline{r}'|)\cap B(\overline{0},s)}|\overline{r}-\overline{r}'|\frac{\partial^2\rho}{\partial^2 t}|_0(\overline{y})+\frac{\partial\rho}{\partial t}|_0(\overline{y})+D\frac{\partial\rho}{\partial t}|_0(\overline{y})$

$\centerdot(\overline{y}-\overline{r}')dS(\overline{y})$

$= \frac{1}{4\pi c|\overline{r}-\overline{r}'|^3}\int_{\delta B(\overline{r}',|\overline{r}-\overline{r}'|)\cap B(\overline{0},s)}|\overline{r}-\overline{r}'|c^2\bigtriangledown^2\rho_0(\overline{y})+\frac{\partial\rho}{\partial t}|_0(\overline{y})+D(\frac{\partial\rho}{\partial t}|_0)(\overline{y})$

$\centerdot(\overline{y}-\overline{r}')dS(\overline{y})$

and;

$$|\overline{E}_{0,2}(\overline{r})| \leq \frac{1}{4\pi\epsilon_0}\int_V |D(\overline{r},\overline{r}')|d\tau' + \frac{1}{4\pi\epsilon_0}|\int_V E(\overline{r},\overline{r}')\hat{(\overline{r}-\overline{r}')}d\tau'| \ (AB)$$

where;

$$D(\overline{r},\overline{r}') = \frac{1}{4\pi c|\overline{r}-\overline{r}'|^3}\int_{\delta B(\overline{r}',|\overline{r}-\overline{r}'|)\cap B(\overline{0},s)} |\overline{r}-\overline{r}'|c^2 \bigtriangledown^2 \rho_0(\overline{y}) + \frac{\partial \rho}{\partial t}|_0(\overline{y})dS(\overline{y})$$

$$E(\overline{r},\overline{r}') = \frac{1}{4\pi c|\overline{r}-\overline{r}'|^3}\int_{\delta B(\overline{r}',|\overline{r}-\overline{r}'|)\cap B(\overline{0},s)} D(\frac{\partial \rho}{\partial t}|_0)(\overline{y}) \centerdot (\overline{y}-\overline{r}')dS(\overline{y})$$

We have that, using Lemma 0.4, that;

$$|D(\overline{r},\overline{r}')| \leq \frac{1}{4\pi c|\overline{r}-\overline{r}'|^3}(|\overline{r}-\overline{r}'|\frac{Cc^2}{|\overline{r}-\overline{r}'|} + 4\pi M s^2)$$

$$= \frac{Cc^2}{4\pi c|\overline{r}-\overline{r}'|^3} + \frac{4\pi M s^2}{4\pi c|\overline{r}-\overline{r}'|^3}$$

$$= \frac{Cc}{4\pi|\overline{r}-\overline{r}'|^3} + \frac{Ms^2}{c|\overline{r}-\overline{r}'|^3}$$

so that, using (AB) and following the method for $\overline{E}_{0,1}(\overline{r})$ above, we have that;

$$|\overline{E}_{0,2}(\overline{r})| \leq \frac{1}{4\pi\epsilon_0}(\frac{Cc}{4\pi} + \frac{Ms^2}{c})\int_V \frac{1}{|\overline{r}-\overline{r}'|^3}d\tau' + \frac{1}{4\pi\epsilon_0}|\int_V |E(\overline{r},\overline{r}')\hat{(\overline{r}-\overline{r}')}d\tau'|$$

$$\leq \frac{1}{4\pi\epsilon_0}(\frac{Cc}{4\pi} + \frac{Ms^2}{c})\frac{4\pi s^3}{3} max_{d\in B(\overline{0},s)}\frac{1}{r_d} + \frac{1}{4\pi\epsilon_0}|\int_V E(\overline{r},\overline{r}')\hat{(\overline{r}-\overline{r}')}d\tau'|$$

$$\leq \frac{H}{|\overline{r}|} + \frac{1}{4\pi\epsilon_0}|\int_V E(\overline{r},\overline{r}')\hat{(\overline{r}-\overline{r}')}d\tau' \ (GH)$$

for some constant $H \in \mathcal{R}_{>0}$. For the decay in the last term, we have that, using lemma 0.4 again;

$$|E(\overline{r},\overline{r}')| \leq \frac{1}{4\pi c|\overline{r}-\overline{r}'|^3}|\int_{\delta B(\overline{r}',|\overline{r}-\overline{r}'|)\cap B(\overline{0},s)} |D(\frac{\partial \rho}{\partial t}|_0)(\overline{y}) \centerdot \overline{y}|dS(\overline{y})$$

$$-\frac{1}{4\pi c|\overline{r}-\overline{r}'|^3}|\int_{\delta B(\overline{r}',|\overline{r}-\overline{r}'|)\cap B(\overline{0},s)} |D(\frac{\partial \rho}{\partial t}|_0)(\overline{y}) \centerdot \overline{r}'|dS(\overline{y})$$

$$\leq \frac{1}{4\pi c|\overline{r}-\overline{r}'|^3} sM area(\delta B(\overline{r}',|\overline{r}-\overline{r}'|)\cap B(\overline{0},s)) + \frac{1}{4\pi c|\overline{r}-\overline{r}'|^3}\sqrt{3}|\overline{r}'|\frac{C}{|\overline{r}-\overline{r}'|}$$

$$\leq \frac{sM.4\pi s^2}{4\pi c|\overline{r}-\overline{r}'|^3} + \frac{C\sqrt{3}|\overline{r}'|}{4\pi c|\overline{r}-\overline{r}'|^4}$$

$$= \frac{Ms}{c|\overline{r}-\overline{r}'|^3} + \frac{C\sqrt{3}|\overline{r}'|}{4\pi c|\overline{r}-\overline{r}'|^4}$$

so that, from (GH), and using the method above;

$|\overline{E}_{0,2}(\overline{r})| \leq \frac{H}{|\overline{r}|} + \frac{1}{4\pi\epsilon_0}\int_V |E(\overline{r},\overline{r}')|d\tau'$

$\leq \frac{H}{|\overline{r}|} + \frac{1}{4\pi\epsilon_0}\int_V (\frac{Ms}{c|\overline{r}-\overline{r}'|^3} + \frac{C\sqrt{3}|\overline{r}'|}{4\pi c|\overline{r}-\overline{r}'|^4})d\tau'$

$\leq \frac{H}{|\overline{r}|} + \frac{1}{4\pi\epsilon_0}\frac{4\pi s^3}{3} max_{d\in B(\overline{0},s)}\frac{1}{r_d} + \frac{1}{4\pi\epsilon_0}\int_V \frac{C\sqrt{3}|\overline{r}'-\overline{r}+\overline{r}|}{4\pi c|\overline{r}-\overline{r}'|^4}d\tau'$

$\leq \frac{K}{|\overline{r}|} + \frac{1}{4\pi\epsilon_0}\int_V \frac{C\sqrt{3}}{4\pi c|\overline{r}-\overline{r}'|^3}d\tau' + \frac{1}{4\pi\epsilon_0}\int_V \frac{C\sqrt{3}|\overline{r}|}{4\pi c|\overline{r}-\overline{r}'|^4}d\tau'$

$\leq \frac{L}{|\overline{r}|} + \frac{1}{4\pi\epsilon_0}\frac{C\sqrt{3}|\overline{r}|}{4\pi c}\int_V \frac{1}{|\overline{r}-\overline{r}'|^4}d\tau'$

$\leq \frac{L}{|\overline{r}|} + \frac{1}{4\pi\epsilon_0}\frac{C\sqrt{3}|\overline{r}|}{4\pi c}\int_V \frac{1}{|\overline{r}-\overline{r}'|^4}d\tau'$

$\leq \frac{L}{|\overline{r}|} + max_{d\in B(\overline{0},s)}\frac{N|\overline{r}|}{r_d^2}$

$\leq \frac{G}{|\overline{r}|}$

where $\{G,H,K,L,N\} \subset \mathcal{R}_{>0}$, so that $\overline{E}_{0,2}(\overline{r})$ is of very moderate decrease.

A similar argument establishes that $\overline{E}_{0,3}(\overline{r})$ is of very moderate decrease, using repeated application of lemma 0.4, so that $\overline{E}_0(\overline{r})$ is of very moderate decrease. By differentiating under the integral sign and using the chain rule, we have that;

$$\frac{\partial\overline{E}}{\partial t}|_0 = \frac{1}{4\pi\epsilon_0}\int_V [\frac{\frac{\partial\rho}{\partial t}(\overline{r}',t_r)}{|\overline{r}-\overline{r}'|^2}(\overline{r}-\overline{r}')\hat{} + \frac{\frac{\partial^2\rho}{\partial t^2}(\overline{r}',t_r)}{c|\overline{r}-\overline{r}'|}(\overline{r}-\overline{r}')\hat{} + \frac{\bigtriangledown(\frac{\partial\rho}{\partial t})(\overline{r}',t_r)}{|\overline{r}-\overline{r}'|}]d\tau'$$

We have that $\{\frac{\partial\rho}{\partial t}, \frac{\partial^2\rho}{\partial^2 t}, \bigtriangledown(\frac{\partial\rho}{\partial t})\}$ satisfy the wave equation, with the initial conditions determined by the corresponding derivatives of the initial conditions $\{\rho_0, \frac{\partial\rho}{\partial t}|_0\}$, noting for example that the initial conditions of $\frac{\partial^2\rho}{\partial^2 t}$ are $\frac{\partial^2\rho}{\partial^2 t}|_0 = c^2\bigtriangledown^2(\rho)|_0$ and $\frac{\partial^3\rho}{\partial^3 t}|_0 = c^2\bigtriangledown^2(\frac{\partial\rho}{\partial t}|_0)$ both of which have compact support. We can then use Kirchoff's formula, as above, and establish the very moderate decrease of $\frac{\partial\overline{E}}{\partial t}|_0(\overline{r})$.

Again, differentiating under the integral sign and using the chain rule, we have that, for example;

$$\frac{\partial\overline{E}}{\partial x}|_0 = \frac{1}{4\pi\epsilon_0}\int_V [-\frac{1}{c}\frac{\rho_t(\overline{r}',t_r)}{|\overline{r}-\overline{r}'|^3}(\overline{r}-\overline{r}')\hat{}(r_1-r_1') + \frac{\rho(\overline{r}',t_r)}{|\overline{r}-\overline{r}'|^3}(1,0,0) - 3\frac{\rho(\overline{r}',t_r)}{|\overline{r}-\overline{r}'|^4}(\overline{r}-\overline{r}')\hat{}$$

$(r_1-r_1')-\frac{1}{c}\frac{\rho_{tt}(\overline{r}',t_r)}{|\overline{r}-\overline{r}'|^2}(\overline{r}-\overline{r}')\hat{}(r_1-r_1')+\frac{\rho_t(\overline{r}',t_r)}{|\overline{r}-\overline{r}'|^2}(1,0,0)-2\frac{\rho_t(\overline{r}',t_r)}{|\overline{r}-\overline{r}'|^3}(\overline{r}-\overline{r}')\hat{}$

$(r_1-r_1')-\frac{1}{c}\frac{\bigtriangledown(\rho_t)(\overline{r}',t_r)}{|\overline{r}-\overline{r}'|^2}(r_1-r_1')-\frac{\bigtriangledown(\rho)(\overline{r}',t_r)}{|\overline{r}-\overline{r}'|^3}(r_1-r_1')]d\tau'$

We can then use the same reasoning as above, that $\{\frac{\partial\rho}{\partial t},\frac{\partial^2\rho}{\partial^2 t},\bigtriangledown(\frac{\partial\rho}{\partial t})\}$ satisfy the wave equation, and use Kirchoff's formula, noting that the decay in $\overline{r}'$, for fixed $\overline{r}$, is greater than in previous cases, to establish the very moderate decrease of $\frac{\partial\overline{E}}{\partial x}|_0$, and similarly for $\{\frac{\partial\overline{E}}{\partial y}|_0,\frac{\partial\overline{E}}{\partial z}|_0\}$. Finally, we use the vector version of Kirchoff's formula for $\overline{E}$, to obtain for $t>0$;

$$\overline{E}(\overline{r},t)=\frac{1}{4\pi c^2t^2}\int_{\delta B(\overline{r},ct)}ct\frac{\partial\overline{E}}{\partial t}|_0(\overline{y})+\overline{E}_0(y)+D\overline{E}_0(\overline{y})\centerdot(\overline{y}-\overline{r})dS(\overline{y})$$

and, for $t<0$;

$$\overline{E}(\overline{r},t)=\frac{1}{4\pi c^2t^2}\int_{\delta B(\overline{r},-ct)}-ct\frac{\partial\overline{E}}{\partial t}|_0(\overline{y})+\overline{E}_0(y)+D\overline{E}_0(\overline{y})\centerdot(\overline{y}-\overline{r})dS(\overline{y})$$

and we can see that, for $t>0$, sufficiently large $|\overline{r}|$, depending on t;

$$|\overline{E}(\overline{r},t)\leq\frac{1}{4\pi c^2t^2}4\pi c^2t^2(ct\frac{C_2}{|\overline{r}|-ct}+\frac{C_1}{|\overline{r}|-ct}+ct\frac{9C_3}{|\overline{r}|-ct})$$

$$\leq\frac{D_t}{|\overline{r}|}$$

for some $\{C_1,C_2,C_3,D_t\}\subset\mathcal{R}_{>0}$, where $\frac{\partial\overline{E}}{\partial t}|_0$ is of very moderate decrease C_2, $\overline{E}_0$ is of very moderate decrease C_1 and $C_3=max_{1\leq i,j,\leq 3}C_{ij}$, where $\frac{\partial e_i}{\partial x_j}$ is of very moderate decrease $C_{ij}\in\mathcal{R}_{>0}$, so that $\overline{E}(\overline{r},t)$ is of very moderate decrease for $t>0$. Similarly, $\overline{E}(\overline{r},t)$ is of very moderate decrease for $t<0$. The proof for $\overline{B}$ is similar, using the fact that $\bigtriangledown\times\overline{J}=\overline{0}$, so that $\square^2(\overline{B})=\overline{0}$, and using Jefimenko's formula for $\overline{B}$. $\square$

Lemma 0.3. *Addendum to Uniqueness of Representation of Arcs Lemma*

Given $\overline{a}\in B(\overline{0},s)$, with $\overline{a}\neq\overline{0}$, there exists, up to a set Bl of measure zero in $B(\overline{0},s)$, a unique $\overline{v}\in V=\bigcup_{d\in B(\overline{0},s)}H_d$, such that $B(\overline{v},|\overline{v}-\overline{r}|)$ passes through $\overline{a}$, with $B(\overline{0},|\overline{a}|)$ and $B(\overline{v},|\overline{v}-\overline{r}|)$ sharing a common tangent plane at $\overline{a}$. It follows that we can define a map $\gamma:B(\overline{0},s)\setminus Bl\rightarrow V\setminus H_{\overline{0}}$ which is a homeomorphism onto its image.

Proof. The proof is straightforward, given a generic $\overline{a}\neq\overline{0}$, the line $l_{0,\overline{a}}$ intersects the hyperplane $H_{\overline{a}}$ in a unique point $\overline{v}$, unless $l_{0,\overline{a}}$ and $H_{\overline{a}}$ are parallel, in which case $\overline{a}\centerdot(\overline{r}-\overline{a})=0$. Letting $\overline{r}=(r_1,r_2,r_3)$, this locus is defined by;

$a_1r_1 + a_2r_2 + a_3r_3 - (a_1^2 + a_2^2 + a_3^2) = 0$

iff $a_1^2 - a_1r_1 + a_2^2 - a_2r_2 + a_3^2 - a_3r_3 = 0$

iff $(a_1 - \frac{r_1}{2})^2 + (a_2 - \frac{r_2}{2})^2 + (a_3 - \frac{r_3}{2})^2 = \frac{r_1^2}{4} + \frac{r_2^2}{4} + \frac{r_3^2}{4}$

which is a sphere centred at $\frac{\overline{r}}{2}$, with radius $\frac{|\overline{r}|}{2}$. Clearly, the intersection of this sphere with $B(\overline{0}, s)$, Bl, is a set of measure zero in $B(\overline{0}, s)$.

For $\overline{y} \in B(\overline{0}, s)$, with $|\overline{y}| = w$, $0 < w \leq s$, we have that, for $\lambda \in \mathcal{R}$;

$|\lambda\overline{y} - \overline{y}| = |\lambda\overline{y} - \overline{r}|$

iff $w|\lambda - 1| = |\lambda\overline{y} - \overline{r}|$

iff $w^2(\lambda - 1)^2 = (\lambda y_1 - r_1)^2 + (\lambda y_2 - r_2)^2 + (\lambda y_3 - r_3)^2$

iff $\lambda^2w^2 - 2\lambda w^2 + w^2 = \lambda^2w^2 - 2\lambda\overline{y} \centerdot \overline{r} + |\overline{r}|^2$

iff $\lambda(-2w^2 + 2\overline{y} \centerdot \overline{r}) = |\overline{r}|^2 - w^2$

iff $\lambda = \frac{|\overline{r}|^2 - w^2}{-2w^2 + 2\overline{y} \centerdot \overline{r}}$

The exceptional locus $Bl \cap \delta B(\overline{0}, w)$ corresponds to the locus;

$2\overline{y} \centerdot \overline{r} - 2w^2 = 0$

iff $\overline{y} \centerdot \overline{r} = w^2$

which is a plane intersecting the sphere $\delta B(\overline{0}, w)$ in a circle C_w. We define the map γ, for $\overline{y} \in B(\overline{0}, s) \setminus Bl$ by;

$\gamma(\overline{y}) = \frac{|\overline{r}|^2 - |\overline{y}|^2}{-2|\overline{y}|^2 + 2\overline{y} \centerdot \overline{r}}\overline{y}$

The fact that γ is bijective and onto $V \setminus H_0$ follows from the original uniqueness of representation of arcs lemma in [4], noting that we excluded the case that an arc passed through the origin $\overline{0} \in B(\overline{0}, s)$. The above argument allows us to define the map γ, which is continuous with a continuous inverse.

$\square$

Lemma 0.4. *First Corollary to Hyperplane Lemma*

We have that there exists $C \in \mathcal{R}_{>0}$ such that for f smooth and having compact support on $B(\overline{0}, s)$;

$|\int_{\delta B(\overline{r}', |\overline{r}-\overline{r}'|) \cap B(\overline{0},s)} \bigtriangledown(f)(\overline{y}) dS(\overline{y})| \leq \frac{C}{|\overline{r}-\overline{r}'|}$

for $\overline{r}$ disjoint from $B(\overline{0}, 3s)$.

Proof. The result is obvious if the arc $\delta B(\overline{r}', |\overline{r} - \overline{r}'|)$ is disjoint from $B(\overline{0}, s)$, so we can assume that there exists $d \in B(\overline{0}, s)$, with $d \in \delta B(\overline{r}', |\overline{r} - \overline{r}'|)$, in particularly;

$|d - \overline{r}'| = |\overline{r} - \overline{r}'|$

so that $\overline{r}' \in H_d$ and H_d is disjoint from $B(\overline{0}, s)$, otherwise we could find $d' \in B(\overline{0}, s)$ with $|d - d'| = |\overline{r} - d'|$, so that $|\overline{r} - d'| < 2s$ contradicting the hypothesis that $\overline{r}$ is disjoint from $B(\overline{0}, 3s)$.

For $\overline{r}' \in H_d$, $d \in B(\overline{0}, s)$, we can use the representation of arcs lemma, see [4], to show that without loss of generality, $l_{\overline{r}',d}$ passes through the origin of $B(\overline{0}, s)$. We let T_d be the tangent plane to $\delta B(\overline{r}', |\overline{r} - \overline{r}'|)$ at d, so that by the hyperplane lemma in [4], ([1]), we have that;

$\int_{T_d \cap B(\overline{0},s)} \bigtriangledown(f)(\overline{y}) d\mu = 0$

where $d\mu$ is Lebesgue measure. Changing coordinates, let $\overline{r}'$ have coordinates $(0, 0, R)$, where $R = |\overline{r} - \overline{r}'|$, let d have coordinates $(0, 0, 0)$, and let T_d correspond to the plane $z = 0$. The hyperplane T_d intersects $\delta B(\overline{0}, s)$ in a circle S of radius $w \leq s$, $S \subset z = 0$ and we have that the original ball $B(\overline{0}, s)$ shifts to $B(\overline{a}, s)$, where $\overline{a} = (0, 0, a)$ and $a \leq s$ with;

$\int_{z=0 \cap B(\overline{a},s)} \bigtriangledown(f)(\overline{y}) d\mu = 0 \ (*)$

Let $\delta B(\overline{r}', R)$ intersect $\delta B(\overline{a}, s)$ in the circle T of radius $w \leq w' \leq s$, and suppose that T is centred at $(0, 0, b)$ with $b \leq a$. We have that;

[1] There we assumed that the hyperplane passed through the origin of a ball B; we can always make this assumption by choosing the ball B to contain the original support V, setting f to be zero on $B \setminus V$ and centering B at a point on the hyperplane.

$b = R - Rcos(\theta)$

where $\theta \leq \phi$ and $tan(\phi) = \frac{s}{R-a}$, with $a \leq s$. For R sufficiently large, $R \geq 2a$;

$tan(\phi) \leq \frac{s}{R-\frac{R}{2}} = \frac{2s}{R}$

and as tan^{-1} is increasing, cos is decreasing for small $\theta > 0$, using Newton's theorem;

$\theta \leq \phi \leq tan^{-1}(\frac{2s}{R}$

$cos(\theta) \geq cos(tan^{-1}(\frac{2s}{R}))$

$b \leq R - Rcos(tan^{-1}(\frac{2s}{R}))$

$= R - \frac{R}{\sqrt{1+(\frac{2s}{R})^2}}$

$= R - R(1 - \frac{1}{2}(\frac{2s}{R})^2 + O(\frac{1}{R^4}))$

$= \frac{2s^2}{R} + O(\frac{1}{R^3})$

$\leq \frac{E}{R}$ $(**)$

for some $E \in \mathcal{R}_{>0}$, $R > 1$. Let pr be the projection from $\mathcal{R}^3$ to $z = 0$ restricted to $\delta B(\overline{r}', R) \cap B(\overline{a}, s)$, then, using $(*), (**)$;

$|\int_{\delta B(\overline{r}',R)\cap B(\overline{a},s)} \bigtriangledown(f)(\overline{y})dS(\overline{y})|$

$= |\int_{\delta B(\overline{r}',R)\cap B(\overline{a},s)} \bigtriangledown(f)(\overline{y})dS(\overline{y}) - \int_{z=0\cap B(\overline{a},s)} \bigtriangledown(f)(\overline{x})d\mu(\overline{x})|$

$= |\int_{z=0\cap B(\overline{a},s)} pr^{-1,*} \bigtriangledown (f)(\overline{x})dpr^{-1,*}S(\overline{x}) - \int_{z=0\cap B(\overline{a},s)} \bigtriangledown(f)(\overline{x})d\mu(\overline{x})|$

$\leq |\int_{z=0\cap B(\overline{a},s)}(pr^{-1,*} \bigtriangledown (f)(\overline{x}) - \bigtriangledown(f)(\overline{x}))d\mu(\overline{x})|$

$+|\int_{z=0\cap B(\overline{a},s)} pr^{-1,*} \bigtriangledown (f)(\overline{x})d(pr^{-1,*}S(\overline{x})) - d\mu(\overline{x})|$ (HH)

For $\{\overline{x}, y\} \subset B(0, s)\}$, we have, by the MVT, that;

$\bigtriangledown(f)(\overline{x} - \bigtriangledown(\overline{y})| \leq (|D(\bigtriangledown(f))_1| + |D(\bigtriangledown(f))_2| + |D(\bigtriangledown(f))_3|)|\overline{x} = \overline{y}|$

$\leq 3max_{\bar{b}\in B(\bar{0},s)}||D(\bigtriangledown(f))(\bar{b})|||\bar{x}-\bar{y}|$

$= 3M|\bar{x}-\bar{y}|$

as $\bigtriangledown(f)$ is smooth with compact support, so that, using $(**)$, for $\bar{x}\in z=0\cap B(\bar{a},s)$, we have that;

$|pr^{-1,*}\bigtriangledown(f)(\bar{x})-\bigtriangledown(f)(\bar{x})|\leq \frac{3ME}{R}$

and;

$|\int_{z=0\cap B(\bar{a},s)}(pr^{-1,*}\bigtriangledown(f)(\bar{x})-\bigtriangledown(f)(\bar{x}))d\mu(\bar{x})|$

$\leq \frac{3ME\pi s^2}{R}$

To compute the change of measure, we use the parametrisation $pr^{-1}: z=0\rightarrow \delta B(\bar{r},R)$;

$pr^{-1}(x,y)=(x,y,R-(R^2-x^2-y^2)^{\frac{1}{2}})$

so that, using Newton's theorem;

$pr_x^{-1}=(1,0,\frac{x}{(R^2-x^2-y^2)^{\frac{1}{2}}})$

$pr_y^{-1}=(0,1,\frac{y}{(R^2-x^2-y^2)^{\frac{1}{2}}})$

$|d(pr^{-1,*}S(\bar{x}))-d\mu(\bar{x})|=|(\frac{1}{|pr_x^{-1}\times pr_y^{-1}|}-1)|dxdy$

$=|\frac{1}{|(-\frac{x}{(R^2-x^2-y^2)^{\frac{1}{2}}},-\frac{y}{(R^2-x^2-y^2)^{\frac{1}{2}}},1)|}-1|dxdy$

$=|\frac{1}{(1+\frac{x^2+y^2}{R^2-x^2-y^2})^{\frac{1}{2}}}-1|dxdy$

$=|1-\frac{1}{2}\frac{x^2+y^2}{R^2-x^2-y^2}+O(\frac{1}{R^4})-1|dxdy$

$=[\frac{1}{2}\frac{x^2+y^2}{R^2-x^2-y^2}+O(\frac{1}{R^4})]dxdy$

$\leq [\frac{s^2}{R^2-2s^s}+O(\frac{1}{R^4})]dxdy$

$\leq \frac{F}{R^2}dxdy$

for some $F \in \mathcal{R}_{>0}$, so that;

$$|\int_{z=0\cap B(\overline{a},s)} pr^{-1,*} \bigtriangledown (f)(\overline{x})d(pr^{-1,*}S(\overline{x})) - d\mu(\overline{x})|$$

$$\leq \frac{\pi s^2 F}{R^2} max_{\overline{b}\in B(\overline{0},s)}| \bigtriangledown (f)(\overline{b})|$$

$$= \frac{H\pi s^2 F}{R^2}$$

and from (HH);

$$|\int_{\delta B(\overline{r}',R)\cap B(\overline{a},s)} \bigtriangledown(f)(\overline{y})dS(\overline{y})| \leq \frac{3ME\pi s^2}{R} + \frac{H\pi s^2 F}{R^2}$$

$$\leq \frac{C}{R}$$

$\square$

Lemma 0.5. *Second Corollary to Hyperplane Lemma*

We have that there exists $C \in \mathcal{R}_{>0}$ such that for f smooth and having compact support on $B(\overline{0}, s)$;

$$|| \int_{\delta B(\overline{r}',|\overline{r}-\overline{r}'|)\cap B(\overline{0},s)} D(\bigtriangledown(f))(\overline{y})dS(\overline{y})|| \leq \frac{C}{|\overline{r}-\overline{r}'|^2}$$

for $\overline{r}$ disjoint from $B(\overline{0}, 3s)$.

Proof. We use the notation in Lemma 0.4. We let;

$$h(x, y) = \sqrt{R^2 + x^2 + y^2} - R = O(\tfrac{1}{R})$$

be defined on T_d correspond to the plane $z = 0$. Using similar triangles, $b(x, y) = h(x, y)cos(\theta)$, where θ is the angle subtended by a line drawn from $\overline{r}'$ to the point (x, y) in the plane $z = 0$. We have that;

$$b(x, y) - h(x, y) = h(x, y)(cos(\theta) - 1)$$

$$= h(x, y)(1 + O(\theta^2) - 1)$$

$$= h(x, y)(O(\tfrac{1}{R^2}))$$

$$= O(\tfrac{1}{R^3})$$

using the fact, proved above, that $\theta=O(\frac{1}{R})$. By Taylor's theorem, we have that, for $(x,y)\in T_d$;

$pr^{-1,*}(D(\bigtriangledown(f)))(x,y)=D(\bigtriangledown(f))(x,y,b(x,y))$

$=D(\bigtriangledown(f))(x,y,0)+b(x,y)\frac{\partial}{\partial z}D(\bigtriangledown(f))(x,y,0)+O(b(x,y)^2)$

$=D(\bigtriangledown(f))(x,y,0)+b(x,y)\frac{\partial}{\partial z}D(\bigtriangledown(f))(x,y,0)+O(\frac{1}{R^2})$

so that;

$pr^{-1,*}(D(\bigtriangledown(f)))(x,y)=D(\bigtriangledown(f))(x,y,0)+h(x,y)\frac{\partial}{\partial z}D(\bigtriangledown(f))(x,y,0)$

$+(b(x,y)-h(x,y))D(\bigtriangledown(f))(x,y,0)+O(\frac{1}{R^2})$

$=D(\bigtriangledown(f))(x,y,0)+h(x,y)\frac{\partial}{\partial z}D(\bigtriangledown(f))(x,y,0)+O(\frac{1}{R^3})+O(\frac{1}{R^2})$

$=D(\bigtriangledown(f))(x,y,0)+h(x,y)\frac{\partial}{\partial z}D(\bigtriangledown(f))(x,y,0)+O(\frac{1}{R^2})$ (AA)

We can now vary R as the coordinate z, and extend $h(x,y)$ to a function $\sqrt{x^2+y^2+z^2}-z$ on $B(\overline{0},s)$. We can then use the original hyperplane lemma to justify integration by parts on the plane T_d, so that;

$\int_{z=0\cap B(\overline{a},s)}D(\bigtriangledown(f))(x,y,0)d\mu(x,y)=\overline{0}$ (BB)

where $\overline{0}$ is the three by three zero matrix, and;

$\int_{z=0\cap B(\overline{a},s)}h(x,y,0)\frac{\partial}{\partial z}D(\bigtriangledown(f))(x,y,0)d\mu(x,y)$

$=\int_{z=0\cap B(\overline{a},s)}\frac{\partial}{\partial z}D(\bigtriangledown(h))(x,y,0)f(x,y,0)d\mu(x,y)$ (CC)

We have that;

$\frac{\partial h}{\partial z}=\frac{z}{\sqrt{x^2+y^2+z^2}}-1=O(1)$

$\frac{\partial^2 h}{\partial z^2}=\frac{1}{\sqrt{x^2+y^2+z^2}}-\frac{z^2}{(x^2+y^2+z^2)^{\frac{3}{2}}}=O(\frac{1}{z})$

$\frac{\partial^3 h}{\partial z^3}=\frac{-z}{(x^2+y^2+z^2)^{\frac{3}{2}}}-\frac{2z}{(x^2+y^2+z^2)^{\frac{3}{2}}}+\frac{3z^3}{(x^2+y^2+z^2)^{\frac{5}{2}}}=O(\frac{1}{z^2})$

$\frac{\partial^2 h}{\partial z\partial x}=\frac{-zx}{(x^2+y^2+z^2)^{\frac{3}{2}}}=O(\frac{1}{z^2})$

$\frac{\partial^2 h}{\partial z\partial y}=\frac{-zy}{(x^2+y^2+z^2)^{\frac{3}{2}}}=O(\frac{1}{z^2})$

$\frac{\partial^3 h}{\partial z\partial^2 x}=\frac{-z}{(x^2+y^2+z^2)^{\frac{3}{2}}}+\frac{3zx^2}{(x^2+y^2+z^2)^{\frac{5}{2}}}=O(\frac{1}{z^2})$

$\frac{\partial^3 h}{\partial z\partial^2 y}=\frac{-z}{(x^2+y^2+z^2)^{\frac{3}{2}}}+\frac{3zy^2}{(x^2+y^2+z^2)^{\frac{5}{2}}}=O(\frac{1}{z^2})$

$\frac{\partial^3 h}{\partial z\partial x\partial y}=\frac{3zxy}{(x^2+y^2+z^2)^{\frac{5}{2}}}=O(\frac{1}{z^4})$

so that, fixing $z=R$;

$||\int_{z=0\cap B(\overline{a},s)}h(x,y,0)\frac{\partial}{\partial z}D(\bigtriangledown(f))(x,y,0)d\mu(x,y)||\leq\frac{E}{R^2}$

and by $(AA),(BB),(CC)$;

$||\int_{z=0\cap B(\overline{a},s)}pr^{-1,*}(D(\bigtriangledown(f)))(x,y)d\mu(x,y)||\leq\frac{F}{R^2}$

for some $\{E,F\}\subset\mathcal{R}_{>0}$.

The change of measure argument is the same as in Lemma 0.4, which adds a correction of $O(\frac{1}{R^2})$, so that we obtain the result.

□

Lemma 0.6. *Let the fields $\{\overline{E},\overline{B}\}$ be as in Lemma 0.2, then $\overline{E}_0(\overline{r})$ and $\overline{B}_0(\overline{r})$ are quasi split normal in the sense of* [2].

Proof. ....Using the method of opposites.

□

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