SOME ARGUMENTS FOR THE WAVE EQUATION IN QUANTUM THEORY 6: TRANSFORMATION METHODS, WAVES, CURRENT AND CHARGE

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Abstract.

Lemma 0.1. *Let $f \in C(\mathcal{R})$ and $\frac{df}{dx} \in C(\mathcal{R})$ be of very moderate decrease, with f and $\frac{df}{dx}$ non-oscillatory, then defining the Fourier transform by;*

$$\mathcal{F}(f)(k) = \frac{1}{(2\pi)^{\frac{1}{2}}} lim_{r\rightarrow\infty} \int_{-r}^{r} f(y)e^{-iky}dy \ (k \neq 0)$$

$$\mathcal{F}(\frac{df}{dx})(k) = \frac{1}{(2\pi)^{\frac{1}{2}}} lim_{r\rightarrow\infty} \int_{-r}^{r} \frac{df}{dx}(y)e^{-iky}dy \ (k \neq 0)$$

we have that $\mathcal{F}(f)$ and $\mathcal{F}(\frac{df}{dx})$ are bounded and there exists a constant $G \in \mathcal{R}_{>0}$, such that;

$$|\mathcal{F}(f)(k)| \leq \frac{G}{|k|^2}$$

for sufficiently large k.

Proof. As f is of very moderate decrease, we have that f is continuous and $lim_{|x|\rightarrow\infty} f(x) = 0$. Similarly, $\frac{df}{dx}$ is continuous and $lim_{|x|\rightarrow\infty} \frac{df}{dx} = 0$. As $lim_{|x|\rightarrow\infty} f(x) = 0$, and f is non-oscillatory, we have that, for $k \neq 0$, the indefinite integral;

$$lim_{r\rightarrow\infty} \int_{-r}^{r} f(y)e^{-iky}dy$$

$$= lim_{r\rightarrow\infty} \int_{-r}^{r} f(y)cos(ky)dy - ilim_{r\rightarrow\infty} \int_{-r}^{r} f(y)sin(ky)dy$$

exists. As f is of very moderate decrease and non-oscillatory, we have that $|f(x)| \leq \frac{D}{|x|}$, for $|x| > E$, and monotone in the intervals $(-\infty, E)$ and (E, ∞). Using the method of [7], letting $K = max(|f||_{[-E,E]})$, we have that;

$|lim_{r\rightarrow\infty}\int_{-r}^{r}f(y)cos(ky)dy| \leq 2KE+2K\int_{E}^{E+\frac{\pi}{2|k|}}\frac{Dcos(|k|(y-E))}{y}dy$

$|lim_{r\rightarrow\infty}\int_{-r}^{r}f(y)sin(ky)dy| \leq 2KE+2K\int_{E}^{E+\frac{\pi}{2|y|}}\frac{Dcos(|k|(y-E))}{y}dy$

so that;

$|lim_{r\rightarrow\infty}\int_{-r}^{r}f(y)e^{-iky}dy| \leq 4KE+4K\int_{E}^{E+\frac{\pi}{2|k|}}\frac{Dcos(|k|(y-E))}{y}dy$

$= 4KE+4KD([\frac{-sin(|k|(y-E))}{|k|}]_{E}^{E+\frac{\pi}{2|k|}}-\int_{E}^{E+\frac{\pi}{2|k|}}\frac{sin(|k|(y-E))}{y^2}dy)$

$= 4KE+4KD(\frac{1}{|k|(E+\frac{\pi}{2|k|})}-\int_{E}^{E+\frac{\pi}{2|k|}}\frac{sin(|k|(y-E))}{y^2}dy)$

$\leq 4KE+4KD(\frac{1}{E|k|+\frac{\pi}{2}}+\int_{E}^{\infty}\frac{1}{y^2}dy)$

$\leq 4KE+4KD(\frac{2}{\pi}+\frac{1}{E}) = N$

so that $\mathcal{F}(f)(k)$ and, similarly, $\mathcal{F}(\frac{df}{dx})(k)$ are bounded, for $k \neq 0$, ([1]). We have, using integration by parts, that;

$\mathcal{F}(\frac{df}{dx})(k) = \frac{1}{(2\pi)^{\frac{1}{2}}}lim_{r\rightarrow\infty}\int_{-r}^{r}\frac{df}{dx}(y)e^{-iky}dy$

$= \frac{1}{(2\pi)^{\frac{1}{2}}}lim_{r\rightarrow\infty}([f(y)e^{-iky}]_{-r}^{r}+ik\int_{-r}^{r}f(y)e^{-iky}dy)$

$= \frac{1}{(2\pi)^{\frac{1}{2}}}[f(y)e^{-iky}]_{-\infty}^{\infty}+ik\frac{1}{(2\pi)^{\frac{1}{2}}}lim_{r\rightarrow\infty}\int_{-r}^{r}f(y)e^{-iky}dy$

$= ik\mathcal{F}(f)(k)$

so that, for $|k| > 1$;

$|\mathcal{F}(f)(k)| \leq \frac{|\mathcal{F}(\frac{df}{dx})(k)|}{|k|}$, (†)

As $\frac{df}{dx}$ is continuous and non-oscillatory, by the proof of Lemma 0.9 in [7], using underflow, for $r \in \mathcal{R}_{>0}$, we can find $\{F_r, G_r\} \subset \mathcal{R}_{>0}$, such that, for all $|k| > F_r$, we have that;

$|\frac{1}{(2\pi)^{\frac{1}{2}}}\int_{-r}^{r}\frac{df}{dx}(y)e^{-iky}dy| < \frac{G_r}{|k|}$, $(**)$

[1] $\mathcal{F}(f)(k)$ and $\mathcal{F}(\frac{df}{dx})(k)$ are differentiable for $k \neq 0$, limit interchange?

It is easy to see from the proof, that $\{F_r, G_r\}$ can be chosen uniformly in r. Then, from $(**)$, we obtain that, for $|k| > F$;

$|\mathcal{F}(\frac{df}{dx})(k)| < \frac{G}{|k|}$, for $|k| > F$

and, from $(\dagger)$, for $|k| > max(F, 1)$, that;

$|\mathcal{F}(f)(k)| \leq \frac{|\mathcal{F}(\frac{df}{dx})(k)|}{|k|} < \frac{G}{|k|^2}$

□

Definition 0.2. *Let $f \in C^3(\mathcal{R})$, with f, f', f'' and f''' bounded, then we define an approximating sequence $\{f_m : m \in \mathcal{N}\}$ by the requirements;*

(i). $f_m \in C^2(\mathcal{R})$, for $m \in \mathcal{N}$.

(ii). $f_m|_{[-m,m]} = f|_{[-m,m]}$.

(iii). f_m is of uniform moderate decay, in the sense that there exists a constant $C \in \mathcal{R}_{>0}$, independent of m, with;

$|f_m(x)| \leq \frac{C}{|x|^2}$, for $x \in (-\infty, -m - \frac{1}{m}) \cup (m + \frac{1}{m}, \infty)$

(iv). There exists constants $\{D, E\} \subset \mathcal{R}_{>0}$, with $\int_{-m-\frac{1}{m}}^{m} |f_m(x)|dx \leq \frac{D}{m}$ and $\int_{m}^{m+\frac{1}{m}} |f_m(x)|dx \leq \frac{D}{m}$.

Lemma 0.3. *Let $f \in C(\mathcal{R})$ and $\frac{df}{dx} \in C(\mathcal{R})$ be of very moderate decrease, with f and $\frac{df}{dx}$ non-oscillatory. Let $\{f_m; m \in \mathcal{N}\}$ be an approximating sequence. Let $\mathcal{F}$ be the ordinary Fourier transform, defined for each f_m, then the sequence $\{\mathcal{F}(f_m) : m \in \mathcal{N}\}$ converges pointwise and uniformly to $\mathcal{F}(f)$ on $\mathcal{R} \setminus \{0\}$, where $\mathcal{F}(f)$ is defined in Lemma 0.1.*

Proof. For $g \in C(\mathcal{R})$ and $n \in \mathcal{N}$, define;

$\mathcal{F}_n(g)(k) = \frac{1}{(2\pi)^{\frac{1}{2}}} \int_{-n}^{n} f(y)e^{-iky}dy$

For $k \in \mathcal{R} \setminus \{0\}$, $\{m, n\} \subset \mathcal{N}$, and $m \geq n$, $\epsilon > 0$,$\delta > 0$, we have;

$|\mathcal{F}(f)(k) - \mathcal{F}(f_m)(k)| \leq |\mathcal{F}(f)(k) - \mathcal{F}_n(f)(k)| + |\mathcal{F}_m(f)(k) - \mathcal{F}_m(f_m)(k)|$

$+|\mathcal{F}_m(f_m)(k)-\mathcal{F}(f_m)(k)|$

$=|\mathcal{F}(f)(k)-\mathcal{F}_m(f)(k)|+|\mathcal{F}_m(f_m)(k)-\mathcal{F}(f_m)(k)|$

$\leq|\mathcal{F}(f)(k)-\mathcal{F}_m(f)(k)|+\int_{-\infty}^{-m}|f_m(x)|dx+|+\int_m^{\infty}|f_m(x)|dx$

$=|\mathcal{F}(f)(k)-\mathcal{F}_m(f)(k)|+\int_{-\infty}^{-m-\frac{1}{m}}|f_m(x)dx+\int_{-m-\frac{1}{m}}^{-m}|f_m(x)|dx$

$+\int_m^{m+\frac{1}{m}}|f_m(x)dx+\int_{m+\frac{1}{m}}^{\infty}|f_m(x)|dx$

$\leq|\mathcal{F}(f)(k)-\mathcal{F}_m(f)(k)|+\frac{D+E}{m}+\int_{-\infty}^{-m-\frac{1}{m}}\frac{C}{x^2}dx+\int_{m+\frac{1}{m}}^{\infty}\frac{C}{x^2}dx$

$\leq|\mathcal{F}(f)(k)-\mathcal{F}_m(f)(k)|+\frac{D+E}{m}+\frac{2C}{m+\frac{1}{m}}$

$\leq|\mathcal{F}(f)(k)-\mathcal{F}_m(f)(k)|+\frac{2C+D+E}{m}$

$\leq\epsilon+\delta$, for $m\geq max(m(\epsilon),\frac{2C+D+E}{\delta})$. As $\epsilon>0$ and $\delta>0$ were arbitrary, we obtain the result.

□

Lemma 0.4. *If $m\in\mathcal{R}_{>0}$ is sufficiently large, $\{a_0,a_1,a_2\}\subset\mathcal{R}$, there exists $h\in\mathcal{R}[x]$ of degree 5, with the property that;*

$h(m)=a_0$, $h'(m)=a_1$, $h''(m)=a_2$, (i)

$h(m+\frac{1}{m})=h'(m+\frac{1}{m})=h''(m+\frac{1}{m})=0$ (ii)

$|h_{[m,m+\frac{1}{m}]}|\leq C$

for some $C\in\mathcal{R}_{>0}$, independent of m sufficiently large, and, if $h'''(m)>0$, $h'''(x)|_{[m,m+\frac{1}{m}]}>0$, if $h'''(m)<0$, $h'''|_{[m,m+\frac{1}{m}]}<0$. In particularly;

$\int_m^{m+\frac{1}{m}}|h'''(x)|dx=|a_2|$

Proof. If $p(x)$ is any polynomial, we have that $h(x)=(x-(m+\frac{1}{m}))^3p(x)$ satisfies condition (ii). Then;

$h'(x)=3(x-(m+\frac{1}{m}))^2p(x)+(x-(m+\frac{1}{m}))^3p'(x)$

$h''(x)=6(x-(m+\frac{1}{m}))p(x)+6(x-(m+\frac{1}{m}))^2p'(x)+(x-(m+\frac{1}{m}))^3p''(x)$

$h'''(x) = 6p(x) + 18(x - (m + \frac{1}{m}))p'(x) + 9(x - (m + \frac{1}{m}))^2 p''(x)$

so we can satisfy (i), by requiring that;

$(a).\ -\frac{p(m)}{m^3} = a_0$

$(b).\ \frac{3p(m)}{m^2} - \frac{p'(m)}{m^3} = a_1$

$(c).\ \frac{-6p(m)}{m} + \frac{6p'(m)}{m^2} - \frac{p''(m)}{m^3} = a_2$

which has the solution;

$p(m) = -a_0 m^3,\ p'(m) = -3a_0 m^4 - a_1 m^3,\ p''(m) = -12a_0 m^5 - 6a_1 m^4 - a_2 m^3$

and can be satisfied by the polynomial;

$p(x) = \frac{1}{2}(-12a_0 m^5 - 6a_1 m^4 - a_2 m^3)(x - m)^2$

$+(-3a_0 m^4 - a_1 m^3)(x - m) + (-a_0 m^3)$

$= \frac{1}{2}(-12a_0 m^5 - 6a_1 m^4 - a_2 m^3)x^2 + (-m(-12a_0 m^5 - 6a_1 m^4 - a_2 m^3)$

$+(-3a_0 m^4 - a_1 m^3))x + (\frac{m^2}{2}(-12a0m^5 - 6a_1 m^4 - a_2 m^3)$

$-m(-3a_0 m^4 - a_1 m^3) - a_0 m^3)$

$= (-6a_0 m^5 - 3a_1 m^4 - \frac{a_2}{2} m^3)x^2 + (12a_0 m^6 + 6a_1 m^5 + a_2 m^4 - 3a_0 m^4$

$-a_1 m^3)x + (-6a_0 m^7 - 3a_1 m^6 - \frac{a_2}{2} m^5 + 3a_0 m^5 + a_1 m^4 - a_0 m^3)$

$= (-6a_0 m^5 - 3a_1 m^4 - \frac{a_2}{2} m^3)x^2 + (12a_0 m^6 + 6a_1 m^5 + (a_2 - 3a_0)m^4$

$-a_1 m^3)x + (-6a_0 m^7 - 3a_1 m^6 + (3a_0 - \frac{a_2}{2})m^5 + 3a_0 m^5 + a_1 m^4 - a_0 m^3)$

$= ax^2 + bx + c\ (*)$

so that;

$h'''(x) = 6(ax^2 + bx + c) + 18(x - (m + \frac{1}{m}))(2ax + b) + 9(x - (m + \frac{1}{m}))^2 2a$

$= (60a)x^2 + (24b - 72a(m+\frac{1}{m}))x + (6c - 18(m+\frac{1}{m})b + 18a(m+\frac{1}{m})^2)$

and, using the computation $(*)$

$h'''(x) = (60(-6a_0m^5) + O(m^4))x^2 + (24.12a_0m^6 - 72m(-6a_0m^5)$

$+O(m^5))x + (6. - 6a_0m^7 - 18m(12a_0m^6) + 18m^2(-6a_0m^5) + O(m^6))$

$= (-360a_0m^5 + O(m^4))x^2 + (740a_0m^6 + O(m^5))x +$

$(-360a_0m^7 + O(m^6))$

which, by the quadratic formula, has roots when;

$x = \frac{-740a_0m^6 + \sqrt{740^2a_0^2m^{12} - 4(-360a_0m^5)(-360a_0m^7)}}{2.-360a_0m^5} + O(1)$

$= \frac{740m}{720} + \frac{170m}{720} + O(1)$

$= \frac{19m}{24} + O(1)$ or $\frac{91m}{72} + O(1)$

We have that $m > \frac{19m}{24}$ and $m + \frac{1}{m} < \frac{91m}{72}$ iff $m > \sqrt{\frac{72}{19}}$, and, clearly, we can ignore the $O(1)$ term for m sufficiently large. In particularly, for sufficiently large m, $h'''(x)$ has no roots in the interval $[m, m+\frac{1}{m}]$, so $h'''|_{[m,m+\frac{1}{m}]} > 0$ or $h'''|_{[m,m+\frac{1}{m}]} < 0$.

We calculate that;

$|h_{[m,m+\frac{1}{m}]}| = |(x-(m+\frac{1}{m}))^3p(x)|_{[m,m+\frac{1}{m}]}|$

$\leq \frac{1}{m^3}|p(x)|_{[m,m+\frac{1}{m}]}$

$= \frac{1}{m^3}|[\frac{1}{2}(-12a_0m^5 - 6a_1m^4 - a_2m^3)(x-m)^2$

$+(-3a_0m^4 - a_1m^3)(x-m) + (-a_0m^3)]|_{[m,m+\frac{1}{m}]}$

$\leq \frac{1}{m^3}[\frac{1}{2}|-12a_0m^5 - 6a_1m^4 - a_2m^3|\frac{1}{m^2} + |-3a_0m^4 - a_1m^3|\frac{1}{m} + |-a_0m^3|]$

$\leq \frac{12|a_0|m^5 + 6|a_1|m^4 + |a_2|m^3|}{m^5} + \frac{3|a_0|m^4 + |a_1|m^3}{m^4} + \frac{|a_0|m^3}{m^3}$

$\leq 12|a_0| + 6|a_1| + |a_2| + 3|a_0| + |a_1| + |a_0|$ $(m > 1)$

$\leq 16|a_0| + 7|a_1| + |a_2|$

For the final claim, we have, as $h'''|_{[m,m+\frac{1}{m}]} > 0$ or $h'''|_{[m,m+\frac{1}{m}]} < 0$, that, using the fundamental theorem of calculus;

$\int_m^{m+\frac{1}{m}} |h'''(x)|dx = |\int_m^{m+\frac{1}{m}} h'''(x)dx|$

$= |h''(m+\frac{1}{m}) - h''(m)| = |-h''(m)| = |a_2|$

□

Lemma 0.5. *If* $m \in \mathcal{R}_{>0}$, $\{a_0, a_1, a_2, a_3\} \subset \mathcal{R}$, *there exists* $h \in C^3(\mathcal{R})$, *with the property that;*

$h(m) = a_0,\ h'(m) = a_1,\ h''(m) = a_2,\ h'''(m) = a_3,\ (i)$

$h(m+\frac{1}{m}) = h'(m+\frac{1}{m}) = h''(m+\frac{1}{m}) = h'''(m+\frac{1}{m}) = 0\ (ii)$

$|h|_{[m,m+1]} \leq C$

where $C \in \mathcal{R}_{>0}$ *is independent of* $m > 1$, *and, if* $a_3 > 0$, $h'''(x)|_{[m,m+\frac{1}{m}]} \geq 0$, $a_3 < 0$, $h'''(x)|_{[m,m+\frac{1}{m}]} \leq 0$. *In particularly;*

$\int_m^{m+\frac{1}{m}} |h'''(x)|dx = |a_2|$

Proof. Let $g(x)$ be a polynomial, then it is clear that the polynomial $h_1(x) = (x - (m+\frac{1}{m})^n g(x)$, for $n \geq 4$, has the property (ii), that $h_1(m+\frac{1}{m}) = h_1'(m+\frac{1}{m}) = h_1''(m+\frac{1}{m}) = h_1'''(m+\frac{1}{m}) = 0$. The condition (i), then amounts to the equations;

$(i)'\ \frac{g(m)}{(-1)^n m^n} = a_0$

$(ii)'\ \frac{ng(m)}{(-1)^{n-1}m^{n-1}} + \frac{g'(m)}{(-1)^n m^n} = a_1$

$(iii)'\ \frac{n(n-1)g(m)}{(-1)^{n-2}m^{n-2}} + \frac{2ng'(m)}{(-1)^{n-1}m^{n-1}} + \frac{g''(m)}{(-1)^n m^n} = a_2$

$(iv)'\ \frac{n(n-1)(n-2)g(m)}{(-1)^{n-3}m^{n-3}} + \frac{3n(n-1)g'(m)}{(-1)^{n-2}m^{n-2}} + \frac{3ng''(m)}{(-1)^{n-1}m^{n-1}} + \frac{g'''(m)}{(-1)^n m^n} = a_3$

which we can solve, by requiring that;

$(i)''\ g(m) = (-1)^n a_0 m^n$

$(ii)''$ $g'(m) = (-1)^n a_1 m^n + (-1)^n a_0 n m^{n+1}$

$(iii)''$ $g''(m) = (-1)^n a_2 m^n + 2(-1)^n n a_1 m^{n+1} + (-1)^n n(n+1) a_0 m^{n+2}$

$(iv)''$ $g'''(m) = (-1)^n a_3 m^n + 3n(-1)^n a_2 m^{n+1} + (-1)^n a_1 n(n+3) m^{n+2}$

$+n(n+1)(n+2)(-1)^n a_0 m^{n+3}$ $(*)$

Let;

$g_1(x) = ((-1)^n a_3 m^n + 3n(-1)^n a_2 m^{n+1} + (-1)^n a_1 n(n+3) m^{n+2}$

$+n(n+1)(n+2)(-1)^n a_0 m^{n+3})(x-m)^3 + ((-1)^n a_2 m^n + 2(-1)^n n a_1 m^{n+1}$

$+(-1)^n n(n+1) a_0 m^{n+2})(x-m)^2 + ((-1)^n a_1 m^n + (-1)^n a_0 n m^{n+1})$

$(x-m) + ((-1)^n a_0 m^n)$

Then $g_1(x)$ satisfies $(*)$, and so does any function of the form $g_2(x) + g_1(x)$ where;

$g_2(m) = g_2'(m) = g_2''(m) = g_2'''(m) = 0$

provided $g_2 \in C^3(\mathcal{R})$. In this case, if;

$h(x) = (x - (m + \frac{1}{m})^n (g_2(x) + g_1(x))$

then h satisfies $(i), (ii)$. We have that;

$|x - (m + \frac{1}{m})^n g_1(x)|_{[m, m+\frac{1}{m}]} \leq \frac{1}{m^n}(|g_2|_{[m,m+\frac{1}{m}]} + |g_1|_{[m,m+\frac{1}{m}]})$

$\leq \frac{1}{m^n}(|g_2|_{[m,m+\frac{1}{m}]} + \frac{1}{m^n}|((-1)^n a_3 m^n + 3n(-1)^n a_2 m^{n+1} + (-1)^n a_1 n(n+3) m^{n+2}$

$+n(n+1)(n+2)(-1)^n a_0 m^{n+3})\frac{1}{m^3} + ((-1)^n a_2 m^n + 2(-1)^n n a_1 m^{n+1}$

$+(-1)^n n(n+1) a_0 m^{n+2})\frac{1}{m^2} + ((-1)^n a_1 m^n + (-1)^n a_0 n m^{n+1})$

$\frac{1}{m} + ((-1)^n a_0 m^n)|$

$= |((-1)^n a_3 m^n + 3n(-1)^n a_2 m^{n+1} + (-1)^n a_1 n(n+3) m^{n+2}$

$+n(n+1)(n+2)(-1)^n a_0 m^{n+3})\frac{1}{m^{n+3}} + ((-1)^n a_2 m^n + 2(-1)^n n a_1 m^{n+1}$

$+(-1)^n n(n+1) a_0 m^{n+2})\frac{1}{m^{n+2}} + ((-1)^n a_1 m^n + (-1)^n a_0 n m^{n+1})$

$\frac{1}{m^{n+1}} + ((-1)^n a_0)|$

$\leq |a_3| + 3n|a_2| + n(n+3)|a_1| + n(n+1)(n+2)|a_0| + |a_2| + 2n|a_1| + n(n+1)|a_0| + |a_1| + n|a_0| + |a_0|$, $(m \geq 1)$

$= \frac{1}{m^n}(|g_2|_{[m,m+\frac{1}{m}]} + (n+1)(n^2+3n+1)|a_0| + (n^2+5n+1)|a_1 + (3n+1)|a_2| + |a_3| = F$ (F)

where $F \in \mathcal{R}_{>0}$ is independent of m. Using the product rule, the condition that $h'''(x) = 0$ in the interval $(m, m+\frac{1}{m})$, is given by;

$n(n-1)(n-2)(x-(m+\frac{1}{m}))^{n-3}(g_2+g_1)(x) + 3n(n-1)(x-(m+\frac{1}{m}))^{n-2}(g_2+g_1)'(x)$

$+3n(x-(m+\frac{1}{m}))^{n-1}(g_2+g_1)''(x) + (x-(m+\frac{1}{m}))^n(g_2+g_1)'''(x) = 0$

which, dividing by $(x-(m+\frac{1}{m}))^{n-3}$, reduces to;

$n(n-1)(n-2)(g_2+g_1)(x) + 3n(n-1)(x-(m+\frac{1}{m}))(g_2+g_1)'(x) +$

$3n(x-(m+\frac{1}{m}))^2(g_2+g_1)''(x) + (x-(m+\frac{1}{m}))^3(g_2+g_1)'''(x) = 0$

and;

$n(n-1)(n-2)g_2(x) + 3n(n-1)(x-(m+\frac{1}{m}))g_2'(x) + 3n(x-(m+\frac{1}{m}))^2 g_2''(x)$

$+(x-(m+\frac{1}{m}))^3 g_2'''(x) = -(n(n-1)(n-2)g_1(x) + 3n(n-1)(x-(m+\frac{1}{m}))g_1'(x)$

$+3n(x-(m+\frac{1}{m}))^2 g_1''(x) + (x-(m+\frac{1}{m}))^3 g_1'''(x))$ (A)

Without loss of generality, assuming that;

$-(n(n-1)(n-2)g_1(x)+3n(n-1)(x-(m+\frac{1}{m}))g_1'(x)+3n(x-(m+\frac{1}{m}))^2g_1''(x)$

$+(x-(m+\frac{1}{m}))^3g_1'''(x))|_m=-(n(n-1)(n-2)a_0-\frac{3n(n-1)a_1}{m}+\frac{3na_2}{m^2}$

$-\frac{a_3}{m^3}\geq 0$

we can choose an analytic function $\phi(x)$ on $[m,m+\frac{1}{m}]$ with;

(a). $\phi(x)\leq -(n(n-1)(n-2)g_1(x)+3n(n-1)(x-(m+\frac{1}{m}))g_1'(x)+3n(x-(m+\frac{1}{m}))^2g_1''(x)$

$+(x-(m+\frac{1}{m}))^3g_1'''(x))$

(b). $\phi(m)=0$

The third order differential equation for g_2;

$n(n-1)(n-2)g_2(x)+3n(n-1)(x-(m+\frac{1}{m}))g_2'(x)+3n(x-(m+\frac{1}{m}))^2g_2''(x)$

$+(x-(m+\frac{1}{m}))^3g_2'''(x)=\phi(x)$, on $[m,1+m]$ (B)

with the requirement that $g_2(m)=g_2'(m)=g_2''(m)=0$, has a solution in $C^3([m,m+\frac{1}{m}))$ by Peano's existence theorem. By the fact (b), we must have that $g_2'''(m)=0$. Writing the power series for ϕ on $[m,m+\frac{1}{m}]$, as;

$\phi(x)=\sum_{j=0}^{\infty}b_j(x-(m+\frac{1}{m}))^j$

we can use the method of equating coefficients, to obtain a particular solution, with;

$g_{2,part}(x)=\sum_{j=0}^{\infty}a_{j,part}(x-(m+\frac{1}{m}))^j$, with;

$a_{j,part}=\frac{b_j}{n(n-1)(n-2)+3n(n-1)j+3nj(j-1)+j(j-1)(j-2)}$, $(j\geq 3)$

$a_{2,part}=\frac{b_2}{n(n-1)(n-2)+6n(n-1)+3n}$ $a_{1,part}=\frac{b_1}{n(n-1)(n-2)+3n(n-1)}$ $a_{0,part}=\frac{b_0}{n(n-1)(n-2)}$

so that $g_{2,part}$ is analytic as $|a_{j,0}| \leq \frac{|b_j|}{n(n-1)(n-2)}$ for $j \geq 0$.

To solve the homogenous Euler equation;

$n(n-1)(n-2)g_2(x) + 3n(n-1)(x-(m+\frac{1}{m}))g_2'(x) + 3n(x-(m+\frac{1}{m}))^2 g_2''(x)$

$+(x-(m+\frac{1}{m}))^3 g_2'''(x) = 0$ on $[m, m+\frac{1}{m}]$

we can make the substitution $y = m + \frac{1}{m} - x$, to reduce to the equation;

$n(n-1)(n-2)g_{2,m}(y) + 3n(n-1)yg_{2,m}'(y) + 3ny^2 g_{2,m}''(y) + y^3 g_{2,m}'''(y) = 0$ on $[0, \frac{1}{m}]$

with $g_{2,m}(y) = g_2(m + \frac{1}{m} - y)$. Making the further substitution $y = e^u$, and letting $r_{2,m}(u) = g_{2,m}(e^u)$, we have that;

$r_{2,m}'(u) = g_{2,m}'(e^u)e^u$

$r_{2,m}''(u) = g_{2,m}''(e^u)e^{2u} + g_{2,m}'(e^u)e^u$

$r_{2,m}'''(u) = g_{2,m}'''(e^{3u}) + 3g_{2,m}''(e^u)e^{2u} + g_{2,m}'(e^u)e^u$

so that;

$n(n-1)(n-2)g_{2,m}(e^u) + 3n(n-1)e^u g_{2,m}'(e^u) + 3ne^{2u}g_{2,m}''(e^u) + e^{3u}g_{2,m}'''(e^u)$

$= n(n-1)(n-2)r_{2,m}(u) + 3n(n-1)e^u(r_{2,m}'(u)e^{-u}) + 3ne^{2u}((r_{2,m}''(u) - g_{2,m}'(e^u)e^u)e^{-2u})$

$+e^{3u}((r_{2,m}'''(u) - 3g_{2,m}''(e^u)e^{2u} - g_{2,m}'(e^u)e^u)e^{-3u})$

$= n(n-1)(n-2)r_{2,m}(u) + 3n(n-1)r_{2,m}'(u) + 3nr_{2,m}''(u) - 3ng_{2,m}'(e^u)e^u + r_{2,m}'''(u) - 3g_{2,m}''(e^u)e^{2u}$

$-g_{2,m}'(e^u)e^u$

$= n(n-1)(n-2)r_{2,m}(u) + 3n(n-1)r'_{2,m}(u) + 3nr''_{2,m}(u) + r'''_{2,m}(u) - (3n+1)g'_{2,m}(e^u)e^u - 3g''_{2,m}(e^u)e^{2u}$

$= n(n-1)(n-2)r_{2,m}(u) + 3n(n-1)r'_{2,m}(u) + 3nr''_{2,m}(u) + r'''_{2,m}(u) - (3n+1)r'_{2,m}(u)$

$-3e^{2u}((r''_{2,m}(u) - g'_{2,m}(e^u)e^u)e^{-2u})$

$= n(n-1)(n-2)r_{2,m}(u) + (3n^2-6n-1)r'_{2,m}(u) + 3nr''_{2,m}(u) + r'''_{2,m}(u) - 3r''_{2,m}(u) + 3g'_{2,m}(e^u)e^u$

$= n(n-1)(n-2)r_{2,m}(u) + (3n^2-6n-1)r'_{2,m}(u) + 3(n-1)r''_{2,m}(u) + r'''_{2,m}(u) + 3r'_{2,m}(u)$

$= n(n-1)(n-2)r_{2,m}(u) + (3n^2-6n+2)r'_{2,m}(u) + (3n-3)r''_{2,m}(u) + r'''_{2,m}(u) = 0$ (C)

We have that;

$(\lambda^3 + 3(n-1)\lambda^2 + (3n^2-6n+2)\lambda + n(n-1)(n-2))' = 3\lambda^2 + 6(n-1)\lambda + (3n^2-6n+2)$

which has roots when $\lambda = -(n-1) + \frac{1}{\sqrt{3}}$, so that, for large n, the characteristic polynomial of (C) has exactly one real root λ_1 and 2 complex conjugate non-real roots, $\{\lambda_2 + i\lambda_3, \lambda_2 - i\lambda_3\}$. It follows, the general solution of (C) is given by;

$r_{2,m}(u) = A_1e^{\lambda_1 u} + A_2e^{\lambda_2 u + i\lambda_3} + A_3e^{\lambda_2 u - i\lambda_3}$

where $\{A_1, A_2, A_3\} \subset \mathcal{C}$, and, we can obtain a real solution, fitting the corresponding initial conditions, of the form;

$r_{2,m}(u) = B_1e^{\lambda_1 u} + B_2e^{\lambda_2 u}cos(\lambda_3 u) + B_3e^{\lambda_2 u}sin(\lambda_3 u)$

where $\{B_1, B_2, B_3\} \subset \mathcal{R}$. It follows that;

$g_{2,m}(y) = r_{2,m}(ln(y))$

$g_2(x) = g_{2,m}(m + \frac{1}{m} - x) + g_{2,part}(x) = r_{2,m}(ln(m + \frac{1}{m} - x)) + g_{2,part}(x)$

$= B_1 e^{\lambda_1 ln(m+\frac{1}{m}-x)} + B_2 e^{\lambda_2 ln(m+\frac{1}{m}-x)} cos(\lambda_3 ln(m+\frac{1}{m}-x))$

$+B_3 e^{\lambda_2 ln(m+\frac{1}{m}-x)} sin(\lambda_3 ln(m+\frac{1}{m}-x)) + g_{2,part}(x)$ (on $[m, m+\frac{1}{m}]$)

We have that;

$\lambda_1|\lambda_2 + i\lambda_3|^2 = -n(n-1)(n-2)$

$\lambda_1 + \lambda_2 + i\lambda_3 + \lambda_2 - i\lambda_3 = \lambda_1 + 2\lambda_2 = -3(n-1)$

Computing the highest degree in n term of the characteristic polynomial, we obtain that, for $\lambda = \alpha n$;

$\alpha^3 n^3 + 3n(\alpha n)^2 + 3n^2(\alpha n) + n^3 = n^3(\alpha+3)^3 = 0$

so that $\alpha = -3$, $\lambda_1 = -3n + O(1)$ and $2\lambda_2 = -3(n-1) - (-3n + O(1)) = 3 - O(1) = O(1)$

Then, if $B_1 = 0$, we can see that $g_2(x)$ has at most a $\frac{1}{x^{O(1)}}$ singularity at $(m+\frac{1}{m})$, which we can achieve with a 2-parameter family choice for the initial conditions of $\{\phi(m), \phi'(m), \phi''(m)\}$. If;

$-(n(n-1)(n-2)a_0 - \frac{3n(n-1)a_1}{m} + \frac{3na_2}{m^2} - \frac{a_3}{m^3} \neq 0$

we can clearly achieve this, while satisfying $(a), (b)$. If;

$-(n(n-1)(n-2)a_0 - \frac{3n(n-1)a_1}{m} + \frac{3na_2}{m^2} - \frac{a_3}{m^3} = 0$

by requiring the the additional property (c);

$\phi'(m) < -(n(n-1)(n-2)g_1(x) + 3n(n-1)(x-(m+\frac{1}{m}))g_1'(x) + 3n(x-(m+\frac{1}{m}))^2 g_1''(x)$

$+(x-(m+\frac{1}{m}))^3 g_1'''(x))'|_m$

we can clearly satisfy $(a), (b)$ as well.

Then, as, for sufficiently large n;

$lim_{x\rightarrow 0}(\frac{B_2 x^n}{x^{O(1)}} sin(\lambda_3 ln(x)) + \frac{B_3 x^n}{x^{O(1)}} cos(\lambda_3 ln(x)))$

$= lim_{x\rightarrow 0}(\frac{B_2x^n}{x^{O(1)}}sin(\lambda_3 ln(x)) + \frac{B_3x^n}{x^{O(1)}}cos(\lambda_3 ln(x)))'$

$= lim_{x\rightarrow 0}(\frac{B_2x^n}{x^{O(1)}}sin(\lambda_3 ln(x)) + \frac{B_3x^n}{x^{O(1)}}cos(\lambda_3 ln(x)))''$

$= lim_{x\rightarrow 0}(\frac{B_2x^n}{x^{O(1)}}sin(\lambda_3 ln(x)) + \frac{B_3x^n}{x^{O(1)}}cos(\lambda_3 ln(x)))''' = 0$

we obtain that $(x-(m+\frac{1}{m}))^n g_2(x)$ extends to a solution in $C^3([m, m+\frac{1}{m}])$, and $(x-(m+\frac{1}{m}))^n(g_2+g_1)(x) \in C^3([m, m+\frac{1}{m}])$. By the fact (a), (A) has no solutions in $(m, m+\frac{1}{m})$, so that $h'''(x) \geq 0$.

We have that;

$|(x-(m+\frac{1}{m})^n g_2(x)|_{[m,m+\frac{1}{m}]} = |(x-(m+\frac{1}{m})^n(B_2 e^{\lambda_2 ln(m+\frac{1}{m}-x)}cos(\lambda_3 ln(m+\frac{1}{m}-x))$

$+B_3 e^{\lambda_2 ln(m+\frac{1}{m}-x)}sin(\lambda_3 ln(m+\frac{1}{m}-x)) + g_{2,part}(x))|$

$\leq |B_2|m^{\lambda_2-n} + |B_3|m^{\lambda_2-n} + m^{-n}|g_{2,part}(x)|$

Noting the right hand side of (a) is bounded by $O(m^n)$ on $[m, m+\frac{1}{m}]$, we can also choose $\phi(x)$ and $g_{2,part}(x)$ to be of $O(m^n)$ on $[m, m+\frac{1}{m}]$, irrespective of the choice of initial conditions $\{\phi(m), \phi'(m), \phi''(m)\}$. We have that $\phi'(m) = O(m^{n+1})$, in the special case, so that choosing $\{B_2, B_3\}$ sufficiently small, noting;

$(x-(m+\frac{1}{m})^n(B_2 e^{\lambda_2 ln(m+\frac{1}{m}-x)}cos(\lambda_3 ln(m+\frac{1}{m}-x))$

$+B_3 e^{\lambda_2 ln(m+\frac{1}{m}-x)}sin(\lambda_3 ln(m+\frac{1}{m}-x)))'|_m = O(max(B_2 m^{n-\lambda_2-1}, B_3 m^{n-\lambda_2-1}))$

we can assume that;

$|(x-(m+\frac{1}{m})^n g_2(x)|_{[m,m+\frac{1}{m}]} \leq D$

where $D \in \mathcal{R}_{>0}$ is independent of m, so that, using (F);

$|h(x)|_{[m,m+\frac{1}{m}]} \leq |(x-(m+\frac{1}{m})^n g_1(x)|_{[m,m+\frac{1}{m}]} + |(x-(m+\frac{1}{m})^n g_2(x)|_{[m,m+\frac{1}{m}]} \leq F + D$

For the final claim, we have, as $h'''|_{[m,m+\frac{1}{m}]} \geq 0$ or $h'''|_{[m,m+\frac{1}{m}]} \leq 0$, that, using the fundamental theorem of calculus, that;

$\int_m^{m+\frac{1}{m}} |h'''(x)|dx = |\int_m^{m+\frac{1}{m}} h'''(x)dx|$

$= |h''(m+\frac{1}{m}) - h''(m)| = |-h''(m)| = |a_2|$

□

Lemma 0.6. *Let f be as in Definition 0.2, then there exists an approximating sequence $\{f_m : m \in \mathcal{N}\}$. Moreover, for sufficiently large m, $|\mathcal{F}(f_m)(k)| \leq \frac{Cm}{|k|^3}$, for $|k| > 1$, where $C \in \mathcal{R}_{>0}$, independent of m.*

Proof. Define f_m by setting;

$f_m(x) = f(x)$ for $x \in [-m, m]$

$f_m(x) = h_{1,m}(x)$, for $x \in [-m - \frac{1}{m}, -m]$

$f_m(x) = h_{2,m}(x)$, for $x \in [m, m + \frac{1}{m}]$

$f_m(x) = 0$, for $x \in (-\infty, -m - \frac{1}{m}]$

$f_m(x) = 0$, for $x \in [m, \infty)$

where $\{h_{1,m}, h_{2,m}\}$ are the polynomials of degree 5, generated by the data $a_{1,m,0} = f(-m)$, $a_{1,m,1} = f'(-m)$, $a_{1,m,2} = f''(-m)$, $a_{2,m,0} = f(m)$, $a_{2,m,1} = f'(m)$, $a_{2,m,2} = f''(m)$, guaranteed by Lemma 0.4 (or Lemma 0.5. By the construction of Lemma 0.4, we have that (i) in Definition 0.2 holds. By the definition, we have (ii). As f_m is identically zero on $-\infty, -m - \frac{1}{m}] \cup [m, \infty)$, we have that (iii) holds. By the proof of Lemma 0.4, we have that;

$max(|h_{1,m}|_{[m,m+\frac{1}{m}]}|, |h_{2,m}|_{[-m-\frac{1}{m},-m]}|) \leq 16||f||_\infty + 7||f'||_\infty + ||f''||_\infty$

It follows that;

$\int_{-m-\frac{1}{m}}^{-m} |f_m(x)|dx \leq (16||f||_\infty + 7||f'||_\infty + ||f''||_\infty)(-m - (-m - \frac{1}{m}))$

$\leq \frac{D}{m}$

$\int_m^{m+\frac{1}{m}} |f_m(x)|dx \leq (16||f||_\infty + 7||f'||_\infty + ||f''||_\infty)((m + \frac{1}{m}) - m)$

$\leq \frac{E}{m}$

where $D = E = (16||f||_\infty + 7||f'||_\infty + ||f''||_\infty)$

proving (iv). For the second claim, we have that;

$\mathcal{F}(f_m''')(k) = \frac{1}{(2\pi)^{\frac{1}{2}}}\int_{-\infty}^{\infty} f_m'''(x)e^{-ikx}dx$

$= \frac{1}{(2\pi)^{\frac{1}{2}}}([f_m''(x)e^{-ikx}]_{-\infty}^{\infty} - ik\int_{-\infty}^{\infty} f_m''(x)e^{-ikx}dx$

$= \frac{-ik}{(2\pi)^{\frac{1}{2}}}([f_m'(x)e^{-ikx}]_{-\infty}^{\infty} - ik\int_{-\infty}^{\infty} f_m'(x)e^{-ikx}dx)$

$= \frac{-k^2}{(2\pi)^{\frac{1}{2}}}([f_m(x)e^{-ikx}]_{-\infty}^{\infty} - ik\int_{-\infty}^{\infty} f_m(x)e^{-ikx}dx)$

$= \frac{ik^3}{(2\pi)^{\frac{1}{2}}}\int_{-\infty}^{\infty} f_m(x)e^{-ikx}dx)$

so that, for $|k| > 1$;

$|\mathcal{F}(f_m)(k)| = |\frac{1}{(2\pi)^{\frac{1}{2}}}\int_{-\infty}^{\infty} f_m(x)e^{-ikx}dx|$

$= \frac{|\frac{1}{(2\pi)^{\frac{1}{2}}}\int_{-\infty}^{\infty} f_m'''(x)e^{-ikx}dx|}{|k|^3}$

$\leq \frac{1}{|k|^3(2\pi)^{\frac{1}{2}}}\int_{-\infty}^{\infty} |f_m'''(x)e^{-ikx}|dx$

$= \frac{1}{|k|^3(2\pi)^{\frac{1}{2}}}\int_{-\infty}^{\infty} |f_m'''(x)|dx$

$= \frac{1}{|k|^3(2\pi)^{\frac{1}{2}}}(\int_{-m-\frac{1}{m}}^{-m} |h_{1,m}'''(x)|dx + \int_{-m}^{m} |f'''(x)|dx + \int_{m}^{m+\frac{1}{m}} |h_{2,m}'''(x)|dx)$

$\leq \frac{1}{|k|^3(2\pi)^{\frac{1}{2}}}(|f''(-m)| + 2m|f'''|_\infty + |f''(m))$

$\leq \frac{1}{|k|^3(2\pi)^{\frac{1}{2}}}(2||f''||_\infty + 2m||f'''||_\infty)$

$\leq \frac{1}{|k|^3(2\pi)^{\frac{1}{2}}}(2m + 2m||f'''||_\infty)$, $(m > ||f''||_\infty$

$= \frac{Cm}{|k|^3}$

where $C = \frac{1}{(2\pi)^{\frac{1}{2}}}(2 + 2||f'''||_\infty)$

□

Lemma 0.7. *Let $f \in C^3(\mathcal{R})$, with f and $\frac{df}{dx}$ non-oscillatory and of very moderate decrease, with $\{f, f', f'', f'''\}$ bounded, then $\mathcal{F}(f) \in L^1(\mathcal{R})$, and we have that;*

$f(x) = \mathcal{F}^{-1}(\mathcal{F}(f))(x)$

where, for $g \in L^1(\mathcal{R})$*;*

$\mathcal{F}^{-1}(g)(x) = \frac{1}{(2\pi)^{\frac{1}{2}}} \int_{-\infty}^{\infty} g(k)e^{ikx}dk$

Proof. By Lemma 0.1, we have that there exists $C \in \mathcal{R}_{>0}$, with $|\mathcal{F}(f)(k)| \leq \frac{C}{|k|^2}$, for sufficiently large k, $(*)$. As f is of very moderate decrease, we have that $|f|^2 \leq \frac{D}{|x|^2}$, for $|x| > 1$, so that, as $f \in C^0(\mathcal{R})$, we have that $f \in L^2(\mathcal{R})$. It follows that $\mathcal{F}(f) \in L^2(\mathcal{R})$, and $\mathcal{F}(f)|_{[-n,n]} \in L^1(\mathcal{R})$, for any $n \in \mathcal{N}$, $(**)$. Combining $(*), (**)$, we obtain that $\mathcal{F}(f) \in L^1(\mathcal{R})$. Let $\{f_m : m \in \mathcal{N}\}$ be the approximating sequence, given by Lemma 0.6, then, as $f_m \in L^1(\mathcal{R})$, $\mathcal{F}(f_m)$ is continuous and, by Lemma 0.3, converges uniformly to $\mathcal{F}(f)$ on $\mathcal{R} \setminus \{0\}$. It follows that $\mathcal{F}(f) \in C^0(\mathcal{R} \setminus \{0\})$. As $f_m \in C^2(\mathcal{R})$ and $f_m'' \in L^1(\mathcal{R})$, we have that there exists constants $D_m \in \mathcal{R}_{>0}$, such that $|\mathcal{F}(f_m)(k) \leq \frac{D_m}{|k|^2}$, for sufficiently large k. Moreover, as $x^n f_m(x) \in L^1(\mathcal{R})$, for $n \in \mathcal{N}$, $\mathcal{F}(f_m) \in C^\infty(\mathcal{R})$. It follows, the Fourier inversion theorem $f_m = \mathcal{F}^{-1}(\mathcal{F}(f_m))$, $(***)$, holds for each f_m, see the proof in [10]. By Lemma 0.3, we have that, for $k \in \mathcal{R} \setminus \{0\}$, $|\mathcal{F}(f)(k) - \mathcal{F}(f_m)(k)| \leq \frac{E}{m}$. Then, for $n \in \mathcal{N}$, $m \in \mathcal{N}$, with $m = n^{\frac{3}{2}}$, using Lemma 0.6, we have, for $x \in \mathcal{R}$, that;

$|\mathcal{F}^{-1}(\mathcal{F}(f))(x) - \mathcal{F}^{-1}(\mathcal{F}(f_m))(x)| = |\mathcal{F}^{-1}(\mathcal{F}(f)(k) - \mathcal{F}(f_m)(k))|$

$= \frac{1}{(2\pi)^{\frac{1}{2}}} | \int_{-n}^{n} (\mathcal{F}(f)(k) - \mathcal{F}(f_m)(k))e^{ikx}dk + \int_{|k|>n} (\mathcal{F}(f)(k) - \mathcal{F}(f_m)(k))e^{ikx}dk|$

$\leq \frac{1}{(2\pi)^{\frac{1}{2}}} (\int_{-n}^{n} |\mathcal{F}(f)(k) - \mathcal{F}(f_m)(k)|dk + \int_{|k|>n} |\mathcal{F}(f)(k)|dk + \int_{|k|>n} |\mathcal{F}(f_m)(k)|dk)$

$\leq \frac{1}{(2\pi)^{\frac{1}{2}}} (\frac{2nE}{m} + \int_{|k|>n} \frac{C}{|k|^2}dk + \int_{|k|>n} \frac{Cm}{|k|^3}dk)$

$\leq \frac{1}{(2\pi)^{\frac{1}{2}}} (\frac{2n}{n^{\frac{3}{2}}} + \frac{2C}{n} + \frac{Cn^{\frac{3}{2}}}{n^2})$

$< \epsilon$

for sufficiently large n, so that, as $\epsilon > 0$ was arbitrary, for $x \in \mathcal{R}$;

$lim_{m \to \infty} \mathcal{F}^{-1}(\mathcal{F}(f_m))(x) = \mathcal{F}^{-1}\mathcal{F}(f)(x)$, $(****)$

and, by Definition 0.2, $(***)$, $(****)$;

$f(x) = lim_{m\rightarrow\infty} f_m(x) = lim_{m\rightarrow\infty}\mathcal{F}^{-1}(\mathcal{F}(f_m))(x) = \mathcal{F}^{-1}\mathcal{F}(f)(x)$

□

Remarks 0.8. *The previous lemma proves an inversion theorem for non-oscillatory functions with moderate decrease. Such functions belong to $L^2(\mathcal{R})$ and an analogous result for Fourier series can be found in* [2]*, where convergence is proved almost everywhere rather than everywhere. The corresponding result for transforms is that;*

If $f \in L^p(\mathcal{R})$, $p \in (1,2]$, then;

$f(x) = lim_{R\rightarrow\infty}\frac{1}{2\pi}\int_{|k|\leq R}\mathcal{F}(f)(k)e^{ixk}dk$

for almost every $x \in \mathcal{R}$.

There is also a converse result, which can be found in [5]*, but is left as an exercise;*

If $f \in L^1(\mathcal{R}) \cap C^0(\mathcal{R})$ and $|\mathcal{F}(f)(k)| \leq \frac{A}{|k|}$, for all $k \neq 0$, and $A \in \mathcal{R}_{\geq 0}$, then;

$f(x) = lim_{R\rightarrow\infty}\frac{1}{2\pi}\int_{|k|\leq R}\mathcal{F}(f)(k)e^{ixk}dk$

for every $x \in \mathcal{R}$.

Lemma 0.9. *Let $f \in C(\mathcal{R}_{>0})$ and $\frac{df}{dx} \in C(\mathcal{R}_{>0})$ be of very moderate decrease, with f and $\frac{df}{dx}$ non-oscillatory, and $lim_{x\rightarrow 0}f(x) = 0$, $lim_{x\rightarrow 0}\frac{df}{dx} = M$, with $M \in \mathcal{R}$, then defining the half Fourier transform $\mathcal{G}$ by;*

$\mathcal{G}(f)(k) = lim_{r\rightarrow\infty}\int_0^r f(y)e^{-iky}dy \ (k \neq 0)$

$\mathcal{G}(\frac{df}{dx})(k) = lim_{r\rightarrow\infty}\int_0^r \frac{df}{dx}(y)e^{-iky}dy \ (k \neq 0)$

we have that $\mathcal{G}(f)$ and $\mathcal{G}(\frac{df}{dx})$ are bounded for $|k| \geq k_0 > 0$, and there exists a constant $G \in \mathcal{R}_{>0}$, such that;

$|\mathcal{G}(f)(k)| \leq \frac{G}{|k|^2}$

for sufficiently large k.

Proof. As f is of very moderate decrease, and $lim_{x\rightarrow 0}f(x)=0$, we have that f is bounded and $lim_{x\rightarrow\infty}f(x)=0$. Similarly, $\frac{df}{dx}$ is bounded and $lim_{x\rightarrow\infty}\frac{df}{dx}=0$. As $lim_{x\rightarrow\infty}f(x)=0$, and f is non-oscillatory, we have that, for $k\neq 0$, the indefinite integral;

$$lim_{r\rightarrow\infty}\int_0^r f(y)e^{-iky}dy$$

$$= lim_{r\rightarrow\infty}\int_0^r f(y)cos(ky)dy - ilim_{r\rightarrow\infty}\int_0^r f(y)sin(ky)dy$$

exists. As f is of very moderate decrease and non-oscillatory, we have that $|f(x)|\leq\frac{D}{x}$, for $x>E$, with $E\in\mathcal{R}_{>0}$, and monotone in the interval (E,∞). Using the method of [7], letting $K=max(|f||_{(0,E]})$, we have that;

$$|lim_{r\rightarrow\infty}\int_{-r}^r f(y)cos(ky)dy|\leq KE+(|\int_E^{\frac{\pi}{2|k|}+\frac{n_k\pi}{|k|}}\frac{Dcos(ky)}{y}dy|+|\int_{\frac{\pi}{2|k|}+\frac{n_k\pi}{|k|}}^{\frac{\pi}{2|k|}+\frac{(n_k+1)\pi}{|k|}}\frac{Dcos(ky)}{y}dy|)$$

$$\leq KE+\frac{D}{E}(\frac{\pi}{2|k|}+\frac{n_k\pi}{|k|}-E)+\int_E^{E+\frac{\pi}{|k|}}\frac{Dsin(|k|(y-E))}{y}dy$$

$$\leq KE+\frac{D\pi}{E|k|}+\int_E^{E+\frac{\pi}{|k|}}\frac{Dsin(|k|(y-E))}{y}dy$$

$$|lim_{r\rightarrow\infty}\int_{-r}^r f(y)sin(ky)dy|\leq KE+(|\int_E^{\frac{m_k\pi}{|k|}}\frac{Dsin(ky)}{y}dy|+|\int_{\frac{m_k\pi}{|k|}}^{\frac{(m_k+1)\pi}{|k|}}\frac{Dsin(ky)}{y}dy|)$$

$$\leq KE+\frac{D}{E}(\frac{m_k\pi}{|k|}-E)+\int_E^{E+\frac{\pi}{|k|}}\frac{Dsin(|k|(y-E))}{y}dy$$

$$\leq KE+\frac{D\pi}{E|k|}+\int_E^{E+\frac{\pi}{|k|}}\frac{Dsin(|k|(y-E))}{y}dy$$

where $n_k=\mu n(\frac{\pi}{2|k|}+\frac{n\pi}{|k|}\geq E:n\in\mathcal{Z}_{\geq 0})$ and $m_k=\mu n(\frac{n\pi}{|k|}\geq E:n\in\mathcal{Z}_{\geq 0})$

so that;

$$|lim_{r\rightarrow\infty}\int_0^r f(y)e^{-iky}dy|\leq 2KE+\frac{2D\pi}{E|k|}+2\int_E^{E+\frac{\pi}{|k|}}\frac{Dsin(|k|(y-E))}{y}dy$$

$$=2KE+\frac{2D\pi}{E|k|}+2D([\frac{-cos(|k|(y-E))}{|k|y}]_E^{E+\frac{\pi}{|k|}}-\int_E^{E+\frac{\pi}{|k|}}\frac{cos(|k|(y-E))}{y^2}dy)$$

$$=2KE+\frac{2D\pi}{E|k|}+2D(\frac{1}{|k|(E+\frac{\pi}{|k|})}+\frac{1}{E|k|}+\int_E^{E+\frac{\pi}{|k|}}\frac{cos(|k|(y-E))}{y^2}dy)$$

$$\leq 2KE+\frac{2D\pi}{E|k|}+2D(\frac{1}{E|k|+\pi}+\frac{1}{E|k|}+\int_E^{\infty}\frac{1}{y^2}dy)$$

$\leq 2KE+\frac{2D\pi}{E|k|}+2D(\frac{2}{E|k|}+\frac{1}{E})=N_k$

Alternatively, letting $F=max(|f||_{(0,\infty)}$, we have that;

$|lim_{r\rightarrow\infty}\int_{-r}^{r}f(y)cos(ky)dy|\leq FE+(|\int_{E}^{\frac{\pi}{2|k|}+\frac{n_k\pi}{|k|}}Fcos(ky)dy|+|\int_{\frac{\pi}{2|k|}+\frac{n_k\pi}{|k|}}^{\frac{\pi}{2|k|}+\frac{(n_k+1)\pi}{|k|}}Fcos(ky)dy|)$

$\leq FE+F(\frac{\pi}{2|k|}+\frac{n_k\pi}{|k|}-E)+\int_{E}^{E+\frac{\pi}{|k|}}Fsin(|k|(y-E))dy$

$\leq FE+\frac{F\pi}{2|k|}+\int_{E}^{E+\frac{\pi}{|k|}}Fsin(|k|(y-E))dy$

$\leq FE+\frac{F\pi}{2|k|}+\int_{E}^{E+\frac{\pi}{|k|}}Fdy$

$\leq FE+\frac{F\pi}{2|k|}+\frac{F\pi}{|k|}$

$=FE+\frac{3F\pi}{2|k|}$

$|lim_{r\rightarrow\infty}\int_{-r}^{r}f(y)sin(ky)dy|\leq FE+(|\int_{E}^{\frac{m_k\pi}{|k|}}Fsin(ky)dy|+|\int_{\frac{m_k\pi}{|k|}}^{\frac{(m_k+1)\pi}{|k|}}Fsin(ky)dy|)$

$\leq FE+F(\frac{m_k\pi}{|k|}-E)+\int_{E}^{E+\frac{\pi}{|k|}}Fsin(|k|(y-E))dy$

$\leq FE+\frac{F\pi}{2|k|}+\int_{E}^{E+\frac{\pi}{|k|}}Fsin(|k|(y-E))dy$

$\leq FE+\frac{F\pi}{2|k|}+\frac{F\pi}{|k|}$

$=FE+\frac{3F\pi}{2|k|}$

so that;

$|lim_{r\rightarrow\infty}\int_{0}^{r}f(y)e^{-iky}dy|\leq 2FE+\frac{3F\pi}{|k|}=M_k$

In either case, $\mathcal{G}(f)(k)$ and, similarly, $\mathcal{G}(\frac{df}{dx})(k)$ are bounded, for $|k|>k_0>0$, ([2]).

We have, using integration by parts, that;

$\mathcal{G}(\frac{df}{dx})(k)=lim_{r\rightarrow\infty}\int_{0}^{r}\frac{df}{dx}(y)e^{-iky}dy$

$=lim_{r\rightarrow\infty}([f(y)e^{-iky}]_{0}^{r}+ik\int_{0}^{r}f(y)e^{-iky}dy)$

[2] $\mathcal{G}(f)(k)$ and $\mathcal{G}(\frac{df}{dx})(k)$ are differentiable for $k\neq 0$, limit interchange?

$= [f(y)e^{-iky}]_0^\infty + iklim_{r\rightarrow\infty} \int_0^r f(y)e^{-iky}dy$

$= ik\mathcal{G}(f)(k)$

so that, for $|k| > 1$;

$|\mathcal{G}(f)(k)| \leq \frac{|\mathcal{G}(\frac{df}{dx})(k)|}{|k|}$, (†)

As $\frac{df}{dx}$ is continuous, non-oscillatory and bounded, by the proof of Lemma 0.9 in [7], using underflow, for $r \in \mathcal{R}_{>0}$, we can find $\{F_r, G_r\} \subset \mathcal{R}_{>0}$, such that, for all $|k| > F_r$, we have that;

$|\int_0^r \frac{df}{dx}(y)e^{-iky}dy| < \frac{G_r}{|k|}$, $(**)$

It is easy to see from the proof, that $\{F_r, G_r\}$ can be chosen uniformly in r. Then, from $(**)$, we obtain that, for $|k| > F$;

$|\mathcal{G}(\frac{df}{dx})(k)| < \frac{G}{|k|}$, for $|k| > F$

and, from (†), for $|k| > max(F, 1)$, that;

$|\mathcal{G}(f)(k)| \leq \frac{|\mathcal{G}(\frac{df}{dx})(k)|}{|k|} < \frac{G}{|k|^2}$

□

Lemma 0.10. *Let f be light symmetrically asymptotic, then defining $\mathcal{F}(f)$ and $\mathcal{F}(\frac{df}{dx})$ as in Lemma 0.1, we have that, for any $\delta > 0$, there exist constants $\{C_\delta, D_\delta\} \subset \mathcal{R}_{>0}$, such that;*

$|\mathcal{F}(f)(k)| \leq \frac{\delta}{|k|} + \frac{C_\delta}{|k|^2}$, *for* $|k| > D_\delta$

Proof. The proof is a simple generalisation of the proofs of Lemmas **??** and 0.1. □

Definition 0.11. *Polars Attempt We say that $g \in C^\infty(\mathcal{R}^3)$ is polar non-oscillatory if, for $0 \leq \theta < \pi$, $-\pi < \phi \leq \pi$, we have that, for $g_{\theta,\phi}$, there exist finitely many point $\{r_{i,\theta,\phi} : 1 \leq i \leq n\} \subset \mathcal{R}_{>0}$, for which $g_{\theta,\phi}|_{(r_{i,\theta,\phi}, r_{i+1,\theta,\phi})}$ is monotone, $2 \leq i \leq n-2$, and $g_{\theta,\phi}|_{(0,r_{1,\theta,\phi})}$ and $g_{\theta,\phi}|_{(r_{n,\theta,\phi},\infty)}$ is monotone. We say that g is polar decaying if, for $0 \leq \theta < \pi$, $-\pi < \phi \leq \pi$, we have that, there exist constants $\{C, D\} \subset \mathcal{R}_{>0}$, such that $|g_{\theta,\phi}(r)| \leq \frac{D}{r^3}$, for $|r| \geq C$.*

Lemma 0.12. *If g is polar non-oscillatory and decaying, we can define;*

$\mathcal{F}(g)(\overline{k})=\frac{1}{(2\pi)^{\frac{3}{2}}}lim_{R\rightarrow\infty}\int_0^R\int_0^\pi\int_{-\pi}^\pi r^2sin(\theta)g(r,\theta,\phi)e^{-ir(k_1sin(\theta)cos(\phi)+k_2sin(\theta)sin(\phi)+k_3cos(\theta))}drd\theta d$

in polar coordinates, $x_1=rsin(\theta)cos(\phi)$, $x_2=rsin(\theta)sin(\phi)$, $x_3=rcos(\theta)$.

where $\mathcal{F}(g)(\overline{k})=\frac{1}{(2\pi)^{\frac{3}{2}}}\int_{\mathcal{R}^3}g(\overline{x})e^{-i\overline{k}\centerdot\overline{x}}d\overline{x}$

is usually defined for $g\in L^1(\mathcal{R}^3)$.

Proof. We can assume that n is minimal with this property, in which case, the points $\{r_{i,\theta,\phi}:1\leq i\leq n\}\subset\mathcal{R}_{>0}$ are local maxima or minima, and, as $g\in C^\infty(\mathcal{R}^3)$, we have that $\frac{\partial g}{\partial r}|_{(r_{i,\theta,\phi},\theta,\phi)}=0$. By the implicit function theorem, we can find smooth maps $\lambda_i:S^2(1)\rightarrow\mathcal{R}^3$, $1\leq i\leq n$, such that $Im(\lambda_i)\subset\frac{\partial f}{\partial r}=0$ and $\{r_{i,\theta,\phi}:0\leq\theta<\pi,-\pi<\phi\leq\pi\}=Im(\lambda_i)$. As $S^2(1)$ is compact, we have that $pr_r(Im(\lambda_i))$ defines a closed bounded interval $I_i\subset mathcal R_{>0}$. Moreover, it is straightforward to see, as each $g_{\theta,\phi}$ is a function, that $I_i\cap I_j=\emptyset$, for $1\leq i<j\leq n$. Let $m=max(\bigcup_{1\leq i\leq n}I_i)$. Then $g_{\theta,\phi}|_{(m,\infty)}$ is monotone and, if $M=max(|f||_{B(\overline{0},m)})$, as g is decaying, we have that $|g|\leq M$, so that g is bounded. Let $\overline{k}\in\mathcal{R}^3$, with $\overline{k}\neq\overline{0}$, then there exists an open $U_{\overline{k}}\subset(0,\pi)\times(-\pi,\pi)$ with $\overline{k}\centerdot(sin(\theta)cos(\phi),sin(\theta)sin(\phi),cos(\theta))=\nu_{\theta,\phi}\neq 0$, for $(\theta,\phi)\in U_{\overline{k}}$. Let $f_{\theta,\phi}=r^2sin(\theta)g_{\theta,\phi}$, then $|f_{\theta,\phi}|\leq\frac{D}{r}$, for $r>C$, and the same remarks above, apply to $f_{\theta,\phi}$, as to $g_{\theta,\phi}$.

As $lim_{r\rightarrow\infty}f_{\theta,\phi}(r)=0$, we have that the indefinite integral;

$lim_{r\rightarrow\infty}\int_0^r f_{\theta,\phi}(r)e^{-ir(k_1sin(\theta)cos(\phi)+k_2sin(\theta)sin(\phi)+k_3cos(\theta))}dr$

$=lim_{r\rightarrow\infty}\int_0^r f_{\theta,\phi}(r)e^{-ir\nu_{\theta,\phi}}dr$

$=lim_{r\rightarrow\infty}\int_0^r f_{\theta,\phi}(r)cos(r\nu_{\theta,\phi})dr-ilim_{r\rightarrow\infty}\int_0^r f_{\theta,\phi}(r)sin(r\nu_{\theta,\phi})dr$

exists. As $f_{\theta,\phi}$ is monotone, and $|f_{\theta,\phi}|(r)\leq\frac{D}{r}$, for $r>max(m,C)=E$, using the method of [7], letting $K=max(|f||_{B(\overline{0},E)})$, we have that;

$|lim_{R\rightarrow\infty}\int_0^R f_{\theta,\phi}(r)cos(r\nu_{\theta,\phi})dr|\leq KE+K\int_E^{E+\frac{\pi}{2|\nu_{\theta,\phi}|}}\frac{Dcos(|\nu_{\theta,\phi}|(r-E))}{r}dr$

$|lim_{R\rightarrow\infty}\int_0^R f_{\theta,\phi}(r)sin(r\nu_{\theta,\phi})dr| \leq KE+K\int_E^{E+\frac{\pi}{2|\nu_{\theta,\phi}|}} \frac{Dcos(|\nu_{\theta,\phi}|(r-E))}{r}dr$

so that;

$|lim_{R\rightarrow\infty}\int_0^r f_{\theta,\phi}(r)e^{-ir\nu_{\theta,\phi}}dr| \leq 2KE+2K\int_E^{E+\frac{\pi}{2|\nu_{\theta,\phi}|}} \frac{Dcos(|\nu_{\theta,\phi}|(r-E))}{r}dr$

$= 2KE + 2KD([\frac{-sin(|\nu_{\theta,\phi}|(r-E))}{|\nu_{\theta,\phi}|}]_E^{E+\frac{\pi}{2|\nu_{\theta,\phi}|}} - \int_E^{E+\frac{\pi}{2|\nu_{\theta,\phi}|}} \frac{sin(|\nu_{\theta,\phi}|(r-E))}{r^2}dr)$

$= 2KE + 2KD(\frac{1}{|\nu_{\theta,\phi}|(E+\frac{\pi}{2|\nu_{\theta,\phi}|})} - \int_E^{E+\frac{\pi}{2|\nu_{\theta,\phi}|}} \frac{sin(|\nu_{\theta,\phi}|(r-E))}{r^2}dr)$

$\leq 2KE + 2KD(\frac{1}{E|\nu_{\theta,\phi}|+\frac{\pi}{2}} + \int_E^\infty \frac{1}{r^2}dr)$

$\leq 2KE + 2KD(\frac{2}{\pi} + \frac{1}{E}) = N$

uniformly, for $(\theta,\phi) \in U_{\overline{k}}$, so that, using the dominated convergence theorem;

$lim_{R\rightarrow\infty}\int_0^R\int_0^\pi\int_{-\pi}^\pi f(r,\theta,\phi)e^{-ir(k_1sin(\theta)cos(\phi)+k_2sin(\theta)sin(\phi)+k_3cos(\theta))}drd\theta d\phi$

exists, and;

$|lim_{R\rightarrow\infty}\int_0^R\int_0^\pi\int_{-\pi}^\pi f(r,\theta,\phi)e^{-ir(k_1sin(\theta)cos(\phi)+k_2sin(\theta)sin(\phi)+k_3cos(\theta))}drdrd\theta d\phi|$

$\leq \int_0^\pi\int_{-\pi}^\pi |(lim_{R\rightarrow\infty}\int_0^R f(r,\theta,\phi)dr|)d\theta d\phi$

$\leq 2N\pi^2$

□

Definition 0.13. *Cartesian We say that* $g \in C^\infty(\mathcal{R}^3)$ *is Cartesian non-oscillatory if, for* $(x,y) \subset \mathcal{R}^2$*, there exist finitely many point* $\{z_{i,x,y} : 1 \leq i \leq n\} \subset \mathcal{R}$*, for which* $g_{x,y}|_{(z_{i,x,y},z_{i+1,x,y})}$ *is monotone,* $2 \leq i \leq n-2$*, and* $g_{x,y}|_{(-\infty,z_{1,x,y})}$ *and* $g_{x,y}|_{(z_{n,x,y},\infty)}$ *is monotone, and, for fixed* $(x,y) \in \mathcal{R}^2$*, with* $(x,y) \neq (0,0)$*, the ordering of* $\{g_{rx,ry}(z_{i,rx,ry}) : 1 \leq i \leq n\}$ *changes, uniformly in* (x,y)*, at most a finite number of times, with* $r \in \mathcal{R}$*. We say that* $g \in C^\infty(\mathcal{R}^3)$ *is slightly decaying if there exists a constant* $C \in \mathcal{R}_{>0}$ *with* $|g(\overline{x})| \leq \frac{C}{|\overline{x}|}$*, for* $|\overline{x}| > 1$*.*

Remarks 0.14. *The components of a causal field* $\overline{E}$*, obtained using Jefimenko's equations, are slightly decaying (and Cartesian non-oscillatory?) if the charge and current* $(\rho,\overline{J})$ *have compact support.*

Definition 0.15. *We say that $f:\mathcal{R}\rightarrow\mathcal{R}$ is analytic at infinity, if $f(\frac{1}{x})$ has a convergent power series expansion for $|x|<\epsilon$, $\epsilon>0$. We say that f is eventually monotone, if there exists $y_0\in\mathcal{R}_{>0}$ such that $f|_{(-\infty,-y_0)}$ and $f|_{(y_0,\infty)}$ are monotone. We say that $f:\mathcal{R}^3\rightarrow\mathcal{R}$ is analytic at infinity, if $f(\frac{x_0}{x},\frac{y_0}{y},\frac{z_0}{z})$ has a convergent power series expansion for $|\overline{x}|<\epsilon_{(x_0,y_0,z_0)}$, $\epsilon_{(x_0,y_0,z_0)}>0$, $(x_0:y_0:z_0)\in P^2(\mathcal{R})$, where $\overline{x}=(x,y,z)$.*

Lemma 0.16. *If $f:\mathcal{R}\rightarrow\mathcal{R}$, $f\neq 0$ is analytic and analytic at infinity, then it has finitely many zeroes. If $f:\mathcal{R}\rightarrow\mathcal{R}$, $\frac{df}{dx}$ is analytic and analytic at infinity, and $f\neq c$, where $c\in\mathcal{R}$, then f is non-oscillatory. If $f:\mathcal{R}\rightarrow\mathcal{R}$, f is analytic for $|x|>a$, where $a\in\mathcal{R}_{\geq 0}$, analytic at infinity, and $f|_{|x|>a}\neq 0$ then f has finitely many zeroes in the region $|x|>a+1$. If $f:\mathcal{R}\rightarrow\mathcal{R}$, $\frac{df}{dx}$ is analytic for $|x|>a$, analytic at infinity, and $f|_{|x|>a}\neq c$, where $c\in\mathcal{R}$, then f is eventually monotone. If $f:\mathcal{R}^3\rightarrow\mathcal{R}$, $f\neq 0$ is analytic and analytic at infinity, then it has finitely many zeroes. If $f:\mathcal{R}^3\rightarrow\mathcal{R}$, $\frac{\partial f}{\partial x}$ is analytic and analytic at infinity, and $\frac{\partial f}{\partial x}\neq 0$, then, for $(y,z)\in\mathcal{R}^2$, $f_{y,z}$ is non-oscillatory. If $f:\mathcal{R}^3\rightarrow\mathcal{R}$, f is analytic for $|\overline{x}|>a$, where $a\in\mathcal{R}_{\geq 0}$, analytic at infinity, and $f|_{|\overline{x}|>a}\neq 0$ then f has finitely many zeroes in the region $|\overline{x}|>a+1$. If $f:\mathcal{R}^3\rightarrow\mathcal{R}$, $\frac{\partial f}{\partial x}$ is analytic for $|\overline{x}|>a$, analytic at infinity, and $\frac{\partial f}{\partial x}|_{|\overline{x}|>a}\neq 0$, then, for $(y,z)\in\mathcal{R}^2$, $f_{y,z}$ is eventually monotone. A similar statement holds for $\{\frac{\partial f}{\partial y},\frac{\partial f}{\partial z}\}$, with $f_{x,z}$ and $f_{x,y}$ replacing $f_{y,z}$ respectively.*

Proof. For the first claim, suppose that f has infinitely many zeroes. Then we can find a sequence $\{y_i;i\in\mathcal{N}\}$ with $f(y_i)=0$. If the sequence is bounded, then by the Bolzano-Weierstrass Theorem, we can find a subsequence $\{y_{i_k};k\in\mathcal{N}\}$, with $f(y_{i_k})=0$, converging to $y\in\mathcal{R}$. By continuity, we have that $f(y)=0$ and y is a limit point of zeroes. As f is analytic, by the identity theorem, it must be identically zero, contradicting the hypothesis. If the sequence is unbounded, then we can find a subsequence $\{y_{i_k};k\in\mathcal{N}\}$, with $f(y_{i_k})=0$, such that $lim_{k\rightarrow\infty}y_{i_k}=\infty$ or $lim_{k\rightarrow\infty}y_{i_k}=-\infty$. As f is analytic at ∞, we can find $\epsilon>0$, such that $f(y)=0$ for $|y|>\frac{1}{\epsilon}$. By the identity theorem again, f is identically zero, contradicting the hypothesis. It follows that f has finitely many zeroes. For the second claim, as $\frac{df}{dx}\neq 0$, by the first part, there exist finitely many points $\{y_1,\ldots,y_n\}$, with $\frac{df}{dx}|_{y_i}=0$, for $1\leq i\leq n$, and with $y_i<y_{i+1}$, for $1\leq i\leq n-1$. In particularly, we have that $f|_{(-\infty,y_1)}$, $f|_{(y_n,\infty)}$ and $f|_{(y_i,y_{i+1})}$ is monotone for

$1 \leq i \leq n-1$, so that f is non-oscillatory. For the third claim, suppose that f has infinitely many zeroes in the region $|x| > a+1$, then we can find a sequence $\{y_i; i \in \mathcal{N}\}$ with $f(y_i) = 0$ and $|y_i > a+1$. As above, if the sequence is bounded, we can find a subsequence $\{y_{i_k}; k \in \mathcal{N}\}$, with $f(y_{i_k}) = 0$, converging to $y \in \mathcal{R}$, with $|y| \geq a+1 > a$. As f is analytic for $|x| > a$, by the identity theorem, it must be identically zero in the region $|x| > a$, contradicting the hypothesis. If the sequence is unbounded, by the same argument as above, f must be identically zero in the region $|x| > a$, contradicting the hypothesis. It follows that f has finitely many zeroes in the region $|x| > a+1$. For the fourth claim, as $\frac{df}{dx}|_{|x|>a} \neq 0$, by the first part, there exist finitely many points $\{y_1, \ldots, y_n\}$, with $\frac{df}{dx}|_{y_i} = 0$, and $|y_i| > a+1$, for $1 \leq i \leq n$. Choose $y_0 > max_{1 \leq i \leq n}(|y_i|)$, then $\frac{df}{dx}|_{|x|>y_0} \neq 0$, so that $f|_{|x|>y_0}$ is monotone. For the fifth claim, suppose that f has infinitely many zeroes. Then, we can find a sequence $\{\overline{y}_i; i \in \mathcal{N}\}$ with $f(\overline{y}_i) = 0$. If the sequence is bounded, then we can apply the Bolzano-Weierstrass Theorem again, and find a subsequence $\{\overline{y}_{i_k}; k \in \mathcal{N}\}$, with $f(\overline{y}_{i_k}) = 0$, converging to $\overline{y} \in \mathcal{R}^3$. By continuity, we have that $f(\overline{y}) = 0$ and $\overline{y}$ is a limit point of zeroes. As f is analytic, by the identity theorem, it must be identically zero, contradicting the hypothesis. If the sequence is unbounded, then we can find a subsequence $\{\overline{y}_{i_k} : k \in \mathcal{N}\}$, with $f(\overline{y}_{i_k}) = 0$, such that $lim_{k \rightarrow \infty}|\overline{y}_{i_k}| = \infty$. Using the cover $\Gamma : \mathcal{R}^3 \setminus \{0\} \rightarrow P^2(\mathcal{R})$, we can see that the sequence $\{\Gamma(\overline{y}_{i_k}) : k \in \mathcal{N}\}$ has an accumulation point $(x_0, y_0, z_0) \in P^2(\mathcal{R})$. Then, as $f(\frac{x_0}{x}, \frac{y_0}{y}, \frac{z_0}{z})$ has a convergent power series expansion for $|\overline{x}| < \epsilon_{x_0,y_0,z_0}$, using the identity theorem again, $f(\frac{x_0}{x}, \frac{y_0}{y}, \frac{z_0}{z}) = 0$, for $|\overline{x}| < \epsilon_{x_0,y_0,z_0}$. Without loss of generality, by a rotation of coordinates, we can assume that $x_0 \neq 0$, $y_0 \neq 0$, $z_0 \neq 0$. Using the fact that the map $\theta : Ann(\frac{\epsilon_{x_0,y_0,z_0}}{2}, \epsilon_{x_0,y_0,z_0}) \rightarrow \mathcal{R}^3$, $\theta(\overline{x}) = (\frac{x_0}{x}, \frac{y_0}{y}, \frac{z_0}{z})$ is a homeomorphism, we can find an open subset $U \subset \mathcal{R}^3$, for which $f|_U = 0$. Applying the identity theorem again, and using the fact that f is analytic, we must have that $f = 0$, contradicting the hypothesis. For the sixth claim, we can apply the previous proof to obtain that $\frac{\partial f}{\partial x}$ has finitely many zeroes. Then, for fixed $\{y, z\} \subset \mathcal{R}$, $\frac{df_{y,z}}{dx}$ has finitely many zeroes, so, by the previous proof $f_{y,z}$ is non-oscillatory. The proof of the seventh claim follows by combining the proofs of the third and fifth claims. For the eighth claim, we have, by the seventh claim, that $\frac{\partial f}{\partial x}|_{|\overline{x}|>a+1}$ has finitely many zeroes. In particular, we can find $b > a+1$ such that $\frac{\partial f}{\partial x}|_{|\overline{x}|>b} \neq 0$. For fixed $\{y, z\} \subset \mathcal{R}$, it follows that $\frac{df_{y,z}}{dx}|_{|x|>b} \neq 0$, so that $f_{y,z}$ is eventually monotone. The final claim follows by symmetry.

□

Definition 0.17. *We say that* $g : \mathcal{R}^3 \rightarrow \mathcal{R}$ *is of very moderate decrease, if, there exists constants* $C \in \mathcal{R}_{>0}$ *and* $s\imath\mathcal{R}_{>0}$*, with;*

$|g(\overline{x})| \leq \frac{C}{|\overline{x}|}$

for $|\overline{x}| > s$, $\overline{x} \in \mathcal{R}^3$

We say that $f : \mathcal{R}^3 \times \mathcal{R} \rightarrow \mathcal{R}$ *is of uniform very moderate decrease, if, for all* $t \in \mathcal{R}$*, there exists a constants* $D \in \mathcal{R}_{>0}$ *and* $s \in \mathcal{R}_0$*, uniform in* t*, with;*

$|f(\overline{x}, t)| \leq \frac{D}{|\overline{x}|}$

for $|\overline{x}| > s$, $\overline{x} \in \mathcal{R}^3$

We say that $f : \mathcal{R}^3 \times \mathcal{R} \rightarrow \mathcal{R}$ *is of very moderate decrease, if, for all* $t \in \mathcal{R}$*, there exists a constants* $D \in \mathcal{R}_{>0}$ *and* $s_t \in \mathcal{R}_0$*, with;*

$|f(\overline{x}, t)| \leq \frac{D}{|\overline{x}|}$

for $|\overline{x}| > s_t$, $\overline{x} \in \mathcal{R}^3$

Lemma 0.18. *The components of the causal fields* $\overline{E}$ *and* $\overline{B}$*, obtained using Jefimenko's equations, are of uniform very moderate decrease and analytic for* $|\overline{x}| > r$*, and analytic at infinity, if, first, the charge and current* $(\rho, \overline{J})$ *are compactly supported and uniformly bounded with* $t \in \mathcal{R}$ *on a volume* $V \subset B(\overline{0}, w)$*, where* $w \in \mathcal{R}_{>0}$*, secondly, the charge* ρ *and the components* j_i *of the current* $\overline{J}$*, for* $1 \leq i \leq 3$*, are smooth and,third, the charge* ρ *and the components* j_i *of the current* $\overline{J}$*, for* $1 \leq i \leq 3$ *are analytic in* t*. If the initial conditions* $\rho_0 \in S(\mathcal{R}^3), \frac{\partial \rho}{\partial t}|_{t=0} \in S(\mathcal{R}^3)$*, with* ρ *defined on* $\mathcal{R}^4$ *by Kirchoff's formula, then* $\rho \in C^\infty(\mathcal{R}^4)$ *and if* ρ_0 *and* $\frac{\partial \rho}{\partial t}|_{t=0}$ *have compact support, then for* $t \in \mathcal{R}$*,* ρ_t *has compact support, in particularly* $\rho_t \in S(\mathcal{R}^3)$*. If the current* $\overline{J}$ *is defined as in* [11]*, with the conditions in the last clause, then, after subtracting a harmonic, time independent, current* $\overline{J}_0(\overline{x})$*, the components* $j_i \in C^\infty(\mathcal{R}^4)$*,* $1 \leq i \leq 3$*, and for each* $t \in \mathcal{R}$*,* $j_{i,t}$ *has compact support and* $j_{i,t} \in S(\mathcal{R}^3)$*. Suppose that the charge* ρ*, obeys the wave equations on*

$\mathcal{R}^4$, with the current $\overline{J}$ defined as in [11], *and with the initial conditions $\rho_0 \in S(\mathcal{R}^3), \frac{\partial\rho}{\partial t}|_{t=0} \in S(\mathcal{R}^3)$ and with compact support. .*

Proof. For the first claim, we have that;

$$|\overline{E}(\overline{r},t)| = \frac{1}{4\pi\epsilon_0}|\int_V \frac{\rho(\overline{r}',t_r)\hat{\mathfrak{r}}}{|\overline{r}-\overline{r}'|^2}d\tau' + \int_V \frac{\dot{\rho}(\overline{r}',t_r)\hat{\mathfrak{r}}}{c|\overline{r}-\overline{r}'|}d\tau' - \int_V \frac{\dot{\overline{J}}(\overline{r}',t_r)}{c^2|\overline{r}-\overline{r}'|}d\tau'|$$

$$\leq \frac{1}{4\pi\epsilon_0}(\int_V \frac{C_1}{|\overline{r}-\overline{r}'|^2}d\tau' + \int_V \frac{C_2}{c|\overline{r}-\overline{r}'|}d\tau' + \int_V \frac{C_3}{c^2|\overline{r}-\overline{r}'|}d\tau')$$

$$= \frac{1}{4\pi\epsilon_0|\overline{r}|}(\int_V \frac{C_1|\overline{r}|}{|\overline{r}-\overline{r}'|^2}d\tau' + \int_V \frac{C_2|\overline{r}|}{c|\overline{r}-\overline{r}'|}d\tau' + \int_V \frac{C_3|\overline{r}|}{c^2|\overline{r}-\overline{r}'|}d\tau')$$

$$= \frac{1}{4\pi\epsilon_0|\overline{r}|}(\int_V \frac{C_1|\overline{r}-\overline{r}'+\overline{r}'|}{|\overline{r}-\overline{r}'|^2}d\tau' + \int_V \frac{C_2|\overline{r}-\overline{r}'+\overline{r}'|}{c|\overline{r}-\overline{r}'|}d\tau' + \int_V \frac{C_3|\overline{r}-\overline{r}'+\overline{r}'|}{c^2|\overline{r}-\overline{r}'|}d\tau')$$

$$\leq \frac{1}{4\pi\epsilon_0|\overline{r}|}(\int_V \frac{C_1}{|\overline{r}-\overline{r}'|}d\tau' + \int_V \frac{C_1|\overline{r}'|}{|\overline{r}-\overline{r}'|^2}d\tau' + \int_V \frac{C_2}{c}d\tau' + \int_V \frac{C_2|\overline{r}'|}{c|\overline{r}-\overline{r}'|}d\tau' + \int_V \frac{C_3}{c^2}d\tau'$$

$$+ \int_V \frac{C_3|\overline{r}'|}{c^2|\overline{r}-\overline{r}'|}d\tau'$$

$$\leq \frac{1}{4\pi\epsilon_0|\overline{r}|}(\int_V \frac{C_1}{w}d\tau' + \int_V \frac{C_1 w}{w^2}d\tau' + \int_V \frac{C_2}{c}d\tau' + \int_V \frac{C_2 w}{cw}d\tau' + \int_V \frac{C_3}{c^2}d\tau'$$

$$+ \int_V \frac{C_3 w}{c^2 w}d\tau')$$

$$\leq \frac{vol(V)}{4\pi\epsilon_0|\overline{r}|}(\frac{C_1}{w} + \frac{C_1}{w} + \frac{C_2}{c} + \frac{C_2}{c} + \frac{C_3}{c^2} + \frac{C_3}{c^2})$$

$$= \frac{D}{\overline{r}}$$

where $\{C_1, C_2, C_3\} \subset \mathcal{R}_{>0}$ are uniform bounds for $\{\rho, \dot{\rho}, |\overline{J}|\}$ on V, $|\overline{r}| > 2w$ and;

$$D = \frac{vol(V)}{4\pi\epsilon_0|}(\frac{2C_1}{w} + \frac{2C_2}{c} + \frac{2C_3}{c^2})$$

We have that, for $1 \leq i \leq 3$, $|e_i| \leq |\overline{E}| \leq \frac{D}{\overline{r}}$, for $|\overline{r}| > 2w$, so the components of $\overline{E}$ are of very moderate decrease.

We have, following the method above, that, for $|\overline{r}| > 2w$;

$$\overline{B}(\overline{r},t) = \frac{\mu_0}{2\pi}|\int_V \frac{\overline{J}(\overline{r}',t_r)\times\hat{\mathfrak{r}}}{|\overline{r}-\overline{r}'|^2}d\tau' + \int_V \frac{\dot{\overline{J}}(\overline{r}',t_r)\times\hat{\mathfrak{r}}}{c|\overline{r}-\overline{r}'|}d\tau|$$

$$\leq \frac{\mu_0}{2\pi}(\int_V \frac{C_3}{|\overline{r}-\overline{r}'|^2}d\tau' + \int_V \frac{C_4}{c|\overline{r}-\overline{r}'|}d\tau')$$

$$\leq \frac{E}{|\overline{r}}$$

where $C_4 \in \mathcal{R}_{>0}$ is a uniform bound for $|\dot{\overline{J}}|$ on V, $|\overline{r} > 2w$, and;

$E = \frac{\mu_0 vol(V)}{2\pi}(\frac{2C_3}{w} + \frac{2C_4}{c})$

Again, we have that, for $1 \leq i \leq 3$, $|b_i| \leq |\overline{B}| \leq \frac{E}{\overline{r}}$, for $|\overline{r}| > 2w$, so the components of $\overline{B}$ are of very moderate decrease.

For the second claim, expand in coordinates (x, y, z) around a point (x_0, y_0, z_0), with $|\overline{x}_0| > w$, and $|\overline{x} - \overline{x}_0| < \frac{|\overline{x}_0| - w}{4}$. Then, using Newton's expansion;

$(1 + y)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} \frac{(-1)^n (2n)!}{2^n n!} y^n$, $|y| < 1$

and the fact that if $|\overline{x} - \overline{x}_0| < \frac{|\overline{x}_0| - w}{4} < \frac{|\overline{x}_0 - \overline{r}'|}{4}$, then $|\overline{x} - \overline{x}_0| < \frac{|\overline{x}_0 - \overline{r}'|}{\sqrt{2}}$, so that;

$|\frac{|\overline{x}-\overline{x}_0|^2}{|\overline{x}_0-\overline{r}'|^2} + \frac{2(\overline{x}-\overline{x}_0)\boldsymbol{.}(\overline{x}_0-\overline{r}')}{|\overline{x}_0-\overline{r}'|^2}| \leq \frac{|\overline{x}-\overline{x}_0|^2}{|\overline{x}_0-\overline{r}'|^2} + |\frac{2(\overline{x}-\overline{x}_0)\boldsymbol{.}(\overline{x}_0-\overline{r}')}{|\overline{x}_0-\overline{r}'|^2}|$

$< \frac{1}{2} + \frac{2|\overline{x}-\overline{x}_0|}{|\overline{x}_0-\overline{r}'|}$

$< \frac{1}{2} + \frac{1}{2} = 1$

it follows;

$\frac{1}{4\pi\epsilon_0} \int_V \frac{\dot{j}_1(\overline{r}', t_r)}{c^2 |\overline{r} - \overline{r}'|} d\tau'$

$= \frac{1}{4\pi\epsilon_0} \int_V \frac{\dot{j}_1(\overline{r}', t_r)}{c^2 [(x-r_1')^2 + (y-r_2')^2 + (z-r_3')^2]^{\frac{1}{2}}} d\tau'$

$= \frac{1}{4\pi\epsilon_0} \int_V \frac{\dot{j}_1(\overline{r}', t_r)}{c^2 [(x-x_0+x_0-r_1')^2 + (y-y_0+y_0-r_2')^2 + (z-z_0+z_0-r_3')^2]^{\frac{1}{2}}} d\tau'$

$= \frac{1}{4\pi\epsilon_0} \int_V \frac{\dot{j}_1(\overline{r}', t_r)}{c^2 [|\overline{x}-\overline{x}_0|^2 + |\overline{x_0}-\overline{r}'|^2 + 2(\overline{x}-\overline{x}_0)\boldsymbol{.}(\overline{x}_0-\overline{r}')]^{\frac{1}{2}}} d\tau'$

$= \frac{1}{4\pi\epsilon_0} \int_V \frac{\dot{j}_1(\overline{r}', t_r)}{c^2 |\overline{x}_0-\overline{r}'| [1 + \frac{|\overline{x}-\overline{x}_0|^2}{|\overline{x}_0-\overline{r}'|^2} + \frac{2(\overline{x}-\overline{x}_0)\boldsymbol{.}(\overline{x}_0-\overline{r}')}{|\overline{x}_0-\overline{r}'|^2}]^{\frac{1}{2}}} d\tau'$

$= \frac{1}{4\pi\epsilon_0 c^2} \int_V \frac{\dot{j}_1(\overline{r}', t_r)}{|\overline{x}_0-\overline{r}'|} (\sum_{n=0}^{\infty} \frac{(-1)^n (2n)!}{2^n n!} (\frac{|\overline{x}-\overline{x}_0|^2}{|\overline{x}_0-\overline{r}'|^2} + \frac{2(\overline{x}-\overline{x}_0)\boldsymbol{.}(\overline{x}_0-\overline{r}')}{|\overline{x}_0-\overline{r}'|^2})^n) d\tau'$

We have that;

$\int_V |\frac{\dot{j}_1(\overline{r}', t_r)}{|\overline{x}_0-\overline{r}'|} (\frac{(-1)^n (2n)!}{2^n n!} (\frac{|\overline{x}-\overline{x}_0|^2}{|\overline{x}_0-\overline{r}'|^2} + \frac{2(\overline{x}-\overline{x}_0)\boldsymbol{.}(\overline{x}_0-\overline{r}')}{|\overline{x}_0-\overline{r}'|^2})^n)| d\tau'$

$$\leq \frac{(2n)!}{2^n n!}\int_V \frac{|\dot{j_1}(\overline{r}',t_r)|}{|\overline{x}_0|-w}\sum_{m=0}^n C_m^n\left(\frac{(\frac{|\overline{x}_0-w}{4})^2}{(|\overline{x}_0|-w)^2}\right)^{n-m}\left(\frac{2(\frac{|\overline{x}_0|-w}{4})}{\overline{x}_0-w}\right)^m d\tau'$$

$$= \frac{(2n)!}{2^n n!}\int_V \frac{|\dot{j_1}(\overline{r}',t_r)|}{|\overline{x}_0|-w}\sum_{m=0}^n C_m^n\left(\frac{1}{16}\right)^{n-m}\left(\frac{1}{2}\right)^m d\tau'$$

$$\leq \frac{(2n)!}{2^n n!(|\overline{x}_0|-w)}\int_V |\dot{j_1}(\overline{r}',t_r)|\left(\frac{9}{16}\right)^n d\tau'$$

$$\leq \frac{C_1(2n)!(\frac{9}{16})^n}{2^n n!(|\overline{x}_0|-w)}$$

and, using Newton's expansion;

$$(1-y)^{-\frac{1}{2}} = \sum_{n=0}^\infty \frac{(2n)!}{2^n n!}y^n,\ |y|<1$$

we have that;

$$\sum_{n=0}^\infty \frac{C_1(2n)!(\frac{9}{16})^n}{2^n n!|\overline{x}_0|-w} = \frac{C_1}{|\overline{x}_0|-w}\frac{1}{(1-\frac{9}{16})^{\frac{1}{2}}} = \frac{4C_1}{\sqrt{7}(|\overline{x}_0|-w)}$$

so that, applying the DCT, we have that;

$$\frac{1}{4\pi\epsilon_0}\int_V \frac{\dot{j_1}(\overline{r}',t_r)}{c^2|\overline{r}-\overline{r}'|}d\tau'$$

$$= \frac{1}{4\pi\epsilon_0 c^2}\sum_{n=0}^\infty \frac{(-1)^n(2n)!}{2^n n!}\int_V \frac{\dot{j_1}(\overline{r}',t_r)}{|\overline{x}_0-\overline{r}'|}\left(\frac{|\overline{x}-\overline{x}_0|^2}{|\overline{x}_0-\overline{r}'|^2}+\frac{2(\overline{x}-\overline{x}_0)\boldsymbol{\cdot}(\overline{x}_0-\overline{r}')}{|\overline{x}_0-\overline{r}'|^2}\right)^n d\tau'\ (\dagger)$$

and integrating the coefficients of $(x_1-x_{1,0})^i(x_2-x_{2,0})^j(x_3-x_{3,0})^k$ $(i,j,k)\in\mathcal{Z}^3_{\geq 0}$, in the expansion $(\dagger)$, to obtain constants $a_{ijk}\in\mathcal{R}$, $(i,j,k)\in\mathcal{Z}^3_{\geq 0}$, we see that the series;

$$\sum_{(i,j,k)\in\mathcal{Z}^3_{\geq 0}} a_{ijk}(x_1-x_{1,0})^i(x_2-x_{2,0})^j(x_3-x_{3,0})^k$$

is absolutely convergent for $|\overline{x}-\overline{x}_0|<\frac{|\overline{x}_0|-w}{4}$.

For the third claim, assuming that $\{x,y,z,x_0,y_0,z_0\}\subset\mathcal{R}\setminus\{0\}$, we have that;

$$\frac{1}{|\overline{r}-\overline{r}'|}\Big|_{(\frac{x_0}{x},\frac{y_0}{y},\frac{z_0}{z})} = \frac{1}{[(\frac{x_0}{x}-r_1')^2+(\frac{y_0}{y}-r_2')^2+(\frac{z_0}{z}-r_3')^2]^{\frac{1}{2}}}$$

$$= \frac{x}{x_0[(1-\frac{r_1'x}{x_0})^2+(\frac{y_0x}{x_0y}-\frac{r_2'x}{x_0})^2+(\frac{z_0x}{x_0z}-\frac{r_3'x}{x_0})^2]^{\frac{1}{2}}}$$

$$= \frac{x}{x_0[1+x[-\frac{2r_1'}{x_0}-\frac{2r_2'}{x_0}(\frac{x}{y})(\frac{y_0}{x_0})-\frac{2r_3'}{x_0}(\frac{x}{z})(\frac{z_0}{x_0})]+x^2(\frac{|\overline{r}'|^2}{x_0^2})+(\frac{x}{y})^2(\frac{y_0}{x_0})^2+(\frac{x}{z})^2(\frac{z_0}{x_0})^2]^{\frac{1}{2}}}\quad (A)$$

$$= \frac{y}{y_0[(\frac{x_0y}{y_0x}-\frac{r_1'y}{y_0})^2+(1-\frac{r_2'y}{y_0})^2+(\frac{z_0y}{y_0z}-\frac{r_3'y}{y_0})^2]^{\frac{1}{2}}}$$

$$= \frac{y}{y_0[1+y(-\frac{2r'_2}{y_0}-\frac{2r'_1}{y_0}(\frac{y}{x})(\frac{x_0}{y_0})-\frac{2r'_3}{y_0}(\frac{y}{z})(\frac{z_0}{y_0}))+y^2(\frac{|\bar{r}'|^2}{y_0^2})+(\frac{y}{x})^2(\frac{x_0}{y_0})^2+(\frac{y}{z})^2(\frac{z_0}{y_0})^2]^{\frac{1}{2}}} \quad (B)$$

$$= \frac{z}{z_0[(\frac{x_0 z}{z_0 x}-\frac{r'_1 z}{z_0})^2+(\frac{y_0 z}{z_0 y}-\frac{r'_2 z}{z_0})^2+(1-\frac{r'_3 z}{z_0})^2]^{\frac{1}{2}}}$$

$$= \frac{z}{z_0[1+z(-\frac{2r'_3}{z_0}-\frac{2r'_1}{z_0}(\frac{z}{x})(\frac{x_0}{z_0})-\frac{2r'_2}{z_0}(\frac{z}{y})(\frac{y_0}{z_0}))+z^2(\frac{|\bar{r}'|^2}{z_0^2})+(\frac{z}{x})^2(\frac{x_0}{z_0})^2+(\frac{z}{y})^2(\frac{y_0}{z_0})^2]^{\frac{1}{2}}} \quad (C)$$

If $0 < \epsilon < 1$, and $|\frac{x}{y}| < \sqrt{1-\epsilon}|\frac{x_0}{y_0}|$ and $|\frac{x}{z}| < \sqrt{\epsilon}|\frac{x_0}{z_0}|$, then

$\alpha(x,y,z) = (\frac{x}{y})^2(\frac{y_0}{x_0})^2 + (\frac{x}{z})^2(\frac{z_0}{x_0})^2 < 1$

and if;

$|x| < min(\frac{|x_0|\sqrt{1-\alpha}}{\sqrt{2}w}, \frac{|x_0|(1-\alpha)}{12w})$

then, in (A);

$x[-\frac{2r'_1}{x_0}-\frac{2r'_2}{x_0}(\frac{x}{y})(\frac{y_0}{x_0})-\frac{2r'_3}{x_0}(\frac{x}{z})(\frac{z_0}{x_0})]+x^2(\frac{|\bar{r}'|^2}{x_0^2})+(\frac{x}{y})^2(\frac{y_0}{x_0})^2+(\frac{x}{z})^2(\frac{z_0}{x_0})^2 < 1$

(D)

Similarly, if $0 < \delta < 1$, and $|\frac{y}{x}| < \sqrt{1-\delta}|\frac{y_0}{x_0}|$ and $|\frac{y}{z}| < \sqrt{\delta}|\frac{y_0}{z_0}|$, then

$\beta(x,y,z) = (\frac{y}{x})^2(\frac{x_0}{y_0})^2 + (\frac{y}{z})^2(\frac{z_0}{y_0})^2 < 1$

and if;

$|y| < min(\frac{|y_0|\sqrt{1-\beta}}{\sqrt{2}w}, \frac{|y_0|(1-\beta)}{12w})$

then, in (B);

$y(-\frac{2r'_2}{y_0}-\frac{2r'_1}{y_0}(\frac{y}{x})(\frac{x_0}{y_0})-\frac{2r'_3}{y_0}(\frac{y}{z})(\frac{z_0}{y_0}))+y^2(\frac{|\bar{r}'|^2}{y_0^2})+(\frac{y}{x})^2(\frac{x_0}{y_0})^2+(\frac{y}{z})^2(\frac{z_0}{y_0})^2 < 1$

(E)

and, if $0 < \theta < 1$, and $|\frac{z}{x}| < \sqrt{1-\theta}|\frac{z_0}{x_0}|$ and $|\frac{z}{y}| < \sqrt{\theta}|\frac{z_0}{y_0}|$, then

$\gamma(x,y,z) = (\frac{x}{y})^2(\frac{y_0}{x_0})^2 + (\frac{x}{z})^2(\frac{z_0}{x_0})^2 < 1$

and if;

$|z| < min(\frac{|z_0|\sqrt{1-\theta}}{\sqrt{2}w}, \frac{|z_0|(1-\theta)}{12w})$

then, in (C);

$z(-\frac{2r'_3}{z_0} - \frac{2r'_1}{z_0}(\frac{z}{x})(\frac{x_0}{z_0}) - \frac{2r'_2}{z_0}(\frac{z}{y})(\frac{y_0}{z_0})) + z^2(\frac{|\overline{r}'|^2}{z_0^2}) + (\frac{z}{x})^2(\frac{x_0}{z_0})^2 + (\frac{z}{y})^2(\frac{y_0}{z_0})^2 < 1$

(F)

In case (D), we can expand (A) using Newton's theorem, as;

$\frac{x}{x_0}\sum_{n=0}^{\infty}\frac{(-1)^n(2n)!}{2^n n!}[x[-\frac{2r'_1}{x_0} - \frac{2r'_2}{x_0}(\frac{x}{y})(\frac{y_0}{x_0}) - \frac{2r'_3}{x_0}(\frac{x}{z})(\frac{z_0}{x_0})] + x^2(\frac{|\overline{r}'|^2}{x_0^2}) + (\frac{x}{y})^2(\frac{y_0}{x_0})^2$

$+(\frac{x}{z})^2(\frac{z_0}{x_0})^2]^n$

$= \sum_{i+j+k\geq 0} a_{ijk}x^i(\frac{x}{y})^j(\frac{x}{z})^k$

with $|\frac{x}{y}| < \sqrt{1-\epsilon}|\frac{x_0}{y_0}|$ and $|\frac{x}{z}| < \sqrt{\epsilon}|\frac{x_0}{z_0}|$, $(*)$. If $|x| > x_1 > 0$, then if;

$|y| > \frac{|x||\frac{y_0}{x_0}|}{\sqrt{1-\epsilon}}$

implies that;

$|y| > \frac{x_1|\frac{y_0}{x_0}|}{\sqrt{1-\epsilon}}$

and, for $m \in \mathcal{N}$, we can obtain an expansion of $\frac{1}{y}$ in the region;

$\frac{x_1|\frac{y_0}{x_0}|}{\sqrt{1-\epsilon}} < |y| < \frac{mx_1|\frac{y_0}{x_0}|}{\sqrt{1-\epsilon}}$

by noting that, with $c = \frac{x_1(1+m)|\frac{y_0}{x_0}|}{2\sqrt{1-\epsilon}}$, $|y-c| < c$, so that;

$\frac{1}{y} = \frac{1}{c+y-c} = \frac{1}{c(1+\frac{(y-c)}{c})}$

$= \frac{1}{c}\sum_{n=0}^{\infty}(-1)^n\frac{(y-c)^n}{c^n}$

$= \frac{1}{\frac{x_1(1+m)|\frac{y_0}{x_0}|}{2\sqrt{1-\epsilon}}}\sum_{n=0}^{\infty}(-1)^n\frac{(y-(\frac{x_1(1+m)|\frac{y_0}{x_0}|}{2\sqrt{1-\epsilon}}))^n}{(\frac{x_1(1+m)|\frac{y_0}{x_0}|}{2\sqrt{1-\epsilon}})^n}$

....

In case (E), we can expand (B) as;

$\frac{y}{y_0}\sum_{n=0}^{\infty}\frac{(-1)^n(2n)!}{2^n n!}[y(-\frac{2r'_2}{y_0}-\frac{2r'_1}{y_0}(\frac{y}{x})(\frac{x_0}{y_0})-\frac{2r'_3}{y_0}(\frac{y}{z})(\frac{z_0}{y_0}))+y^2(\frac{|\overline{r}'|^2}{y_0^2})+(\frac{y}{x})^2(\frac{x_0}{y_0})^2$

$+(\frac{y}{z})^2(\frac{z_0}{y_0})^2]^n$

$=\sum_{i+j+k\geq 0} b_{ijk}y^i(\frac{y}{x})^j(\frac{y}{z})^k$

with $|\frac{y}{x}|<\sqrt{1-\delta}|\frac{y_0}{x_0}|$ and $|\frac{y}{z}|<\sqrt{\delta}|\frac{y_0}{z_0}|$

In case (F), we can expand (C) as;

$\frac{z}{z_0}\sum_{n=0}^{\infty}\frac{(-1)^n(2n)!}{2^n n!}[z(-\frac{2r'_3}{z_0}-\frac{2r'_1}{z_0}(\frac{z}{x})(\frac{x_0}{z_0})-\frac{2r'_2}{z_0}(\frac{z}{y})(\frac{y_0}{z_0}))+z^2(\frac{|\overline{r}'|^2}{z_0^2})+(\frac{z}{x})^2(\frac{x_0}{z_0})^2$

$+(\frac{z}{y})^2(\frac{y_0}{z_0})^2]^n$

$=\sum_{i+j+k\geq 0} c_{ijk}z^i(\frac{z}{x})^j(\frac{z}{y})^k$

with $|\frac{z}{x}|<\sqrt{1-\theta}|\frac{z_0}{x_0}|$ and $|\frac{z}{y}|<\sqrt{\theta}|\frac{z_0}{y_0}|$

....

For the fourth claim, suppose the initial conditions $\rho_0\in S(\mathcal{R}^3),\frac{\partial\rho}{\partial t}|_{t=0}\in S(\mathcal{R}^3)$, have compact support, with ρ defined on $\mathcal{R}^4$ by Kirchoff's formula;

For $t>0$;

$\rho(\overline{x},t)=\frac{1}{4\pi c^2t^2}\int_{\delta B(\overline{x},ct)}(tg(\overline{y})+\rho_0(\overline{y})+D\rho_0(\overline{y})\cdot(\overline{y}-\overline{x}))dS(\overline{y})$

and, for $t<0$;

$\rho(\overline{x},t)=\frac{1}{4\pi c^2t^2}\int_{\delta B(\overline{x},-ct)}(tg(\overline{y})+\rho_0(\overline{y})+D\rho_0(\overline{y})\cdot(\overline{y}-\overline{x}))dS(\overline{y})$

then, see [**?**] and the construction in [11], we have that, for $\overline{x}\in\mathcal{R}^3$;

$lim_{t\rightarrow 0+}\rho(\overline{x},t)=\rho(\overline{x},0)$

$lim_{t\rightarrow 0+}\frac{\partial\rho}{\partial t}(\overline{x},t)=g(\overline{x})$

$lim_{t\rightarrow 0+}\rho(\overline{x},-t)=\rho(\overline{x},0)$

$lim_{t\rightarrow 0+}\frac{\partial\rho}{\partial t}(\overline{x},-t)=-g(\overline{x})$

where $g((\overline{x})) = \frac{\partial\rho}{\partial t}|_{t=0}$, so that;

$lim_{t\rightarrow 0-}\rho(\overline{x},t) = \rho(\overline{x},0)$

$lim_{t\rightarrow 0-}\frac{\partial\rho}{\partial t}(\overline{x},t) = lim_{t\rightarrow 0+} - \frac{\partial\rho}{\partial t}(\overline{x},-t)$

$= - - g(\overline{x})$

$= g(\overline{x})$

In particular;

$lim_{t\rightarrow 0}\rho(\overline{x},t) = \rho(\overline{x},0)$

$lim_{t\rightarrow 0}\frac{\partial\rho}{\partial t}(\overline{x},t) = g(\overline{x})$

Using the fact that $\rho_0 \in S(\mathcal{R}^3)$, $g(\overline{x}) \in S(\mathcal{R}^3)$, the transform method, see Lemma **??** and uniqueness of the wave equation solution, given the 2 initial conditions, we have for $t > 0$;

$\rho(\overline{x},t) = \frac{1}{(2\pi)^{\frac{3}{2}}}\int_{\mathcal{R}^3}(b(\overline{k})e^{ikct} + d(\overline{k})e^{-ikct})e^{i\overline{k}.\overline{x}}d\overline{k}$

$\rho(\overline{x},-t) = \frac{1}{(2\pi)^{\frac{3}{2}}}\int_{\mathcal{R}^3}(b^-(\overline{k})e^{ikct} + d^-(\overline{k})e^{-ikct})e^{i\overline{k}.\overline{x}}d\overline{k}\ (X)$

where;

$b(\overline{k}) = \frac{1}{2}(\mathcal{F}(\rho_0)(\overline{k}) + \frac{1}{ikc}\mathcal{F}(g)(\overline{k}))$

$d(\overline{k}) = \frac{1}{2}(\mathcal{F}(\rho_0)(\overline{k}) - \frac{1}{ikc}\mathcal{F}(g)(\overline{k}))$

$b^-(\overline{k}) = \frac{1}{2}(\mathcal{F}(\rho_0)(\overline{k}) + \frac{1}{ikc}\mathcal{F}(-g)(\overline{k}))$

$= \frac{1}{2}(\mathcal{F}(\rho_0)(\overline{k}) - \frac{1}{ikc}\mathcal{F}(g)(\overline{k}))$

$d^-(\overline{k}) = \frac{1}{2}(\mathcal{F}(\rho_0)(\overline{k}) - \frac{1}{ikc}\mathcal{F}(-g)(\overline{k}))$

$= \frac{1}{2}(\mathcal{F}(\rho_0)(\overline{k}) + \frac{1}{ikc}\mathcal{F}(g)(\overline{k}))$

see also earlier in the paper, so that, for $t < 0$;

$\rho(\overline{x},t)=\frac{1}{(2\pi)^{\frac{3}{2}}}\int_{\mathcal{R}^3}(b^-(\overline{k})e^{-ikct}+d^-(\overline{k})e^{ikct})e^{i\overline{k}.\overline{x}}d\overline{k}$ (Y)

Differentiating under the integral sign in (X), we have that, for $t>0$;

$\frac{\partial^{i+j+k}\rho}{\partial x^i\partial y^j\partial z^k}(\overline{x},t)=\frac{1}{(2\pi)^{\frac{3}{2}}}\int_{\mathcal{R}^3}((ik_1)^i(ik_2)^j(ik_3)^k b(\overline{k})e^{ikct}+(ik_1)^i(ik_2)^j(ik_3)^k d(\overline{k})e^{-ikct})e^{i\overline{k}.\overline{x}}d\overline{k}$

where $(ik_1)^i(ik_2)^j(ik_3)^k b(\overline{k})\in S(\mathcal{R}^3)$ and $(ik_1)^i(ik_2)^j(ik_3)^k d(\overline{k})\in S(\mathcal{R}^3)$, so that;

$lim_{t\rightarrow 0+}\frac{\partial^{i+j+k}\rho}{\partial x^i\partial y^j\partial z^k}(\overline{x},t)$

$=lim_{t\rightarrow 0+}\frac{1}{(2\pi)^{\frac{3}{2}}}\int_{\mathcal{R}^3}((ik_1)^i(ik_2)^j(ik_3)^k b(\overline{k})e^{ikct}+(ik_1)^i(ik_2)^j(ik_3)^k d(\overline{k})e^{-ikct})e^{i\overline{k}.\overline{x}}d\overline{k}$

$=\frac{1}{(2\pi)^{\frac{3}{2}}}\int_{\mathcal{R}^3}((ik_1)^i(ik_2)^j(ik_3)^k b(\overline{k})+(ik_1)^i(ik_2)^j(ik_3)^k d(\overline{k}))e^{i\overline{k}.\overline{x}}d\overline{k}$

$=\frac{1}{(2\pi)^{\frac{3}{2}}}\int_{\mathcal{R}^3}(ik_1)^i(ik_2)^j(ik_3)^k\mathcal{F}(\rho_0)(\overline{k})e^{i\overline{k}.\overline{x}}d\overline{k}$

$=\frac{\partial^{i+j+k}\rho}{\partial x^i\partial y^j\partial z^k}(\overline{x},0)$ $(X)'$

Similarly, differentiating under the integral sign in (Y), using the fact that $b^-(\overline{k})+d^-(\overline{k})=\mathcal{F}(\rho_0)(\overline{k})$;

$lim_{t\rightarrow 0-}\frac{\partial^{i+j+k}\rho}{\partial x^i\partial y^j\partial z^k}(\overline{x},t)=\frac{\partial^{i+j+k}\rho}{\partial x^i\partial y^j\partial z^k}(\overline{x},0)$ (Y')

and combining $(X)',(Y)'$, we obtain that;

$lim_{t\rightarrow 0}\frac{\partial^{i+j+k}\rho}{\partial x^i\partial y^j\partial z^k}(\overline{x},t)=\frac{\partial^{i+j+k}\rho}{\partial x^i\partial y^j\partial z^k}(\overline{x},0)$

By a similar argument, differentiating under the integral sign, and using the facts that $b(\overline{k})ikc-d(\overline{k}ikc=\mathcal{F}(g)(\overline{k})$ $-ikcb^-(\overline{k})+ikcd^-(\overline{k})=\mathcal{F}(g)(\overline{k})$;
$lim_{t\rightarrow 0}\frac{\partial^{i+j+k+1}\rho}{\partial x^i\partial y^j\partial z^k\partial t}(\overline{x},t)=\frac{\partial^{i+j+k}g}{\partial x^i\partial y^j\partial z^k}(\overline{x},0)$

Similarly, using the fact that $\rho_0\in S(\mathcal{R}^3)$, $\{b(\overline{k}),d(\overline{k})\}\subset L^1(\mathcal{R}^3)$, so we can apply the inversion theorem, we have that;

$lim_{t\rightarrow 0+}\frac{\partial^{i+j+k+2}\rho}{\partial x^i\partial y^j\partial z^k\partial t^2}(\overline{x},t)$

$=lim_{t\rightarrow 0+}\frac{1}{(2\pi)^{\frac{3}{2}}}\int_{\mathcal{R}^3}(ik_1)^i(ik_2)^j(ik_3)^k(-k^2c^2)b(\overline{k})e^{ikct}$

$+(ik_1)^i(ik_2)^j(ik_3)^k(-k^2c^2)d(\overline{k})e^{-ikct})e^{i\overline{k}.\overline{x}}d\overline{k}$

$= \frac{1}{(2\pi)^{\frac{3}{2}}}\int_{\mathcal{R}^3}(ik_1)^i(ik_2)^j(ik_3)^k(-k^2c^2)(b(\overline{k})+d(\overline{k}))e^{i\overline{k}.\overline{x}}d\overline{k}$

$= \frac{1}{(2\pi)^{\frac{3}{2}}}\int_{\mathcal{R}^3}(ik_1)^i(ik_2)^j(ik_3)^k(-k^2c^2)(\mathcal{F}(\rho_0)(\overline{k})e^{i\overline{k}.\overline{x}}d\overline{k}$

$= \frac{1}{(2\pi)^{\frac{3}{2}}}\int_{\mathcal{R}^3}c^2(\mathcal{F}(\frac{\partial^{i+j+k}\bigtriangledown^2(\rho_0)}{\partial x^i\partial y^j\partial z^k})(\overline{k})e^{i\overline{k}.\overline{x}}d\overline{k}$

$= c^2\frac{\partial^{i+j+k}\bigtriangledown^2(\rho_0)}{\partial x^i\partial y^j\partial z^k}(\overline{x})$

and;

$$lim_{t\rightarrow 0-}\frac{\partial^{i+j+k+2}\rho}{\partial x^i\partial y^j\partial z^k\partial t^2}(\overline{x},t) = \frac{\partial^{i+j+k}c^2\bigtriangledown^2(\rho_0)}{\partial x^i\partial y^j\partial z^k}(\overline{x})$$

As $\rho|_{t>0}$, $\rho|_{t<0}$ obey the wave equation, so do the partial derivatives $\frac{\partial^{i+j+k+l}}{\partial x^i\partial y^j\partial z^k\partial t^l}|_{t>0}$, so that, for $l\geq 1$, l even, $t\neq 0$;

$$\frac{\partial^{i+j+k+l}\rho}{\partial x^i\partial y^j\partial z^k\partial t^l}|_{t\neq 0} = c^l(\bigtriangledown^2)^{\frac{l}{2}}(\frac{\partial^{i+j+k}\rho}{\partial x^i\partial y^j\partial z^k})|_{t\neq 0}$$

and, for $l\geq 1$, l odd, $t\neq 0$;

$$\frac{\partial^{i+j+k+l}\rho}{\partial x^i\partial y^j\partial z^k\partial t^l}|_{t\neq 0} = c^{l-1}(\bigtriangledown^2)^{\frac{l-1}{2}}(\frac{\partial^{i+j+k+1}\rho}{\partial x^i\partial y^j\partial z^k\partial t})|_{t\neq 0}$$

and, using the above, for l even;

$$lim_{t\rightarrow 0}\frac{\partial^{i+j+k+l}\rho(\overline{x},t)}{\partial x^i\partial y^j\partial z^k\partial t^l} = c^l(\bigtriangledown^2)^{\frac{l}{2}}(\frac{\partial^{i+j+k}\rho_0}{\partial x^i\partial y^j\partial z^k})$$

and, for l odd;

$$lim_{t\rightarrow 0}\frac{\partial^{i+j+k+l}\rho(\overline{x},t)}{\partial x^i\partial y^j\partial z^k\partial t^l} = c^{l-1}(\bigtriangledown^2)^{\frac{l-1}{2}}(\frac{\partial^{i+j+k}g}{\partial x^i\partial y^j\partial z^k})$$

In particularly, as all the partial derivatives of ρ extend continuously to the boundary $t=0$, we have that $\rho\in C^\infty(\mathcal{R}^4)$, and the wave equation is satisfied at $t=0$, $\frac{\partial^2\rho}{\partial t^2} = c^2\bigtriangledown^2(\rho)$, ([3]). By Kirchoff's formula,

[3] It is relatively straightforward calculation to check, using the integral representation of a solution to the wave equation, $\bigtriangledown^2(f)-\frac{1}{c^2}\frac{\partial^2 f}{\partial t^2}=0$ in $\mathcal{R}^3\times[0,\infty)$, generated by the initial data (g,h), that $lim_{t\rightarrow 0+}\frac{\partial^{i+j+k+l}f_t}{\partial x^i\partial x^j\partial z^k\partial t^l} = (c^2\bigtriangledown^2)^{\frac{i}{2}}\frac{\partial^{i+j+k+l}g}{\partial x^i\partial x^j\partial z^k}$ for i even and that $lim_{t\rightarrow 0+}\frac{\partial^{i+j+k+l}f_t}{\partial x^i\partial x^j\partial z^k\partial t^l} = (c^2\bigtriangledown^2)^{\frac{i-1}{2}}\frac{\partial^{i+j+k+l}h}{\partial x^i\partial x^j\partial z^k}$ for i odd. By uniqueness of the wave equation with specified initial conditions (g,h), the same must be true for Kirchoff's representation. The same result holds for the backward wave equation with initial data $(g,-h)$, so the limit of the partial derivatives is

same for $t > 0$ as $t < 0$. We have, if;

$\rho(\overline{x},t) = \frac{1}{4\pi c^2t^2}\int_{\delta B(\overline{x},ct)}(tg(\overline{y}) + \rho_0(\overline{y}) + D\rho_0(\overline{y}) \centerdot (\overline{y}-\overline{x}))dS(\overline{y})$ $(t > 0)$

$\rho(\overline{x},t) = \frac{1}{4\pi c^2t^2}\int_{\delta B(\overline{x},-ct)}(tg(\overline{y}) + \rho_0(\overline{y}) + D\rho_0(\overline{y}) \centerdot (\overline{y}-\overline{x}))dS(\overline{y})$ $(t < 0)$

Then, for $t > 0$, $\rho(\overline{x},t) = \rho(\overline{x},-t)$ iff;

$\frac{1}{4\pi c^2t^2}\int_{\delta B(\overline{x},ct)}(tg(\overline{y}) + \rho_0(\overline{y}) + D\rho_0(\overline{y}) \centerdot (\overline{y}-\overline{x}))dS(\overline{y})$

$= \frac{1}{4\pi c^2t^2}\int_{\delta B(\overline{x},ct)}(-tg(\overline{y}) + \rho_0(\overline{y}) + D\rho_0(\overline{y}) \centerdot (\overline{y}-\overline{x}))dS(\overline{y})$

iff $\frac{1}{4\pi c^2t^2}\int_{\delta B(\overline{x},ct)} 2tg(\overline{y})dS(\overline{y}) = 0$

iff $\int_{\delta B(\overline{x},ct)} g(\overline{y})dS(\overline{y}) = 0$

iff $g(\overline{y}) = 0$

as if $g(\overline{y}_0) \neq 0$, without loss of generality, by continuity, we can choose $t_0 > 0$ sufficiently small with $g|_{\delta B(\overline{y}_0,ct)} > 0$, so that $\int_{\delta B(\overline{y}_0,ct_0)} g(\overline{y})dS(\overline{y}) > 0$

and, for $t > 0$, $\rho(\overline{x},t) = -\rho(\overline{x},-t)$ iff;

$\frac{1}{4\pi c^2t^2}\int_{\delta B(\overline{x},ct)}(tg(\overline{y}) + \rho_0(\overline{y}) + D\rho_0(\overline{y}) \centerdot (\overline{y}-\overline{x}))dS(\overline{y})$

$= \frac{1}{4\pi c^2t^2}\int_{\delta B(\overline{x},ct)}(tg(\overline{y}) - \rho_0(\overline{y}) - D\rho_0(\overline{y}) \centerdot (\overline{y}-\overline{x}))dS(\overline{y})$

iff $\frac{1}{4\pi c^2t^2}\int_{\delta B(\overline{x},ct)} 2[\rho_0(\overline{y}) + D\rho_0(\overline{y}) \centerdot (\overline{y}-\overline{x})]dS(\overline{y}) = 0$

iff $\int_{\delta B(\overline{x},ct)}[\rho_0(\overline{y}) + D\rho_0(\overline{y}) \centerdot (\overline{y}-\overline{x})]dS(\overline{y}) = 0$

iff $\int_{\delta B(\overline{x},ct)} \rho_0(\overline{y})dS(\overline{y}) + ct\int_{\delta B(\overline{x},ct)} \bigtriangledown(\rho_0) \centerdot d\overline{S} = 0$

iff $\int_{\delta B(\overline{x},ct)} \rho_0(\overline{y})dS(\overline{y}) + ct\int_{B(\overline{x},ct)} div(\bigtriangledown(\rho_0))dV(\overline{y}) = 0$

iff $\int_{\delta B(\overline{x},ct)} \rho_0(\overline{y})dS(\overline{y}) + ct\int_{B(\overline{x},ct)} \bigtriangledown^2(\rho_0)dV(\overline{y}) = 0$

iff $\rho_0(\overline{y}) = 0$

as if $\rho_0(\overline{y}_0) \neq 0$, by continity, without loss of generality, there exists $\epsilon > 0$, such that, for sufficiently small t_0;

$\int_{\delta B(\overline{y}_0,ct_0)} \rho_0(\overline{y})dS(\overline{y}) > 4\pi\epsilon c^2t_0^2$

and, if M is a uniform bound on $\bigtriangledown^2(\rho_0)$

$|ct_0\int_{B(\overline{y}_0,ct_0)} \bigtriangledown^2(\rho_0)dV(\overline{y})| < \frac{4M\pi c^4t_0^4}{3}$

see [3] and [11] and the above, we had for $t > 0$;

$\rho(\overline{x},t) = \frac{1}{4\pi c^2 t^2}\int_{\delta B(\overline{x},ct)}(tg(\overline{y}) + \rho_0(\overline{y}) + D\rho_0(\overline{y}) \cdot (\overline{y} - \overline{x}))dS(\overline{y})$

and, for $t < 0$;

$\rho(\overline{x},t) = \frac{1}{4\pi c^2 t^2}\int_{\delta B(\overline{x},-ct)}(tg(\overline{y}) + \rho_0(\overline{y}) + D\rho_0(\overline{y}) \cdot (\overline{y} - \overline{x}))dS(\overline{y})$

In particular, for fixed $t_0 \in \mathcal{R}$, as ρ_0 and g have compact support, we can see that $\delta B(\overline{x}, c|t_0|) \cap Supp(\rho_0, g, D\rho_0) = \emptyset$, for $|\overline{x}_0| > C_{t_0}$, where $C_{t_0} \in \mathcal{R}_{>0}$, so that ρ_{t_0} has compact support as well. As $\rho_{t_0} \in C^\infty(\mathcal{R}^3)$, we then have that $\rho_{t_0} \in S(\mathcal{R}^3)$.

For the fifth claim, with;

$\overline{J}(\overline{x},t) = -c^2\int_{-\infty}^t \bigtriangledown(\rho)ds$

see [11] for the existence of the integral. We have, differentiating under the integral sign, and using the fundamental theorem of calculus, that, for $(i,j,k) \in \mathcal{Z}^3_{\geq 0}$;

$\frac{\partial^{i+j+k} j_1}{\partial x^i \partial y^j \partial z^k} = -c^2\int_{-\infty}^t \frac{\partial^{i+j+k+1}\rho}{\partial x^{i+1}\partial y^j \partial z^k} ds\ (Z)$

$\frac{\partial^{i+j+k+1} j_1}{\partial x^i \partial y^j \partial z^k \partial t} = -c^2 \frac{\partial^{i+j+k+1}\rho}{\partial x^{i+1}\partial y^j \partial z^k}$

and for $l \geq 2$;

$\frac{\partial^{i+j+k+l} j_1}{\partial x^i \partial y^j \partial z^k \partial t^l} = -c^2 \frac{\partial^{i+j+k+1}\rho}{\partial x^{i+1}\partial y^j \partial z^k \partial t^{l-1}}$

As $(\frac{\partial^{i+j+k}\rho}{\partial x^i \partial y^j \partial z^k})_0 \in S(\mathcal{R}^3)$, and $\frac{\partial^{i+j+k}\rho}{\partial x^i \partial y^j \partial z^k}$ satisfies the wave equation on $\mathcal{R}^4$, by the proof in [11], we have that the integral (Z) is well defined. Then, as $\rho \in C^\infty(\mathcal{R}^4)$, we have that $j_1 \in C^\infty(\mathcal{R}^4)$. A similar argument shows that the components $\{j_2, j_3\} \subset C^\infty(\mathcal{R}^4)$. By the fundamental

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so that, if $4\pi\epsilon c^2 t_0^2 > \frac{4M\pi c^4 t_0^4}{3}$ iff $\frac{3\epsilon}{Mc^2} > t_0^2$, we can choose $0 < t_0 < \frac{(3\epsilon)^{\frac{1}{2}}}{\sqrt{M}c}$, to obtain;

$\int_{\delta B(\overline{y}_0,ct_0)} \rho_0(\overline{y})dS(\overline{y}) + ct_0\int_{B(\overline{y}_0,ct_0)} \bigtriangledown^2(\rho_0)dV(\overline{y}) > 0$

In either case, we can reflect a solution for $t \geq 0$ to obtain a smooth solution on $\mathcal{R}^4$.

theorem of calculus, we have that;

$\frac{\partial \overline{J}}{\partial t}=-c^2\bigtriangledown(\rho)$

By the previous claim, for $t_0\in\mathcal{R}$, ρ_{t_0} has compact support, so that $(\bigtriangledown(\rho))_{t_0}$ has compact support and $(\frac{\partial \overline{J}}{\partial t})_{t_0}$ has compact support. It is clear from the above that the compact support V_t of ρ_t and $(\bigtriangledown(\rho))_t$ varies continuously with t, so on the interval $(t_0-\epsilon,t_0+\epsilon)$, $(\frac{\partial \overline{J}}{\partial t})|_{(t_0-\epsilon,t_0+\epsilon)}$ has compact support $W_{t_0,\epsilon}$ in $\mathcal{R}^4$.

$\overline{J}$ satisfies the wave equation on $\mathcal{R}^4$, as, using the fundamental theorem of calculus and the fact that $\bigtriangledown(\rho)$ satisfies the wave equation;

$\square^2(\overline{J})=\bigtriangledown^2(\overline{J})+\frac{1}{c^2}\frac{\partial^2\overline{J}}{\partial t^2}$

$=-c^2(\int_{-\infty}^t\bigtriangledown^2(\bigtriangledown(\rho))ds)+\frac{1}{c^2}(-c^2\frac{\partial\bigtriangledown(\rho)}{\partial t})$

$=-c^2(\int_{-\infty}^t-\frac{1}{c^2}\frac{\partial^2\bigtriangledown(\rho)}{\partial t^2}ds)-\frac{\partial\bigtriangledown(\rho)}{\partial t}$

$=\frac{\partial\bigtriangledown(\rho)}{\partial t})-\frac{\partial\bigtriangledown(\rho)}{\partial t}$

$=\overline{0}$

By the connecting relation;

$\bigtriangledown\rho+\frac{1}{c^2}\frac{\partial\overline{J}}{\partial t}=\overline{0}$

we have that $\frac{\partial\overline{J}}{\partial t}$ vanishes outside $Supp(\rho_t)$, and for any $\overline{x}\in\mathcal{R}^3$, there exists two uniformly bounded intervals $[t_{1,\overline{x},-},t_{2,\overline{x},-}]$, $[t_{1,\overline{x},+},t_{2,\overline{x},+}]$, for which $\overline{x}\in Supp(\rho_t)$, for $t\in[t_{1,\overline{x},-},t_{2,\overline{x},-}]\cup[t_{1,\overline{x},+},t_{2,\overline{x},+}]$. Using the fact that $Supp(\rho_t)$ is moving and $\bigtriangledown(\rho)$ satisfies the wave equation, so uniformly bounded, we can define;

$\overline{J}_0(\overline{x})=\int_{t_{1,\overline{x},-}}^{t_{2,\overline{x},-}}\frac{\partial\overline{J}}{\partial t}dt+\int_{t_{1,\overline{x},+}}^{t_{2,\overline{x},+}}\frac{\partial\overline{J}}{\partial t}dt$

$=\int_{-\infty}^{\infty}\frac{\partial\overline{J}}{\partial t}dt$ (the ultimate value of $\overline{J}(\overline{x},t)$)

with $\overline{J}_0$ bounded. On any ball $B(\overline{0},r)$, we have that $\overline{J}-\overline{J}_0$ eventually vanishes, and, as $div(\overline{J})-div(\overline{J}_0)=0$ ultimately on the ball, and $div(\overline{J})=-\frac{\partial\rho}{\partial t}=0$, ultimately, otherwise charge would build up, we have that $div(\overline{J}_0)=0$. It follows that $(\rho,\overline{J}-\overline{J}_0)$ satisfies the continuity equation., and the linkage relation;

$$\bigtriangledown\rho+\frac{1}{c^2}\frac{\partial(\overline{J}-\overline{J}_0)}{\partial t}=\overline{0}$$

is still satisfied, as $\overline{J}_0$ is time independent. On any ball $B(\overline{0},r)$, we have that ultimately $\overline{J}-\overline{J}_0=\overline{0}$, so that, as $\square^2(\overline{J})=\overline{0}$ and $\overline{J}_0$ is time independent, ultimately;

$$\bigtriangledown^2(\overline{J}_0)=\square^2(\overline{J}_0)=\square^2(\overline{J})=\overline{0}$$

and $\overline{J}_0$ is harmonic. As the components $\bigtriangledown(\rho)_i$, for $1\leq i\leq 3$, satisfy the wave equation, we have that that there exists constants $C_i\in\mathcal{R}_{>0}$, for which $|\bigtriangledown(\rho)_i(\overline{x},t)|\leq\frac{C_i}{|t|}$ for $1\leq i\leq 3$, so that;

$$|\bigtriangledown(\rho)(\overline{x},t)|\leq\frac{\sqrt{C_1^2+C_2^2+C_3^2}}{|t|}$$

and;

$$|\overline{J}_0(\overline{x})|=|\int_{t_{1,\overline{x},-}}^{t_{2,\overline{x},-}}-c^2\bigtriangledown(\rho)dt+\int_{t_{1,\overline{x},+}}^{t_{2,\overline{x},+}}-c^2\bigtriangledown(\rho)dt|$$

$$\leq c^2[(t_{2,\overline{x},-}-t_{1,\overline{x},-})+(t_{2,\overline{x},+}-t_{1,\overline{x},+})]|\bigtriangledown(\rho)|_{[t_{1,\overline{x},-},t_{2,\overline{x},-}]\cup[t_{1,\overline{x},-},t_{2,\overline{x},-}]}|$$

$$\leq c^2(t_{2,\overline{x},-}-t_{1,\overline{x},-})\frac{\sqrt{C_1^2+C_2^2+C_3^2}}{|t_{1,\overline{x},-}|}+c^2(t_{2,\overline{x},+}-t_{1,\overline{x},+})\frac{\sqrt{C_1^2+C_2^2+C_3^2}}{|t_{1,\overline{x},+}|}$$

$$\leq\frac{C}{|\overline{x}|}$$

as the intervals $[t_{1,\overline{x},-},t_{2,\overline{x},-}]$, $[t_{1,\overline{x},+},t_{2,\overline{x},+}]$ are uniformly bounded, and the hitting times $\{t_{1,\overline{x},-},t_{1,\overline{x},+}\}$ are proportional to the distance $\overline{x}$. It follows, as bounded harmonic functions are constant, that $\overline{J}_0=\overline{0}$, and $\overline{J}$ has compact supports.

.... If $t_1<t_2$, with $\{t_1,t_2\}\subset\mathcal{R}$, and $\{V_{t_1},V_{t_2}\}$ denote the compact supports of $\{\rho_{t_1},\rho_{t_2}\}$, then as the supports vary continuously, and $\overline{J}_t$ and ρ_t are compactly supported for each $t\in[t_1,t_2]$, $\overline{J}_t$ and ρ_t are uniformly compacted supported for $t\in[t_1,t_2]$ in a ball $B(\overline{0},p)$, for some $p\in\mathcal{R}_{>0}$. In particularly;

$$\int_{V_{t_1}}\rho_{t_1}dV=\int_{B(\overline{0},p)}\rho_{t_1}dV$$

$$\int_{V_{t_2}}\rho_{t_2}dV=\int_{B(\overline{0},p)}\rho_{t_2}dV$$

For $t \in [t_1, t_2]$, using the continuity equation, the divergence theorem and the fact $\overline{J}_t$ is uniformly compacted supported for $t \in [t_1, t_2]$ in $B(\overline{0}, p)$, we have that;

$\frac{d}{dt}(\int_{B(\overline{0},p)} \rho_t dV) = \int_{B(\overline{0},p)} \frac{\partial \rho}{\partial t} dV$

$= \int_{B(\overline{0},p)} div(\overline{J})_t dV$

$= \int_{\delta B(\overline{0},p)} \overline{J}_t \centerdot d\overline{S} dV$

$= 0$

so that;

$\int_{B(\overline{0},p)} \rho_{t_1} dV = \int_{B(\overline{0},p)} \rho_{t_2} dV$

$\int_{V_{t_1}} \rho_{t_1} dV = \int_{V_{t_2}} \rho_{t_2} dV$

In particularly, $\frac{d}{dt}(\int_{V_t} \rho_t dV) = 0$, ([4]). The same argument applies for $\frac{\partial \rho}{\partial t}$, with associated current $\overline{J}_1 = -c^2 \bigtriangledown (\rho)$ and compact supports W_t, $t \in \mathcal{R}$, obeying the wave equation $\square^2(\overline{J}_1) = \overline{0}$. It follows from the Reynold's transport theorem, ([5]), the divergence theorem and the fact that $\overline{J}_1$ vanishes outside W_t and V_t, that;

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In fact, the result is true for $(\rho, \overline{J})$ satisfying the continuity equation, when $\overline{J}$ fails to have compact support, and the components j_i, for $1 \leq i \leq 3$, are uniformly of rapid decay, in the sense, that for any finite interval $[t_1, t_2]$, there exists constants $C_{1,2,i,n} \in \mathcal{R}_{>0}$ such that $|j_i(\overline{x}, t)| \leq \frac{C_{1,i,2,n}}{|\overline{x}|^n}$ for $t \in [t_1, t_2]$ and $|\overline{x}| > 1$. In order to see this, suppose that on a finite interval (t_1, t_2), ρ is supported uniformly on $B(\overline{0}, p)$. and $\frac{d}{dt}\int_{V_t} \rho dV \neq 0$, for some $t \in [t_1, t_2]$. Then there exists an interval $(t_0 - \epsilon, t_0 + \epsilon) \subset (t_1, t_2)$, such that, without loss of generality, $\frac{d}{dt}\int_{V_t} \rho dV|_{(t_0-\epsilon, t_0+\epsilon)} > 0$, and, by the intermediate value theorem, we can assume that $\int_{V_t} \rho dV|_{(t_0-\epsilon, t_0+\epsilon)}$ is strictly increasing, with $\int_{V_{t_0+\epsilon}} \rho_{t_0+\epsilon} dV - \int_{V_{t_0}} \rho_{t_0} dV > \delta > 0$, $(*)$. Using the hypotheses on $\overline{J}$, we can choose $r > p$ sufficienly large such that for $t \in (t_0 - \epsilon, t_0 + \epsilon)$, $|\int_{\delta B(\overline{0}, r)} \overline{J}_t \centerdot d\overline{S}| < \delta_1$, and by the continuity equation, for $t \in (t_0 - \epsilon, t_0 + \epsilon)$;

$|\frac{d}{dt}\int_{B(\overline{0},r)} p dV| = |\int_{B(\overline{0},r)} \frac{\partial \rho}{\partial t} dV|$

$= | - \int_{B(\overline{0},r)} div(\overline{J}) dV|$

$= |\int_{\delta B(\overline{0},r)} \overline{J} \centerdot d\overline{S}|$

$< \delta_1$

and the intermediate value theorem;

$|\int_{B(\overline{0},r)} p_{t_0+\epsilon} dV - \int_{B(\overline{0},r)} p_{t_0} dV| < \delta_1 \epsilon$

so choosing $\delta_1 = \frac{\delta}{2\epsilon}$, we obtain that;

$|\int_{B(\overline{0},r)} p_{t_0+\epsilon} dV - \int_{B(\overline{0},r)} p_{t_0} dV| = |\int_{V_{t_0+\epsilon}} p_{t_0+\epsilon} dV - \int_{V_{t_0}} p_{t_0} dV|$

$< \frac{\delta}{2}$

which contradicts $(*)$.

[5] The Reynolds transport theorem is true in this case, but is not the usual form, as, due to the failure of analyticity, there can be jumps in the support. There is also an issue with using the formula $\rho\overline{v} = \overline{J}$, when substituting for the velocity of the area element. This could be resolved in [12].

$\int_{V_t}\bigtriangledown^2(\rho)dV=\frac{1}{c^2}\int_{V_t}\frac{\partial^2\rho}{\partial t^2}dV$

$=\frac{1}{c^2}(\frac{d}{dt}(\int_{V_t}\frac{\partial\rho}{\partial t}dV)-\int_{V_t}div(\overline{J}_1))$

$=-\frac{1}{c^2}\int_{V_t}div(\overline{J}_1)dV$

$=-\frac{1}{c^2}\int_{\delta V_t}\overline{J}_1\centerdot d\overline{S}$

$=0$

In particular, at $t=0$, we can assume that;

$\int_{V_0}\bigtriangledown^2(\rho_0)dV=\int_{V_0}(\sum_{i=1}^3(\frac{\partial^2\rho}{\partial x_i^2})_0)dV=0$ (O), ([6]).

We can define antiderivatives, by letting;

$p^a(\overline{x},t)=\int_{-\infty}^t p(\overline{x},s)ds$

$\overline{J}^a(\overline{x},t)=\int_{-\infty}^t\overline{J}(\overline{x},s)ds$ (if the integral exists)

As is easily checked, if $p\in C^\infty(\mathcal{R}^4)$ and the components $j_i\in C^\infty(\mathcal{R}^4)$, $1\leq i\leq 3$, then $\rho^a\in C^\infty(\mathcal{R}^4)$ and the components $j_i^a\in C^\infty(\mathcal{R}^4)$, for $1\leq i\leq 3$. The wave equation holds for ρ^a and $\overline{J}^a$, as, using the fundamental theorem of calculus, differentiating under the integral sign, the result about he left hand limit in [11], and using the fact that ρ satisfies the wave equation;

$\square^2(\rho^a)=\int_{-\infty}^t\bigtriangledown^2(\rho)ds-\frac{1}{c^2}\frac{\partial\rho}{\partial t}$

$=\int_{-\infty}^t\frac{1}{c^2}\frac{\partial^2\rho}{\partial t^2}ds-\frac{1}{c^2}\frac{\partial\rho}{\partial t}$

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[6] Note that you can also deduce this, using the divergence theorem, and the fact that $\bigtriangledown(\rho_0)$ vanishes on δV_0;

$\int_{V_0}\bigtriangledown^2(\rho_0)dV=\int_{\delta V_0}\bigtriangledown\centerdot(\bigtriangledown(\rho_0))dV$

$=\int_{\delta V_0}\bigtriangledown(\rho_0)\centerdot d\overline{S}$

$=0$

$= \frac{1}{c^2}\frac{\partial \rho}{\partial t} - \frac{1}{c^2}\frac{\partial \rho}{\partial t}$

$= 0$

and;

$\square^2(\overline{J}^a) = \int_{-\infty}^{t} \bigtriangledown^2(\overline{J})ds - \frac{1}{c^2}\frac{\partial \overline{J}}{\partial t}$

$= \int_{-\infty}^{t} \frac{1}{c^2}\frac{\partial^2 \overline{J}}{\partial t^2}ds - \frac{1}{c^2}\frac{\partial \overline{J}}{\partial t}$

$= \frac{1}{c^2}\frac{\partial \overline{J}}{\partial t} - \frac{1}{c^2}\frac{\partial \overline{J}}{\partial t}$

$= \overline{0}$

Differentiating under the integral sign and using the fundamental theorem of calculus, the fact that the continuity equation holds for $(\rho, \overline{J})$, the continuity equation holds as;

$\frac{\partial \rho^a}{\partial t} + \bigtriangledown \cdot \overline{J}^a$

$= \rho + \int_{-\infty}^{t} \bigtriangledown \cdot \overline{J}ds$

$= \rho + \int_{-\infty}^{t} + \int_{-\infty}^{t} -\frac{\partial \rho}{\partial s}ds$

$= \rho - \rho = 0$

and, differentiating under the integral sign, using the fundamental calculus of calculus and the connecting relation for $(\rho, \overline{J})$, the connecting relation holds;

$\bigtriangledown(\rho^a) + \frac{1}{c^2}\frac{\partial \overline{J}^a}{\partial t}$

$= \int_{-\infty}^{t} \bigtriangledown(\rho)ds + \frac{1}{c^2}\overline{J}$

$= \int_{-\infty}^{t} -\frac{1}{c^2}\frac{\partial \overline{J}}{\partial t}ds + \frac{1}{c^2}\overline{J}$

$= -\frac{1}{c^2}\overline{J} + \frac{1}{c^2}\overline{J}$

$=\overline{0}$,([7])

.... Then the fields $\{\overline{E},\overline{B}\}$ are well defined by Jefimenko's equations and the components are of uniform very moderate decrease.

$\square$

Lemma 0.19. *Cartesian method*

If $g:\mathcal{R}^3\rightarrow\mathcal{R}$ is analytic for $|\overline{x}|>r$, where $r\in\mathcal{R}_{>0}$, analytic at infinity, of very moderate decrease, and continuous, with $\{\frac{\partial g}{\partial x},\frac{\partial g}{\partial y},\frac{\partial g}{\partial z}\}$ analytic for $|\overline{x}|>r$ and analytic at infinity, we can define, for $\overline{k}\in\mathcal{R}^3$, with $k_1\neq 0$, $k_2\neq 0$, $k_3\neq 0$;

$$\mathcal{F}(g)(\overline{k})=\frac{1}{(2\pi)^{\frac{3}{2}}}lim_{r_1\rightarrow\infty}lim_{r_2\rightarrow\infty}lim_{r_3\rightarrow\infty}\int_{-r_1}^{r_1}\int_{-r_2}^{r_2}\int_{-r_3}^{r_3}g(\overline{x})e^{-i\overline{k}..\overline{x}}dx_1dx_2dx_3$$

Moreover, for $k_1\neq 0$, $k_2\neq 0$, $k_3\neq 0$, we have that;

$$\mathcal{F}(g)(\overline{k})=\frac{1}{(2\pi)^{\frac{3}{2}}}lim_{r\rightarrow\infty}\int_{0\leq\theta\leq\pi,-\pi\leq\phi\leq\pi}g(r,\theta,\phi)e^{-ik_1rsin(\theta)cos(\phi)}$$

$$e^{-ik_2rsin(\theta)sin(\phi)}e^{-ik_3rcos(\theta)}r^2sin(\theta)drd\theta d\phi$$

---

[7] We don't necessarily have that $(\rho^a,\overline{J}^a)$ has compact supports. On a finite interval $[t_1,t_2]$, for sufficiently large $\overline{x}$, we have $\frac{\partial\rho^a}{\partial t}=\rho=0$, and;

$$\bigtriangledown^2(\rho_a)=\frac{1}{c^2}\frac{\partial^2\rho^a}{\partial^2 t}$$

$$=0$$

Let $h(\overline{x})$ define ρ^a for sufficiently large $\overline{x}$, then, as $\mathcal{R}^3=\bigcup_{t\in\mathcal{R}}Supp(\rho_t)^c$;

$$\bigtriangledown^2(h(\overline{x}))=\square^2(h(\overline{x}))=0$$

everywhere. We can repeat the argument for the antiderivative $\overline{J}^a$ to obtain $\overline{c}(\overline{x})$ defining $\overline{J}^a$ for sufficiently large $\overline{x}$. so, as $\mathcal{R}^3=\bigcup_{t\in\mathcal{R}}Supp(\overline{J}_t)^c$, we have that $\bigtriangledown^2(\overline{c}(\overline{x}))=\square^2(\overline{c}(\overline{x}))=\overline{0}$, and, clearly, for the pair $(h(\overline{x}),\overline{c}(\overline{x}))$, we have that;

$$div(\overline{c}(\overline{x}))=-\frac{\partial h}{\partial t}=0$$

$$\bigtriangledown(h)(\overline{x})=-c^2\frac{\partial\overline{c}(\overline{x})}{\partial t}$$

$$=\overline{0}$$

and $(\rho^a-h(\overline{x}),\overline{J}^a-\overline{c}(\overline{x}))$ has compact supports and inherits all the properties above for $(\rho^a,\overline{J}^a)$.

Proof. Let C_r be the cube defined by $C_r = \{(x,y,z) \in \mathcal{R}^3 : |x| \leq r, |y| \leq r, |z| \leq r\}$, then, as g is analytic for $|\overline{x}| > r$, we have that g is analytic on $\mathcal{R}^3 \setminus C_r$, with global power series expansion $\sum_{(i,j,k)\in\mathcal{Z}^3_{\geq 0}} a_{ijk}x^i y^j z^k$, convergent on $\mathcal{R}^3 \setminus C_r$. As g is continuous, it is bounded on C_r and we can define;

$$f(\overline{k}) = \int_{\overline{x}\in C_r} g(\overline{x})e^{-i\overline{k}\cdot\overline{x}}d\overline{x}$$

For $(x,y) \in \mathcal{R}^2$, we have that $g_{x,y}(z)$ is analytic for $|z| > r$, analytic at infinity, and of very moderate decrease. In particularly, by Lemma 0.16, using the fact that $g_{x,y}$ is also of very moderate decrease, $g_{x,y}$ is eventually monotone, and for $k_3 \neq 0$, we can define;

$$g_3(x,y,k_3) = lim_{r_3\to\infty}\int_{r<|z|<r_3} g_{x,y}(z)e^{-ik_3 z}dz$$

$$= lim_{r_3\to\infty}\int_{r<|z|<r_3}(\sum_{(i,j,k)\in\mathcal{Z}^3_{\geq 0}} a_{ijk}x^i y^j z^k)(\sum_{m=0}^{\infty}\frac{(-ik_3 z)^m}{m!})dz$$

$$= lim_{r_3\to\infty}\int_{r<|z|<r_3}(\sum_{(i,j,k,m)\in\mathcal{Z}^4_{\geq 0}}\frac{(-ik_3)^m a_{ijk}}{m!}x^i y^j z^{k+m})dz$$

$$= lim_{r_3\to\infty}\sum_{(i,j,k,m)\in\mathcal{Z}^4_{\geq 0}}\frac{(-ik_3)^m a_{ijk}}{m!}x^i y^j\int_{r<|z|<r_3} z^{k+m}dz$$

$$= lim_{r_3\to\infty}\sum_{(i,j,k,m)\in\mathcal{Z}^4_{\geq 0}}\frac{(-ik_3)^m a_{ijk}}{m!}x^i y^j([\frac{z^{k+m+1}}{k+m+1}]_{-r_3}^{-r} + [\frac{z^{k+m+1}}{k+m+1}]_r^{r_3})$$

$$= lim_{r_3\to\infty}\sum_{(i,j,k,m)\in\mathcal{Z}^4_{\geq 0},k+m+1 odd}\frac{2(-ik_3)^m a_{ijk}}{m!(k+m+1)}x^i y^j(r_3^{k+m+1} - r^{k+m+1})$$

$$= lim_{r_3\to\infty}\sum_{(i,j,k,m)\in\mathcal{Z}^4_{\geq 0},k+m+1 odd}\frac{2(-ik_3)^m r_3^{k+m+1}a_{ijk}}{m!(k+m+1)}x^i y^j$$

$$-\sum_{(i,j,k,m)\in\mathcal{Z}^4_{\geq 0},k+m+1 odd}\frac{2(-ik_3)^m a_{ijk}r^{k+m+1}}{m!(k+m+1)}x^i y^j$$

$$= \sum_{(i,j)\in\mathcal{Z}^2_{\geq 0}}(b_{ij} - c_{ij})x^i y^j$$

where;

$$b_{ij} = lim_{r_3\to\infty}\sum_{(i,j,k,m)\in\mathcal{Z}^4_{\geq 0},k+m+1 odd}\frac{2(-ik_3)^m r_3^{k+m+1}a_{ijk}}{m!(k+m+1)}$$

$$c_{ij} = \sum_{(i,j,k,m)\in\mathcal{Z}^4_{\geq 0},k+m+1 odd}\frac{2(-ik_3)^m a_{ijk}r^{k+m+1}}{m!(k+m+1)}$$

so that $g_3(x,y,k_3)$ is analytic for $(x,y) \in \mathcal{R}^2$, in particularly continuous.

For $(x_0 : y_0) \in P^1(\mathcal{R})$, we have that $(x_0 : y_0 : 1) \in P^2(\mathcal{R})$ and, as g is analytic at infinity, there exists $\epsilon_{x_0,y_0,1}$, such that $g(\frac{x_0}{x}, \frac{y_0}{y}, \frac{1}{z})$ is defined by a convergent power series $\sum_{(i,j,k)\in\mathcal{Z}^3} d_{ijk}x^iy^jz^k$ in the region $x^2+y^2+z^2<\epsilon^2_{x_0,y_0,1}$. Without loss of generality, assuming that $x_0\neq 0$, $y_0\neq 0$, as g is analytic for $|\frac{x_0}{x}|>r$, $|\frac{y_0}{y}|>r$, $|\frac{1}{z}|>r$, $|x|<\frac{|x_0|}{r}$, $|y|<\frac{|y_0|}{r}$, $|z|<\frac{1}{r}$, by uniqueness of power series, we can replace the region $x^2+y^2+z^2<\epsilon^2_{x_0,y_0,1}$, by the region $|x|<\frac{|x_0|}{r}$, $|y|<\frac{|y_0|}{r}$, $|z|<\frac{1}{r}$. Then;

$$g_3(\tfrac{x_0}{x}, \tfrac{y_0}{y}, k_3) = lim_{r_3\rightarrow\infty}\int_{r<|z|<r_3} g(\tfrac{x_0}{x}, \tfrac{y_0}{y}, w)e^{-ik_3w}dw$$

$$= lim_{r_3\rightarrow\infty}\int_{r<|z|<r_3} g(\tfrac{x_0}{x}, \tfrac{y_0}{y}, \tfrac{1}{z})e^{\frac{-ik_3}{z}} - \tfrac{dz}{z^2}dz \ (z=\tfrac{1}{w}, z\neq 0)$$

$$= lim_{r_3\rightarrow\infty}\int_{r<|z|<r_3}(\textstyle\sum_{(i,j,k)\in\mathcal{Z}^3_{\geq 0}} d_{ijk}x^iy^jz^k)(\sum_{m=0}^{\infty}\frac{-(-ik_3)^m}{z^{m+2}m!})dz$$

$$= lim_{r_3\rightarrow\infty}\int_{r<|z|<r_3}(\textstyle\sum_{(i,j,k,m)\in\mathcal{Z}^4_{\geq 0}}\frac{-(-ik_3)^m d_{ijk}}{m!}x^iy^jz^{k-m-2})dz$$

$$= lim_{r_3\rightarrow\infty}\textstyle\sum_{(i,j,k,m)\in\mathcal{Z}^4_{\geq 0}}\frac{-(-ik_3)^m d_{ijk}}{m!}x^iy^j\int_{r<|z|<r_3} z^{k-m-2}dz$$

$$= lim_{r_3\rightarrow\infty}\textstyle\sum_{(i,j,k,m)\in\mathcal{Z}^4_{\geq 0},k\neq m+1}\frac{-(-ik_3)^m d_{ijk}}{m!}x^iy^j([\frac{z^{k-m-1}}{k-m-1}]^{-r}_{-r_3}+[\frac{z^{k-m-1}}{k-m-1}]^{r_3}_{r})$$

$$+lim_{r_3\rightarrow\infty}\textstyle\sum_{(i,j,k,m)\in\mathcal{Z}^4_{\geq 0},k=m+1}\frac{-(-ik_3)^m d_{ijk}}{m!}x^iy^j([-ln(z)]^{r_3}_{r}+[ln(z)]^{r_3}_{r})$$

$$= lim_{r_3\rightarrow\infty}\textstyle\sum_{(i,j,k,m)\in\mathcal{Z}^4_{\geq 0},k\neq m+1}\frac{-(-ik_3)^m d_{ijk}}{m!}x^iy^j([\frac{z^{k-m-1}}{k-m-1}]^{-r}_{-r_3}+[\frac{z^{k-m-1}}{k-m-1}]^{r_3}_{r})$$

$$= lim_{r_3\rightarrow\infty}\textstyle\sum_{(i,j,k,m)\in\mathcal{Z}^4_{\geq 0},k\neq m+1,k-m-1 odd}\frac{-2(-ik_3)^m d_{ijk}}{m!(k-m-1)}x^iy^j(r_3^{k-m-1}-r^{k-m-1})$$

$$= lim_{r_3\rightarrow\infty}\textstyle\sum_{(i,j,k,m)\in\mathcal{Z}^4_{\geq 0},k\neq m+1,k-m-1 odd}\frac{-2(-ik_3)^m r_3^{k-m-1} d_{ijk}}{m!(k-m-1)}x^iy^j$$

$$-\textstyle\sum_{(i,j,k,m)\in\mathcal{Z}^4_{\geq 0},k\neq m+1,k-m-1 odd}\frac{-2(-ik_3)^m d_{ijk}r^{k-m-1}}{m!(k-m-1)}x^iy^j$$

$$= \textstyle\sum_{(i,j)\in\mathcal{Z}^2_{\geq 0}}(k_{ij}-l_{ij})x^iy^j$$

where;

$$k_{ij} = lim_{r_3\rightarrow\infty}\textstyle\sum_{(i,j,k,m)\in\mathcal{Z}^4_{\geq 0},k\neq m+1,k-m-1 odd}\frac{-2(-ik_3)^m r_3^{k-m-1} d_{ijk}}{m!(k-m-1)}$$

$$l_{ij} = \textstyle\sum_{(i,j,k,m)\in\mathcal{Z}^4_{\geq 0},k\neq m+1,k-m-1 odd}\frac{-2(-ik_3)^m d_{ijk}r^{k-m-1}}{m!(k-m-1)}$$

We can then take $\epsilon_{x_0,y_0} = \frac{min(\frac{|x_0|}{r},\frac{|y_0|}{r})}{\sqrt{2}}$, so that as $(x_0 : y_0) \in P^2(\mathcal{R})$ was arbitrary, $g_3(x,y,k_3)$ is analytic at infinity.

As g is of very moderate decrease, we have that;

$|g_{xy}(z)| = |g(x,y,z)| \leq \frac{C}{(x^2+y^2+z^2)^{\frac{1}{2}}}$

$= \frac{|z|}{(x^2+y^2+z^2)^{\frac{1}{2}}}\frac{C}{|z|}$

$\leq \frac{C}{|z|}$ (A)

for $(x^2+y^2+z^2)^{\frac{1}{2}} > |z| > s$. As $\frac{\partial g}{\partial z}$ is analytic for $|\overline{x}| > r$ and analytic at infinity, it has finitely many zeroes, so that $g_{xy}(z)$ is eventually monotone in the interval $|z| > E$, for some $E \in \mathcal{R}_{>0}$, uniformly in (x,y), (B), and we can achieve both $(A),(B)$, for $|z| > v = max(s,E)$. Without loss of generality, we can assume that $v > E > r$. We also have that, for $(x^2+y^2+z^2)^{\frac{1}{2}} \geq (x^2+y^2)^{\frac{1}{2}} > s$;

$|g_{xy}(z)| \leq \frac{C}{(x^2+y^2+z^2)^{\frac{1}{2}}}$

$= \frac{C}{(x^2+y^2)^{\frac{1}{2}}}\frac{(x^2+y^2)^{\frac{1}{2}}}{(x^2+y^2+z^2)^{\frac{1}{2}}}$

$\leq \frac{C}{(x^2+y^2)^{\frac{1}{2}}}$

Then, by a simple generalisation of Lemma 0.9, for $|(x,y)| \geq s$, we have that;

$|g_3(x,y,k_3)| = |limr_3 \rightarrow \infty \int_{r<|z|<r_3} g(x,y,z)e^{-ik_3z}dz|$

$\leq \frac{4Cv}{|(x,y)|} + \frac{6C\pi}{|(x,y)|k_3|}$

$= \frac{W}{|(x,y)|}$ (F)

where $W = 4Cv + \frac{6C\pi}{|k_3|}$, so that $g_3(x,y,k_3)$ is of very moderate decrease.

As $g_3(x,y,k_3$ is analytic for $(x,y) \in \mathcal{R}^2$ and analytic at infinity, so is $\frac{\partial g_3}{\partial y}$, so that, for fixed $x \in \mathcal{R}$, $g_{3,x,k_3}(y)$ is eventually monotone and of very moderate decrease, so that, for $k_2 \neq 0$, we can define;

$g_2(x,k_2,k_3)=lim_{r_2\rightarrow\infty}\int_{-r_2}^{r_2}g_3(x,y,k_3)e^{-ik_2y}dy$

As $g_3(x,y,k_3$ is analytic for $(x,y)\in\mathcal{R}^2$ and analytic at infinity, using $(1:1)\in P^1(\mathcal{R})$, $g_3(\frac{1}{x},\frac{1}{y},k_3)$ is defined by a convergent power series $\sum_{(i,j)\in\mathcal{Z}^2_{\geq 0}}s_{ij}x^iy^j$, valid for $(x,y)\in\mathcal{R}^2$, so that, for $x_0\neq 0$;

$g_2(x_0,k_2,k_3)=lim_{r_2\rightarrow\infty}\int_{-r_2}^{r_2}g_3(x_0,w,k_3)e^{-ik_2w}dw$

$=lim_{r_2\rightarrow\infty}\int_{-r_2}^{r_2}g_3(\frac{1}{x},\frac{1}{y},k_3)e^{\frac{-ik_2}{y}}-\frac{1}{y^2}dy$ $(w=\frac{1}{y},\ x_0=\frac{1}{x})$

$=lim_{r_2\rightarrow\infty,\epsilon\rightarrow 0}\int_{\epsilon<|y|<r_2}(\sum_{(i,j)\in\mathcal{Z}^2_{\geq 0}}s_{ij}x^iy^j)(\sum_{m=0}^{\infty}\frac{-(-ik_2)^m}{y^{m+2}m!})dy$

$=lim_{r_2\rightarrow\infty,\epsilon\rightarrow 0}\int_{\epsilon<|y|<r_2}(\sum_{(i,j,m)\in\mathcal{Z}^3_{\geq 0}}\frac{-(-ik_2)^m s_{ij}}{m!}x^iy^{j-m-2})dy$

$=lim_{r_2\rightarrow\infty,\epsilon\rightarrow 0}\sum_{(i,j,m)\in\mathcal{Z}^3_{\geq 0}}\frac{-(-ik_2)^m s_{ij}}{m!}x^i\int_{\epsilon<|y|<r_2}y^{j-m-2}dy$

$=lim_{r_2\rightarrow\infty,\epsilon\rightarrow 0}\sum_{(i,j,m)\in\mathcal{Z}^3_{\geq 0},j\neq m+1}\frac{-(-ik_2)^m s_{ij}}{m!}x^i([\frac{y^{j-m-1}}{j-m-1}]_{-r_2}^{-\epsilon}+[\frac{y^{j-m-1}}{j-m-1}]_{\epsilon}^{r_2})$

$+lim_{r_2\rightarrow\infty,\epsilon\rightarrow 0}\sum_{(i,j,m)\in\mathcal{Z}^3_{\geq 0},j=m+1}\frac{-(-ik_3)^m s_{ij}}{m!}x^i([-ln(z)]_{\epsilon}^{r_2}+[ln(z)]_{\epsilon}^{r_2})$

$=lim_{r_2\rightarrow\infty,\epsilon\rightarrow 0}\sum_{(i,j,m)\in\mathcal{Z}^3_{\geq 0},j\neq m+1}\frac{-(-ik_3)^m s_{ij}}{m!}x^i([\frac{y^{j-m-1}}{j-m-1}]_{-r_2}^{-\epsilon}+[\frac{y^{j-m-1}}{j-m-1}]_{\epsilon}^{r_2})$

$=lim_{r_2\rightarrow\infty,\epsilon\rightarrow 0}\sum_{(i,j,m)\in\mathcal{Z}^3_{\geq 0},j\neq m+1,j-m-1odd}\frac{-2(-ik_2)^m s_{ij}}{m!(j-m-1)}x^i(r_2^{j-m-1}-\epsilon^{j-m-1})$

$=lim_{r_2\rightarrow\infty}\sum_{(i,j,m)\in\mathcal{Z}^3_{\geq 0},j\neq m+1,j-m-1odd}\frac{-2(-ik_2)^m r_2^{j-m-1}s_{ij}}{m!(j-m-1)}x^i$

$-lim_{\epsilon\rightarrow 0}\sum_{(i,j,m)\in\mathcal{Z}^3_{\geq 0},j\neq m+1,j-m-1odd}\frac{-2(-ik_2)^m s_{ij}\epsilon^{k-m-1}}{m!(j-m-1)}x^i$

$=\sum_{i\in\mathcal{Z}_{\geq 0}}(\alpha_i-\beta_i)x^i$

$=\sum_{i\in\mathcal{Z}_{\geq 0}}(\alpha_i-\beta_i)x^i$

$=\sum_{i\in\mathcal{Z}_{\geq 0}}(\alpha_i-\beta_i)(\frac{1}{x_0})^i$

where;

$\alpha_i=lim_{r_2\rightarrow\infty}\sum_{(i,j,m)\in\mathcal{Z}^3_{\geq 0},j\neq m+1,j-m-1odd}\frac{-2(-ik_2)^m r_2^{j-m-1}s_{ij}}{m!(j-m-1)}$

$\beta_i=lim_{\epsilon\rightarrow 0}\sum_{(i,j,m)\in\mathcal{Z}^3_{\geq 0},j\neq m+1,j-m-1odd}\frac{-2(-ik_2)^m s_{ij}\epsilon^{k-m-1}}{m!(j-m-1)}$

It follows that $g_2(x,k_2,k_3)$ is analytic at infinity, and, as $\frac{1}{x_0}$ is analytic for $x_0 \neq 0$, and the composition of analytic functions is analytic, $g_2(x,k_2,k_3)$ is analytic for $x \neq 0$.

By the same reasoning as above, we have that $g_2(x,k_2,k_3)$ is of very moderate decrease, and using the fact that $\frac{dg_2}{dx}$ is analytic for $x \neq 0$, and analytic at infinity , using Lemma 0.16, $g_2(x,k_2,k_3)$ is eventually monotone, so , for $k_3 \neq 0$, we can define;

$g_1(k_1,k_2,k_3) = lim_{r_1 \rightarrow \infty} \int_{-r_1}^{r_1} g_2(x,k_2,k_3)e^{-ik_1x}dx$

For $|z| < r$, $x \in \mathcal{R}$, by the usual arguments, we can define;

$h_2(x,z,k_2) = lim_{r_2 \rightarrow \infty} \int_{r<|y|<r_2} g(x,y,z)e^{-ik_2y}dy$

As above, as g is analytic in the region $|y| > r$ and analytic at infinity, we can show that h_2 is analytic for $x \neq 0$ and $z \neq 0$, analytic at infinity, and of very moderate decrease. By the usual arguments, we can then define, for $|z| < r$;

$h_1(z,k_1,k_2) = lim_{r_1 \rightarrow \infty} \int_{|x|<r_1} g(x,y,z)e^{-ik_1x}dx$

and show that h_1 is analytic for $0 < |z| < r$, and smooth at 0 (extra argument here), in particular, bounded. Then , for $k_3 \neq 0$, let;

$h_3(k_1,k_2,k_3) = \int_{|z|<r} h_1(z,k_1,k_2)e^{-ik_3z}dz$

For $|z| < r$, $|y| < r$, define;

$s_1(y,z,k_1) = lim_{r_1 \rightarrow \infty} \int_{r<|x|<r_1} g(x,y,z)e^{-ik_1x}dx$

As above, as g is analytic in the region $|x| > r$ and analytic at infinity, we can show that s_1 is analytic for $y \neq 0$ and $z \neq 0$, analytic at infinity, and of very moderate decrease. We can also show that s_1 is smooth along on the locus $((y = 0 \cup z = 0) \cap (|y| < r \cap |z| < r)) \subset \mathcal{R}^2$ (extra argument here). Then, by the usual arguments, we can define;

$s_{2,3}(k_1,k_2,k_3) = \int_{|y|<r,|z|<r} s_1(y,z,k_1)e^{-ik_2y}e^{-ik_3z}dydz$

Let $m(k_1,k_2,k_3) = h_3(k_1,k_2,k_3) + s_{2,3}(k_1,k_2,k_3)$, for $k_1 \neq 0$, $k_2 \neq 0$, $k_3 \neq 0$. Then, for $k_1 \neq 0$, $k_2 \neq 0$, $k_3 \neq 0$, it is clear, totalling the volumes, that we have;

$$\mathcal{F}(g)(\bar{k}) = \frac{1}{(2\pi)^{\frac{3}{2}}}(f(\bar{k}) + g_1(\bar{k}) + m(\bar{k}))$$

□

Lemma 0.20. *Let g and all its partial derivatives $\{\frac{\partial^{(i_1,i_2,i_3)}g}{\partial^{i_1}x\partial^{i_2}y\partial^{i_3}z} : 0 \leq i_1 + i_2 + i_3 \leq 4\}$ satisfy the hypotheses of the previous lemma. Then, for $k_1 \neq 0$, $k_2 \neq 0$, $k_3 \neq 0$, with $|k_1|, |k_2|, |k_3|$, sufficiently large, there exists constants $C_{i_1,i_2,i_3} \in \mathcal{R}_{>0}$, with;*

$$|\mathcal{F}(\frac{\partial^{(i_1,i_2,i_3)}g}{\partial^{i_1}x\partial^{i_2}y\partial^{i_3}z})(\bar{k})| \leq \frac{C_{i_1,2,i_3}}{|k_1||k_2||k_3|}$$

and $D \in \mathcal{R}_{>0}$, with;

$$|\mathcal{F}(g)(\bar{k})| \leq \frac{D}{|k_1||k_2||k_3||\bar{k}|^4}$$

We have that, for $r > 0$, $\mathcal{F}(g)|_{B(\bar{0},r)} \in L^1(B(\bar{0}, r))$, $\mathcal{F}(g)|_V \in L^1(V)$, $\mathcal{F}(g)|_{V_i} \in L^1(V_i)$, for $1 \leq i \leq 3$, $\mathcal{F}(g)|_{V_{ij}} \in L^1(V_{ij})$, $1 \leq i < j \leq 3$, where;

$$V = \{(k_1, k_2, k_3) : |k_1| \geq E_1, |k_2| \geq E_2, |k_3| \geq E_3\}.$$

$$V_i = \{(k_1, k_2, k_3) : |k_i| < E_i, |k_l| \geq E_l, l \neq i, 1 \leq l \leq 3\}.$$

$$V_{ij} = \{(k_1, k_2, k_3) : |k_i| < E_i, |k_j| < E_j, |k_l| \geq E_l, l \neq i, l \neq j,$$

$$1 \leq l \leq 3\}.$$

In particularly, $\mathcal{F}(g) \in L^1(\mathcal{R}^3)$.

Proof. For the first claim, let;

$$a_3(x, y, k_3) = lim_{r_3\to\infty} \int_{-r_3}^{r_3} g(x, y, z)e^{-ik_3z}dz$$

for $k_3 \neq 0$. (Then for fixed x, k_3, $a_3(x, y, k_3)$ is of very moderate decrease in y and oscillatory for sufficiently large y.

Then, we can define;

$$a_2(x, k_2, k_3) = lim_{r_2\to\infty} \int_{-r_2}^{r_2} a_3(x, y, k_3)e^{-ik_2y}dy$$

for $k_2 \neq 0$. (For, fixed k_2, k_3, $a_2(x, k_2, k_3)$ is of very moderate decrease in y and oscillatory, for sufficiently large x).

so we can define, for $k_1 \neq 0$;

$a_1(k_1, k_2, k_3) = lim_{r_1 \rightarrow \infty} \int_{-r_1}^{r_1} a_2(x, k_2, k_3) e^{-ik_1 y} dy$

$\mathcal{F}(g)(\overline{k}) = \frac{1}{(2\pi)}^{\frac{3}{2}} a_1(k_1, k_2, k_3)$

Using the end of the proof of Lemma 0.9, we can show that that there exists $C \in \mathcal{R}_{>0}$, independent of x, y, with;

$|a_3(x, y, k_3)| \leq \frac{C||g||_\infty}{|k_3|}$

for sufficiently large $|k_3| \geq C_3$. Similarly, for sufficiently large $|k_2| \geq C_2$;

$|a_2(x, k_2, k_3)| \leq \frac{C||a_3|_{|k_3| \geq C_3}||_\infty}{|k_2}$

$\leq \frac{C^2||g||_\infty}{|k_2||k_3|}$

and, for sufficiently large $|k_1| \geq C_1$;

$|a_1(k_1, k_2, k_3)| \leq \frac{C||a_2|_{|k_2| \geq C_2, |k_3| \geq C_3}||_\infty}{|k_1}$

$\leq \frac{C^2||a_3|_{|k_3| \geq C_3}||_\infty}{|k_2||k_3|}$

$\leq \frac{C^3||g||_\infty}{|k_1||k_2||k_3|}$

so that, for $|k_1| \geq C_1|$, $|k_2| \geq C_2$, $|k_3| \geq C_3$;

$||\mathcal{F}(g)(\overline{k})||_\infty \leq \frac{1}{(2\pi)^{\frac{3}{2}}} \frac{C^3||g||_\infty}{|k_1||k_2||k_3|}$

$= \frac{C_{0,0,0}}{|k_1||k_2||k_3|}$

where $C_{0,0,0} = \frac{1}{(2\pi)^{\frac{3}{2}}} C^3||g||_\infty$

Similarly, for $|k_1|, |k_2|, |k_3|$ sufficiently large, we can find constants $C_{i_1,i_2,i_3} \in \mathcal{R}_{>0}$, for $i_1 + i_2 + i_3 \geq 4$, such that;

$|\mathcal{F}(\frac{\partial^{(i_1,i_2,i_3)}g}{\partial^{i_1}x\partial^{i_2}y\partial^{i_3}z})(\overline{k})| \leq \frac{C_{i_1,2,i_3}}{|k_1||k_2||k_3|}$

For the second claim, we have, for $k_1 \neq 0$, $k_2 \neq 0$, $k_3 \neq 0$, using repeated integration by parts, that;

$\mathcal{F}(\frac{\partial^4 g}{\partial x^4} + \frac{\partial^4 g}{\partial y^4} + \frac{\partial^4 g}{\partial z^4} + 2\frac{\partial^4 g}{\partial x^2\partial y^2} + 2\frac{\partial^4 g}{\partial x^2\partial z^2} + 2\frac{\partial^4 g}{\partial y^2\partial z^2})(\overline{k})$

$= \frac{1}{(2\pi)^{\frac{3}{2}}} lim_{r_1\rightarrow\infty} lim_{r_2\rightarrow\infty} lim_{r_3\rightarrow\infty} \int_{-r_1}^{r_1}\int_{-r_2}^{r_2}\int_{-r_3}^{r_3}(\frac{\partial^4 g}{\partial x^4} + \frac{\partial^4 g}{\partial y^4} + \frac{\partial^4 g}{\partial z^4} + 2\frac{\partial^4 g}{\partial x^2\partial y^2}$

$+2\frac{\partial^4 g}{\partial x^2\partial z^2} + 2\frac{\partial^4 g}{\partial y^2\partial z^2})e^{-ik_1x}e^{-ik_2y}e^{-ik_3z}dxdydz$

$= (k_1^4+k_2^4+k_3^4+2k_1^2k_2^2+2k_1^2k_3^2+2k_2^2k_3^2)\frac{1}{(2\pi)^{\frac{3}{2}}} lim_{r_1\rightarrow\infty} lim_{r_2\rightarrow\infty} lim_{r_3\rightarrow\infty}$

$\int_{-r_1}^{r_1}\int_{-r_2}^{r_2}\int_{-r_3}^{r_3} g(x,y,z)e^{-ik_1x}e^{-ik_2y}e^{-ik_3z}dxdydz$

$= |\overline{k}|^4\mathcal{F}(g)(\overline{k})$

so that, using the first claim, for sufficiently large $|k|_1, |k|_2, |k|_3$;

$|\mathcal{F}(g)(\overline{k})| \leq \frac{\mathcal{F}(g)(\frac{\partial^4 g}{\partial x^4} + \frac{\partial^4 g}{\partial y^4} + \frac{\partial^4 g}{\partial z^4} + 2\frac{\partial^4 g}{\partial x^2\partial y^2} + 2\frac{\partial^4 g}{\partial x^2\partial z^2} + 2\frac{\partial^4 g}{\partial y^2\partial z^2})(\overline{k})}{|\overline{k}|^4}$

$\leq \frac{C_{4,0,0}+C_{0,4,0}+C_{0,0,4}+2C_{2,2,0}+2C_{2,0,2}+2C_{0,2,2}}{|k_1||k_2||k_3||\overline{k}|^4}$

$= \frac{D}{|k_1||k_2||k_3||\overline{k}|^4}$

where $D = C_{4,0,0} + C_{0,4,0} + C_{0,0,4} + 2C_{2,2,0} + 2C_{2,0,2} + 2C_{0,2,2}$.

For the next claim, we have as g is of very moderate decrease, that $|g| \leq \frac{D}{|x|}$, for $|x| \geq s$, and, as g is continuous, that $|g| \leq C$, for $|\overline{x}| \leq s$. Using polar coordinates (R,θ,ϕ), we have;

$\int_{\mathcal{R}^3}|g|^4 d\overline{x}$

$= \int_{B(\bar{0},s)}|g|^4 d\overline{x} + \int_{\mathcal{R}^3\setminus B(\bar{0},s)}|g|^4 d\overline{x}$

$\leq \frac{4C^4\pi s^3}{3} + \int_{\mathcal{R}^3\setminus B(\bar{0},s)}\frac{D}{|x|^4}d\overline{x}$

$\frac{\leq \frac{4C^4\pi s^3}{3} + \int_s^\infty |DR^2 sin(\theta)}{R^4|dR}$

$\leq \frac{4C^4\pi s^3}{3} + D\int_s^\infty \frac{dR}{R^2}$

$= \frac{4C^4\pi s^3}{3} + \frac{D}{s}$

so that $g \in L^4(\mathcal{R}^3)$. Letting $p = 4$, and $\frac{1}{p} + \frac{1}{q} = 1$, so that $q = \frac{4}{3}$, and generalising the Haussdorff-Young inequality, see [15], we have that $\mathcal{F}(g) \in L^{\frac{4}{3}}(\mathcal{R}^3)$, and we can find $F \in \mathcal{R}_{>0}$, with;

$||\mathcal{F}(g)||_{L^{\frac{4}{3}}(\mathcal{R}^3)} \leq F||g||_{L^4(\mathcal{R}^3)}$

$\leq F(\frac{4C^4\pi s^3}{3} + \frac{D}{s})^{\frac{1}{2}}$

By Holders's inequality, we have that for $r > 0$, $\mathcal{F}(g)|_{B(\overline{0},r)} \in L^1(B(\overline{0},r))$, and;

$||\mathcal{F}(g)(\overline{k})||_{L^1(B(\overline{0},r))}$

$\leq ||\mathcal{F}(g)(\overline{k})||_{L^{\frac{4}{3}}(B(\overline{0},r))}||1||_{L^4(B(\overline{0},r))}$

$\leq F(\frac{4C^4\pi s^3}{3} + \frac{D}{s})^{\frac{1}{2}}(\frac{4\pi r^3}{3})^{\frac{1}{2}}$

Using the second claim, we have that there exist constants $\{E_1, E_2, E_3\} \subset \mathcal{R}_{>0}$, such that, for $|k_1| \geq E_1, |k_2| \geq E_2, |k_3| \geq E_3$;

$|\mathcal{F}(g)(\overline{k})| \leq \frac{D}{|k_1||k_2||k_3||\overline{k}|^4}$

$\leq \frac{D}{E_1E_2E_3|\overline{k}|^4}$

$= \frac{F}{|\overline{k}|^4}$

where $F = \frac{D}{E_1E_2E_3|}$. Then, using polar coordinates, $k_1' = rsin(\theta)cos(\phi)$, $k_2' = rsin(\theta)sin(\phi)$, $k_3' = rcos(\theta)$, $0 \leq \theta \leq \pi$, $-\pi \leq \phi \leq \pi$;

$\int_{k_1 \geq E_1, k_2 \geq E_2, k_3 \geq E_3} |\mathcal{F}(g)(\overline{k})| d\overline{k}$

$= \int_{k_1' \geq 0, k_2' \geq 0, k_3' \geq 0} |\mathcal{F}(g)(k_1' + E_1, k_2' + E_2, k_3' + E_3)| d\overline{k}'$

$\leq \int_{k_1' \geq 0, k_2' \geq 0, k_3' \geq 0} \frac{F}{|(k_1'+E_1, k_2'+E_2, k_3'+E_3)|^4} d\overline{k}'$

$= \int_{0 \leq \theta \leq \frac{\pi}{2}, 0 \leq \phi \leq \frac{\pi}{2}} \int_0^\infty \frac{F}{|(k_1'+E_1, k_2'+E_2, k_3'+E_3)|^4} r^2 sin(\theta) dr d\theta d\phi$

$= \int_{0\leq\theta\leq\frac{\pi}{2},0\leq\phi\leq\frac{\pi}{2}}\int_0^\infty \frac{F}{|(rsin(\theta)cos(\phi)+E_1,rsin(\theta)sin(\phi)+E_2,rcos(\theta)+E_3)|^4}r^2sin(\theta)drd\theta d\phi$

$\leq \int_{0\leq\theta\leq\frac{\pi}{2},0\leq\phi\leq\frac{\pi}{2}}\int_0^{r_0} \frac{F}{|(rsin(\theta)cos(\phi)+E_1,rsin(\theta)sin(\phi)+E_2,rcos(\theta)+E_3)|^4}r^2sin(\theta)drd\theta d\phi$

$+\int_{0\leq\theta\leq\frac{\pi}{2},0\leq\phi\leq\frac{\pi}{2}}\int_{r_0}^\infty \frac{Fr^2}{r^4}\frac{1}{|(sin(\theta)cos(\phi)+\frac{E_1}{r},sin(\theta)sin(\phi)+\frac{E_2}{r},cos(\theta)+\frac{E_3}{r})|^4}drd\theta d\phi$

$\leq \int_{0\leq\theta\leq\frac{\pi}{2},0\leq\phi\leq\frac{\pi}{2}}\int_0^{r_0} \frac{Fr_0^2}{|(E_1,E_2,E_3)|^4}$

$+\int_{0\leq\theta\leq\frac{\pi}{2},0\leq\phi\leq\frac{\pi}{2}}\int_0^{r_0} \frac{F}{r^2}\frac{1}{(1+\frac{2sin(\theta)cos(\phi)E_1}{r}+\frac{2sin(\theta)sin(\phi)E_2}{r}+\frac{2cos(\theta)E_3}{r}+\frac{E_1^2}{r^2}+\frac{E_2^2}{r^2}+\frac{E_3^2}{r^2})^2}drd\theta d\phi$

$\leq r_0(\frac{\pi}{2})^2\frac{Fr_0^2}{|(E_1,E_2,E_3)|^4}$

$+\int_{0\leq\theta\leq\frac{\pi}{2},0\leq\phi\leq\frac{\pi}{2}}\int_{r_0}^\infty \frac{F}{r^2}\frac{1}{(1+\frac{2sin(\theta)cos(\phi)E_1}{r}+\frac{2sin(\theta)sin(\phi)E_2}{r}+\frac{2cos(\theta)E_3}{r}+\frac{E_1^2}{r^2}+\frac{E_2^2}{r^2}+\frac{E_3^2}{r^2})^2}drd\theta d\phi$

$\leq \frac{\pi^2r_0^3F}{4(E_1^2+E_2^2+E_3^2)^2}+\int_{0\leq\theta\leq\frac{\pi}{2},0\leq\phi\leq\frac{\pi}{2}}\int_{r_0}^\infty \frac{4F}{r^2}drd\theta d\phi$

$= \frac{\pi^2r_0^3F}{4(E_1^2+E_2^2+E_3^2)^2}+\frac{4F(\frac{\pi}{2})^2}{r_0}$

$= \frac{\pi^2r_0^3F}{4(E_1^2+E_2^2+E_3^2)^2}+\frac{F\pi^2}{r_0}$

for $r_0\geq 12max(E_1,E_2,E_3)$.

Similarly, repeating the calculation for all the finitely many connected regions in $|k_1|\geq E_1,|k_2|\geq E_2,|k_3|\geq E_3$, we obtain that $\mathcal{F}(g)|_V\in L^1(V)$, where;

$V=\{(k_1,k_2,k_3):|k_1|\geq E_1,|k_2|\geq E_2,|k_3|\geq E_3\}$.

Using the same argument as above, for $a_2(x,k_2,k_3)$, we have, without loss of generality, that for $|k_2|\geq E_2$ and for $|k_3|\geq E_3$, there exists $D\in\mathcal{R}_{>0}$, with;

$|\mathcal{F}(\frac{\partial^4 g}{\partial y^4}+\frac{\partial^4 g}{\partial z^4}+2\frac{\partial^4 g}{\partial y^2\partial z^2})(x,k_2,k_3)|\leq\frac{D||\frac{\partial^4 g}{\partial y^4}+\frac{\partial^4 g}{\partial z^4}+2\frac{\partial^4 g}{\partial y^2\partial z^2}||_\infty}{|k_2||k_3|}$

We have, for $k_2\neq 0$, $k_3\neq 0$, that;

$\mathcal{F}(\frac{\partial^4 g}{\partial y^4}+\frac{\partial^4 g}{\partial z^4}+2\frac{\partial^4 g}{\partial y^2\partial z^2})(x,k_2,k_3)$

$=\frac{1}{(2\pi)^{\frac{3}{2}}}lim_{r_2\rightarrow\infty}lim_{r_3\rightarrow\infty}\int_{-r_2}^{r_2}\int_{-r_3}^{r_3}(\frac{\partial^4 g}{\partial y^4}+\frac{\partial^4 g}{\partial z^4}+2\frac{\partial^4 g}{\partial y^2\partial z^2})e^{-ik_2y}e^{-ik_3z}dydz$

$= (k_2^4 + k_3^4 + 2k_2^2 k_3^2)\frac{1}{(2\pi)^{\frac{3}{2}}} lim_{r_2\rightarrow\infty} lim_{r_3\rightarrow\infty}$

$\int_{-r_2}^{r_2}\int_{-r_3}^{r_3} g(x,y,z)e^{-ik_2 y}e^{-ik_3 z}dydz$

$= |(k_2,k_3)|^4\mathcal{F}(g)(x,k_2,k_3)$

so that, for $x \in \mathcal{R}$, $|k_2| \geq E_2$, $|k_3| \geq E_3$;

$|\mathcal{F}(g)(x,k_2,k_3)| \leq \frac{D||\frac{\partial^4 g}{\partial y^4}+\frac{\partial^4 g}{\partial z^4}+2\frac{\partial^4 g}{\partial y^2\partial z^2}||_\infty}{|k_2||k_3||(k_2,k_3)|^4}$

$\leq \frac{D||\frac{\partial^4 g}{\partial y^4}+\frac{\partial^4 g}{\partial z^4}+2\frac{\partial^4 g}{\partial y^2\partial z^2}||_\infty}{C_2 C_3|(k_2,k_3)|^4}$

$= \frac{E}{|(k_2,k_3)|^4}$

where $E = \frac{D||\frac{\partial^4 g}{\partial y^4}+\frac{\partial^4 g}{\partial z^4}+2\frac{\partial^4 g}{\partial y^2\partial z^2}||_\infty}{C_2 C_3}$.

As above, we have that;

$\int_{k_2\geq E_2, k_3\geq E_3} |\mathcal{F}(g)(x,k_2,k_3)|dk_2 dk_3$

$= \int_{k_2'\geq 0, k_3'\geq 0} |\mathcal{F}(g)(k_2'+E_2, k_3'+E_3)|dk_2' dk_3'$

$\leq \int_{k_2'\geq 0, k_3'\geq 0} \frac{E}{|k_2'+E_2, k_3'+E_3)|^4}dk_2' dk_3' < \infty$

so clearly, for $x \in \mathcal{R}$, $\mathcal{F}(g)(x,k_2,k_3) \in L^1(S)$, where $S = \{(k_2,k_3) \in \mathcal{R}^2, |k_2| \geq E_2, |k_3| \geq E_3\}$. Let;

$\theta(x) = \int_{|k_2|\geq E_2, |k_3|\geq E_3} \mathcal{F}(g)(x,k_2,k_3)dk_2 dk_3$

As above, we have for sufficiently large x, $\theta(x)$ is non oscillatory and of very moderate decrease.

Interchanging limits, we have that;

$\int_{V_1} \mathcal{F}(g)(k_1,k_2,k_3)dk_1 dk_2 dk_3$

$= \int_{|k_1|<E_1}\int_{|k_2|\geq E_2, |k_3|\geq E_3} \mathcal{F}(g)(k_1,k_2,k_3)dk_1 dk_2 dk_3$

$= \int_{|k_1|<E_1}\int_{|k_2|\geq E_2, |k_3|\geq E_3}(lim_{r_1\rightarrow\infty}\int_{-r_1}^{r_1}\mathcal{F}(g)(x,k_2,k_3)e^{-ik_1 x}dx)dk_1 dk_2 dk_3$

$= \int_{|k_1|<E_1} lim_{r_1\to\infty} \int_{-r_1}^{r_1} (\int_{|k_2|\geq E_2,|k_3|\geq E_3} \mathcal{F}(g)(x,k_2,k_3)dk_2dk_3)e^{-ik_1x}dxdk_1$

$= \int_{|k_1|<E_1} (lim_{r_1\to\infty} \int_{-r_1}^{r_1} \theta(x)e^{-ik_1x}dx)dk_1$

$= \int_{|k_1|<E_1} \mathcal{F}_1(\theta)(k_1)dk_1$

where F_1 is the Fourier transform for non-oscillatory functions of very moderate decrease in one variable. As above, we have that $\mathcal{F}_1(\theta) \in L^2(\mathcal{R})$, so that $\mathcal{F}_1(\theta)|_{|k_1|<E_1} \in L^1(|k_1| < E_1)$. It follows that;

$\int_{|k_1|<E_1} \mathcal{F}_1(\theta)(k_1)dk_1 < \infty$

and $\mathcal{F}(g)(k_1,k_2,k_3) \in L^1(V_1)$. Similarly, we can show that;

$\mathcal{F}(g)(k_1,k_2,k_3) \in (\bigcap_{1\leq i\leq 3} L^1(V_i) \cap \bigcap_{1\leq i<j\leq 3} L^1(V_{ij}))$

As $\mathcal{R}^3 \setminus (\bigcup_{1\leq i\leq 3} V_i \cup \bigcup_{1\leq i<j\leq 3} V_{ij}) = C_{E_1,E_2,E_3}$, where;

$C_{E_1,E_2,E_3} = \{(k_1,k_2,k_3) \in \mathcal{R}^3 : |k_1| < E_1, |k_2| < E_2, |k_3| < E_3\}$

and $C_{E_1,E_2,E_3} \subset B(\bar{0},r)$, where $r = max(E_1,E_2,E_3)$, we have that $\mathcal{F}(g)(k_1,k_2,k_3) \in L^1(C_{E_1,E_2,E_3})$ and $\mathcal{F}(g)(k_1,k_2,k_3) \in L^1(\mathcal{R}^3)$.

□

Definition 0.21. *Let $f \in C^{14}(\mathcal{R}^2)$ with $\frac{\partial^{i_1+i_2}f}{\partial x^{i_1}\partial y^{i_2}}$ bounded for $0 \leq i_1+i_2 \leq 14$. Let $C_n = \{(x,y) \in \mathcal{R}^2 : |x| \leq n, |y| \leq n\}$. Then we define an inflexionary approximation sequence $\{f_m : m \in \mathcal{N}\}$ by the requirements;*

(i). $f_m \in C^{14}(\mathcal{R}^2)$

(ii). $f_m|_{C_m} = f|_{C_m}$

(iii) $f_m|_{(\mathcal{R}^2\setminus C_{m+\frac{1}{m}})} = 0$

(iv). For $|x| \leq m$, for $0 \leq i \leq 13$;

$\frac{\partial^i f_m}{\partial y^i}|_{(x,m)} = \frac{\partial^i f}{\partial y^i}|_{(x,m)}$

$\frac{\partial^i f_m}{\partial y^i}|_{(x,-m)} = \frac{\partial^i f}{\partial y^i}|_{(x,-m)}$

$\frac{\partial^i f_m}{\partial y^i}|_{(x,m+\frac{1}{m})} = 0$

$\frac{\partial^i f_m}{\partial y^i}|_{(x,-m-\frac{1}{m})} = 0$

(v). For $|x| \leq m$

if $\frac{\partial^{14} f}{\partial y^{14}}|_{(x,m)} > 0,\ \frac{\partial^{14} f_m}{\partial y^{14}}|_{V_{x,m}} \geq 0$

if $\frac{\partial^{14} f}{\partial y^{14}}|_{(x,m)} < 0,\ \frac{\partial^{14} f_m}{\partial y^{14}}|_{V_{x,m}} \leq 0$

if $\frac{\partial^{14} f}{\partial y^{14}}|_{(x,-m)} > 0,\ \frac{\partial^{14} f_m}{\partial y^{14}}|_{V_{x,-m}} \geq 0$

if $\frac{\partial^{14} f}{\partial y^{14}}|_{(x,-m)} < 0,\ \frac{\partial^{14} f_m}{\partial y^{14}}|_{V_{x,-m}} \leq 0$

(vi). For $0 \leq |y| \leq m + \frac{1}{m},\ 0 \leq i \leq 13$

$\frac{\partial^i f_m}{\partial x^i}|_{(x,y)} = \frac{\partial^i f_m}{\partial x^i}|_{(m,y)},\ m \leq x \leq m + \frac{1}{m}$

$\frac{\partial^i f_m}{\partial x^i}|_{(x,y)} = \frac{\partial^i f_m}{\partial x^i}|_{(-m,y)},\ -m - \frac{1}{m} \leq x \leq -m$

$\frac{\partial^i f_m}{\partial x^i}|_{(m+\frac{1}{m},y)} = 0$

$\frac{\partial^i f_m}{\partial x^i}|_{(-m-\frac{1}{m},y)} = 0$

(vii) For $m \leq |x| \leq m + \frac{1}{m},\ 0 \leq |y| \leq m + \frac{1}{m}$

if $\frac{\partial^{14} f}{\partial x^{14}}|_{(m,y)} > 0,\ \frac{\partial^{14} f_m}{\partial x^{14}}|_{H_{m,y}} \geq 0$

if $\frac{\partial^{14} f}{\partial x^{14}}|_{(m,y)} < 0,\ \frac{\partial^{14} f_m}{\partial x^{14}}|_{H_{m,y}} \leq 0$

if $\frac{\partial^{14} f}{\partial x^{14}}|_{(-m,y)} > 0,\ \frac{\partial^{14} f_m}{\partial x^{14}}|_{H_{-m,y}} \geq 0$

if $\frac{\partial^{14} f}{\partial x^{14}}|_{(-m,y)} < 0,\ \frac{\partial^{14} f_m}{\partial x^{14}}|_{H_{-m,y}} \leq 0$

where;

$V_{x,m} = \{(x,y) \in \mathcal{R}^2 : y \in (m, m + \frac{1}{m})\}$

$V_{x,-m} = \{(x,y) \in \mathcal{R}^2 : y \in (-m - \frac{1}{m}, -m)\}$

$H_{m,y}=\{(x,y)\in\mathcal{R}^2:x\in(m,m+\frac{1}{m})\}$

$H_{-m,y}=\{(x,y)\in\mathcal{R}^2:x\in(-m-\frac{1}{m},-m)\}$

Definition 0.22. *Let $f\in C^{14}(\mathcal{R}^3)$ with $\frac{\partial^{i_1+i_2+i_3}f}{\partial x^{i_1}\partial y^{i_2}\partial z^{i_3}}$ bounded for $0\leq i_1+i_2+i_3\leq 14$. Let $W_n=\{(x,y,z)\in\mathcal{R}^3:|x|\leq n,|y|\leq n,|z|\leq n\}$. Then we define an inflexionary approximation sequence $\{f_m:m\in\mathcal{N}\}$ by the requirements;*

(i). $f_m\in C^{14}(\mathcal{R}^3)$

(ii). $f_m|_{W_m}=f|_{W_m}$

(iii) $f_m|_{(\mathcal{R}^3\setminus W_{m+\frac{1}{m}})}=0$

(iv). *For* $0\leq|y|\leq m,0\leq|z|\leq m$, *for* $0\leq i\leq 13$;

$\frac{\partial^i f_m}{\partial x^i}|_{(m,y,z)}=\frac{\partial^i f}{\partial x^i}|_{(m,y,z)}$

$\frac{\partial^i f_m}{\partial x^i}|_{(-m,y,z)}=\frac{\partial^i f}{\partial x^i}|_{(-m,y,z)}$

$\frac{\partial^i f_m}{\partial x^i}|_{(m+\frac{1}{m},y,z)}=0$

$\frac{\partial^i f_m}{\partial x^i}|_{(-m-\frac{1}{m},y,z)}=0$

(v). *For* $0\leq|y|\leq m,0\leq|z|\leq m$

if $\frac{\partial^{14}f}{\partial x^{14}}|_{(m,y,z)}>0$, $\frac{\partial^{14}f_m}{\partial x^{14}}|_{H_{m,y,z}}\geq 0$

if $\frac{\partial^{14}f}{\partial y^{14}}|_{(m,y,z)}<0$, $\frac{\partial^{14}f_m}{\partial x^{14}}|_{H_{m,y,z}}\leq 0$

if $\frac{\partial^{14}f}{\partial y^{14}}|_{(-m,y,z)}>0$, $\frac{\partial^{14}f_m}{\partial x^{14}}|_{H_{-m,y,z}}\geq 0$

if $\frac{\partial^{14}f}{\partial y^{14}}|_{(-m,y,z)}<0$, $\frac{\partial^{14}f_m}{\partial x^{14}}|_{H_{-m,y,z}}\leq 0$

(vi). *For* $0\leq|x|\leq m+\frac{1}{m}$ $0\leq|z|\leq m$, $0\leq i\leq 13$

$\frac{\partial^i f_m}{\partial y^i}|_{(x,y,z)}=\frac{\partial^i f_m}{\partial y^i}|_{(x,m,z)}$, $m\leq y\leq m+\frac{1}{m}$

$\frac{\partial^i f_m}{\partial y^i}|_{(x,y,z)}=\frac{\partial^i f_m}{\partial y^i}|_{(x,-m,z)}$, $-m-\frac{1}{m}\leq y\leq -m$

$\frac{\partial^i f_m}{\partial y^i}|_{(x,m+\frac{1}{m},z)}=0$

$\frac{\partial^i f_m}{\partial y^i}|_{(x,-m-\frac{1}{m},z)}=0$

(vii) *For* $0\leq|x|\leq m+\frac{1}{m}$, $0\leq|z|\leq m$

if $\frac{\partial^{14} f_m}{\partial y^{14}}|_{(x,m,z)}>0$, $\frac{\partial^{14} f_m}{\partial y^{14}}|_{V_{x,m,z}}\geq 0$

if $\frac{\partial^{14} f_m}{\partial y^{14}}|_{(x,m,z)}<0$, $\frac{\partial^{14} f_m}{\partial y^{14}}|_{V_{x,m,z}}\leq 0$

if $\frac{\partial^{14} f_m}{\partial y^{14}}|_{(x,-m,z)}>0$, $\frac{\partial^{14} f_m}{\partial y^{14}}|_{V_{x,-m,z}}\geq 0$

if $\frac{\partial^{14} f_m}{\partial y^{14}}|_{(x,-m,z)}<0$, $\frac{\partial^{14} f_m}{\partial y^{14}}|_{V_{x,-m,z}}\leq 0$

$(viii)$. *For* $0\leq|x|\leq m+\frac{1}{m}$ $0\leq|y|\leq m+\frac{1}{m}$, $0\leq i\leq 13$

$\frac{\partial^i f_m}{\partial z^i}|_{(x,y,z)}=\frac{\partial^i f_m}{\partial z^i}|_{(x,y,m)}$, $m\leq z\leq m+\frac{1}{m}$

$\frac{\partial^i f_m}{\partial z^i}|_{(x,y,z)}=\frac{\partial^i f_m}{\partial z^i}|_{(x,y,-m)}$, $-m-\frac{1}{m}\leq z\leq -m$

$\frac{\partial^i f_m}{\partial z^i}|_{(x,y,m+\frac{1}{m})}=0$

$\frac{\partial^i f_m}{\partial z^i}|_{(x,y,-m-\frac{1}{m})}=0$

(ix) *For* $0\leq|x|\leq m+\frac{1}{m}$, $0\leq|y|\leq m+\frac{1}{m}$

if $\frac{\partial^{14} f_m}{\partial z^{14}}|_{(x,y,m)}>0$, $\frac{\partial^{14} f_m}{\partial z^{14}}|_{D_{x,y,m}}\geq 0$

if $\frac{\partial^{14} f}{\partial z^{14}}|_{(x,y,m)}<0$, $\frac{\partial^{14} f_m}{\partial z^{14}}|_{D_{x,y,m}}\leq 0$

if $\frac{\partial^{14} f}{\partial z^{14}}|_{(x,y,-m)}>0$, $\frac{\partial^{14} f_m}{\partial z^{14}}|_{D_{x,y,-m}}\geq 0$

if $\frac{\partial^{14} f}{\partial z^{14}}|_{(x,y,-m)}<0$, $\frac{\partial^{14} f_m}{\partial z^{14}}|_{D_{x,y,-m}}\leq 0$

where;

$H_{m,y,z}=\{(x,y,z)\in\mathcal{R}^3 : x\in(m,m+\frac{1}{m})\}$

$H_{-m,y,z}=\{(x,y,z)\in\mathcal{R}^3 : x\in(-m-\frac{1}{m},-m)\}$

$V_{x,m,z}=\{(x,y,z)\in\mathcal{R}^3:y\in(m,m+\frac{1}{m})\}$

$V_{x,-m,z}=\{(x,y,z)\in\mathcal{R}^3:y\in(-m-\frac{1}{m},-m)\}$

$D_{x,y,m}=\{(x,y,z)\in\mathcal{R}^3:z\in(m,m+\frac{1}{m})\}$

$D_{x,y,-m}=\{(x,y,z)\in\mathcal{R}^3:z\in(-m-\frac{1}{m},-m)\}$

Lemma 0.23. *If $[a,b]\subset\mathcal{R}$, with a,b finite, and $\{g,g_1,g_2\}\subset C^\infty([a,b])$, then, if $m\in\mathcal{R}_{>0}$ is sufficiently large, there exists $h\in C^\infty([m,m+\frac{1}{m}]\times[a,b])$, with the property that;*

$h(m,y)=g(y)$, $\frac{\partial h}{\partial x}|_{(m,y)}=g_1(y)$, $\frac{\partial^2 h}{\partial x^2}|_{(m,y)}=g_2(y)$, $y\in[a,b]$, (i)

$h(m+\frac{1}{m},y)=\frac{\partial h}{\partial x}(m+\frac{1}{m},y)=\frac{\partial^2 h}{\partial x^2}(m+\frac{1}{m},y)=0$, $y\in[a,b]$, (ii)

$|h|_{[m,m+\frac{1}{m}]\times[a,b]}|\leq C$

for some $C\in\mathcal{R}_{>0}$, independent of m sufficiently large, and, if $\frac{\partial^3 h}{\partial x^3}(m,y)>0$, $\frac{\partial^3 h}{\partial x^3}(x,y)>0$, for $x\in[m,m+\frac{1}{m}]$, and if $\frac{\partial^3 h}{\partial x^3}(m,y)<0$, $\frac{\partial^3 h}{\partial x^3}(x,y)<0$, for $x\in[m,m+\frac{1}{m}]$, $(*)$. In particularly;

$\int_m^{m+\frac{1}{m}}|\frac{\partial^3 h}{\partial x^3}|_{(x,y)}|dx=|g_2(y)|$

Moreover, for $i\in\mathcal{N}$, $\frac{\partial^i h}{\partial y^i}$ has the property that;

$\frac{\partial^i h}{\partial y^i}(m,y)=g^{(i)}(y)$, $\frac{\partial^{i+1} h}{\partial y^i\partial x}|_{(m,y)}=g_1^{(i)}(y)$, $\frac{\partial^{i+2} h}{\partial y^i\partial x^2}|_{(m,y)}=g_2^{(i)}(y)$

$y\in[a,b]$, $(i)'$

$\frac{\partial^i h}{\partial y^i}(m+\frac{1}{m},y)=\frac{\partial^{i+1} h}{\partial y^i\partial x}(m+\frac{1}{m},y)=\frac{\partial^{i+2} h}{\partial y^i\partial x^2}(m+\frac{1}{m},y)=0$

$y\in[a,b]$, $(ii)'$

$|\frac{\partial^i h}{\partial y^i}|_{[m,m+\frac{1}{m}]\times[a,b]}|\leq C_i$

for some $C_i\in\mathcal{R}_{>0}$, independent of m sufficiently large, and, if $\frac{\partial^{i+3} h}{\partial y^i\partial x^3}(m,y)>0$, $\frac{\partial^{i+3} h}{\partial y^i\partial x^3}(x,y)>0$, for $x\in[m,m+\frac{1}{m}]$, and if $\frac{\partial^{i+3} h}{\partial y^i\partial x^3}(m,y)<0$, $\frac{\partial^{i+3} h}{\partial y^i\partial x^3}(x,y)<0$, for $x\in[m,m+\frac{1}{m}]$, $(**)$. In particularly;

$\int_m^{m+\frac{1}{m}} |\frac{\partial^{i+3}h}{\partial y^i \partial x^3}|_{(x,y)}|dx = |g_2^{(i)}(y)|$

Proof. For the construction of h in the first part, just use the proof of Lemma 0.4, replacing the constant coefficients $\{a_0, a_1, a_2\} \subset \mathcal{R}$ with the data $\{g(y), g_1(y), g_2(y)\}$. The properties $(i), (ii)$ are then clear. Noting that $[a, b]$ is a finite interval and $\{g, g_1, g_2\} \subset C^\infty([a, b])$, by continuity, there exists a constant D, with $max(|g(y)|, |g_1(y)|, |g_2(y)| : y \in [a, b]) \leq D$, so, as in the proof of Lemma 0.4, we can use the bound $C = 16D + 7D + D = 24D$, for $m > 1$. The proof of $(*)$ follows uniformly in y, as in the proof of 0.4, for sufficiently large m, again using the fact that the data $\{g(y), g_1(y), g_2(y) : y \in [a, b]\}$ is bounded. The next claim is just the FTC again. For the second part, when we calculate $\frac{\partial^i h}{\partial y^i}$, for $i \in \mathcal{N}$, we are just differentiating the coefficients which are linear in the data $\{g(y), g_1(y), g_2(y)\}$, so we obtain a function which fits the data $\{g^{(i)}(y), g_1^{(i)}(y), g_2^{(i)}(y)\}$ and $(i)', (ii)'$ follow. Noting that, for $i \in \mathcal{N}$, $\{g^{(i)}, g_1^{(i)}, g_2^{(i)}\} \subset C^\infty([a, b])$, again by continuity, there exists constants D_i, with $max(|g^{(i)}(y)|, |g_1^{(i)}(y)|, |g_2^{(i)}(y)| : y \in [a, b]) \leq D_i$, so, again, as in the proof of Lemma 0.4, we can use the bound $C_i = 16D_i + 7D_i + D_i = 24D_i$, for $m > 1$. The proof of $(**)$ follows uniformly in y, for each $i \in \mathcal{N}$, as in the proof of Lemma 0.4, for sufficiently large m, again using the fact that the data $\{g^{(i)}(y), g_1^{(i)}(y), g_2^{(i)}(y) : y \in [a, b]\}$ is bounded. The last claim is again just the FTC. □

Lemma 0.24. *Conjecture*

Fix $n \in \mathcal{N}$, with $n \geq 3$. If $m \in \mathcal{R}_{>0}$ is sufficiently large, $\{a_i : 0 \leq i \leq n-1\} \subset \mathcal{R}$, there exists $h \in \mathcal{R}[x]$ of degree $2n-1$, with the property that;

$h^{(i)}(m) = a_i$, $0 \leq i \leq n-1$ (i)

$h^{(i)}(m + \frac{1}{m}) = 0$, $0 \leq i \leq n-1$ (ii)

$|h|_{[m,m+\frac{1}{m}]}| \leq C$

for some $C \in \mathcal{R}_{>0}$, independent of m sufficiently large, and, if $h^{(n)}(m) > 0$, $h^{(n)}(x)|_{[m,m+\frac{1}{m}]} > 0$, if $h^{(n)}(m) < 0$, $h^{(n)}|_{[m,m+\frac{1}{m}]} < 0$. In particularly;

$\int_m^{m+\frac{1}{m}} |h^{(n)}(x)|dx = |a_{n-1}|$, ([8])

Proof. We sketch a proof based on the special case $n = 3$, which was shown in Lemma 0.4, leaving the details to the reader, ([9]). We have that $h(x) = (x - (m + \frac{1}{m}))^n p(x)$ where $p(x)$ is a polynomial satisfies condition (ii). Computing the derivatives $h^{(i)}(m)$, for $0 \leq i \leq n-1$, we obtain n linear equations involving the unknowns $p^{(i)}(m)$, $0 \leq i \leq n-1$, of the form;

$$\sum_{k=0}^{i} \frac{d_{ik}p^{(k)}(m)}{m^{n-i+k}} = a_i, \ (0 \leq i \leq n-1) \ (*)$$

which we can solve for $p^{(i)}(m)$, $0 \leq i \leq n-1$, using the fact that the matrix $(d_{ik})_{0\leq i\leq n-1, 0\leq k\leq i}$ is lower triangular and $|d_{ii}| = 1$, for $0 \leq i \leq n-1$. Then we can take;

$$p(x) = \sum_{i=0}^{n-1} p^{(i)}(m)(x-m)^i$$

so that h has degree $n + (n-1) = 2n - 1$. It is clear from $(*)$, that we have;

$$p^{(i)}(m) = \sum_{k=0}^{i} c_{ik} a_{i-k} m^{n+k}, \ (0 \leq i \leq n-1)$$

[8] If $a_0 > 0$, $a_1 > 0$, there does not exist a smooth function h on the interval $(m, m+\frac{1}{m})$, with $h(m) = a_0$, $h'(m) = a_1$, $h(m+\frac{1}{m}) = 0$, $h'(m+\frac{1}{m}) = 0$, such that $h'' > 0$ or $h'' < 0$. To see this, if $h'' > 0$, using the MVT, we have that $h'(x) > h'(m) > 0$, for $x \in (m, m+\frac{1}{m})$, contradicting the fact that $h'(m+\frac{1}{m}) = 0$. If $h'' < 0$, and $h'(x)$ has no roots in the interval $(m, m+\frac{1}{m})$, then as $h'(m) > 0$, $h'(x) > 0$ on $(m, m+\frac{1}{m})$, and h is increasing on $(m, m+\frac{1}{m})$, so that $h(m+\frac{1}{m}) > h(m) = a_0 > 0$, contradicting the fact that $h(m+\frac{1}{m}) = 0$. Otherwise, if $h'(x)$ has a root in the interval $(m, m+\frac{1}{m})$, as $h'' < 0$, it attains a maximum at $x_0 \in (m, m+\frac{1}{m})$. Using the MVT again, we must have that for $y \in (x_0, m+\frac{1}{m})$, $h'(y) < h'(x_0) = 0$, so that $h'(m+\frac{1}{m}) < 0$, contradicting the fact that $h'(m+\frac{1}{m}) = 0$.

[9] One step requires the verification that for a computable polynomial r_n of degree $n-1$, $r_n(1) \neq 0$, which is highly unlikely on generic grounds and the fact that $r_3(1) \neq 1$, although $r_2(1) = 1$, see footnote 8. The geometric idea is that allowing for inflexionary type curves, where we can have points $x_{0,i} \in (m, m+\frac{1}{m})$ for which $h^{(i)}(x_{0,i}) = 0$, where $2 \leq i \leq n-1$, the end conditions can be satisfied while still having $h^{(n)}|_{(m,m+\frac{1}{m})} > 0$ or $h^{(n)}|_{(m,m+\frac{1}{m})} < 0$. However, you still need to do a concrete calculation, which in the case of verifying the conjecture for all $n \in \mathcal{N}$, $n \geq 3$, would involve finding the exact pattern in the coefficients obtained in the proof of Lemma 0.4. We actually only need the result for some $n \geq 14$ in the rest of this paper.

where $(c_{ik})_{0\leq i\leq n-1,0\leq k\leq i}$ is a real matrix, so that $p(x)$ has the form;

$p(x)=\sum_{i=0}^{n-1}v_i x^i$ $(**)$

where;

$v_{n-1-i}=\sum_{k=0}^{n-1}r_{ik}m^{n+k}+\sum_{l=0}^{i}s_{il}m^{2n-1+l}$, $(0\leq i\leq n-1)$

for real matrices $(r_{ik})_{0\leq i\leq n-1,0\leq k\leq n-1}$ and $(s_{il})_{0\leq i\leq n-1,0\leq l\leq i}$.

It is then clear, using the product rule and $(**)$, that;

$h^{(n)}(x)=\sum_{k=0}^{n-1}w_k x^k$

where $w_k=z_k a_0 m^{3n-2-k}+O(m^{3n-3-k})$, $(0\leq k\leq n-1)$

By homogeneity, it is then clear that the real roots of $h^{(n)}(x)$ are of the form $t_{s_0}m+O(1)$, where $t_{s_0}\in\mathcal{R}$, $1\leq s_0\leq n-1$, and t_{s_0} satisfies a polynomial $r(x)$ of degree $n-1$, which is effectively computable for given n. We can exclude any roots in the interval $[m,m+\frac{1}{m}]$, for sufficiently large m, provided $t_{s_0}\neq 1$, for $1\leq s_0\leq n-1$, which we can check by showing that $r(1)\neq 0$. We have that;

$|h|_{(m,m+\frac{1}{m})}|=|(x-(m+\frac{1}{m}))^n p(x)|$

$\leq\frac{1}{m^n}|\sum_{i=0}^{n-1}p^{(i)}(m)(x-m)^i|$

$\leq\frac{1}{m^n}\sum_{i=0}^{n-1}\frac{|p^{(i)}(m)|}{m^i}$

$\leq\sum_{i=0}^{n-1}\sum_{k=0}^{i}|c_{ik}|a_{i-k}|\frac{m^{n+k}}{m^{n+i}}$

$\leq\sum_{i=0}^{n-1}\sum_{k=0}^{i}|c_{ik}|a_{i-k}|=C$, $(m>1)$

The last claim is just the FTC.

$\square$

Lemma 0.25. *If $[a,b]\subset\mathcal{R}$, with a,b finite, $n\geq 3$, and $\{g_j:0\leq j\leq n-1\}\subset C^\infty([a,b])$, then, if $m\in\mathcal{R}_{>0}$ is sufficiently large, there exists $h\in C^\infty([m,m+\frac{1}{m}]\times[a,b])$, with the property that;*

$\frac{\partial^{(j)}h}{\partial x^j}|_{(m,y)}=g_j(y)$, $y\in[a,b]$, (i)

$\frac{\partial h^j}{\partial x^j}(m+\frac{1}{m},y)=0$, $y\in[a,b]$, (ii)

$|h|_{[m,m+\frac{1}{m}]\times[a,b]}|\leq C$

for some $C\in\mathcal{R}_{>0}$*, independent of* m *sufficiently large, and, if* $\frac{\partial^n h}{\partial x^n}(m,y)>0$*,* $\frac{\partial^n h}{\partial x^n}(x,y)>0$*, for* $x\in[m,m+\frac{1}{m}]$*, and if* $\frac{\partial^n h}{\partial x^n}(m,y)<0$*,* $\frac{\partial^n h}{\partial x^n}(x,y)<0$*, for* $x\in[m,m+\frac{1}{m}]$*,* $(*)$*. In particularly;*

$\int_m^{m+\frac{1}{m}}|\frac{\partial^n h}{\partial x^n}|_{(x,y)}|dx=|g_{n-1}(y)|$

Moreover, for $i\in\mathcal{N}$*,* $\frac{\partial^i h}{\partial y^i}$ *has the property that;*

$\frac{\partial^{i+j}h}{\partial x^j\partial y^i}(m,y)=g_j^{(i)}(y)$, $y\in[a,b]$, $(i)'$

$\frac{\partial^{i+j}h}{\partial x^j\partial y^i}(m+\frac{1}{m},y)=0$, $y\in[a,b]$, $(ii)'$

$|\frac{\partial^i h}{\partial y^i}|_{[m,m+\frac{1}{m}]\times[a,b]}|\leq C_i$

for some $C_i\in\mathcal{R}_{>0}$*, independent of* m *sufficiently large, and, if* $\frac{\partial^{i+n}h}{\partial y^i\partial x^n}(m,y)>0$*,* $\frac{\partial^{i+n}h}{\partial y^i\partial x^n}(x,y)>0$*, for* $x\in[m,m+\frac{1}{m}]$*, and if* $\frac{\partial^{i+n}h}{\partial y^i\partial x^n}(m,y)<0$*,* $\frac{\partial^{i+n}h}{\partial y^i\partial x^n}(x,y)<0$*, for* $x\in[m,m+\frac{1}{m}]$*,* $(**)$*. In particularly;*

$\int_m^{m+\frac{1}{m}}|\frac{\partial^{i+n}h}{\partial y^i\partial x^n}|_{(x,y)}|dx=|g_{n-1}^{(i)}(y)|$

Proof. For the construction of h in the first part, just use the proof of Lemma 0.24, replacing the constant coefficients $\{a_j:0\leq j\leq n-1\}\subset\mathcal{R}$ with the data $\{g_j(y):0\leq j\leq n-1\}$. The properties $(i),(ii)$ are then clear. Noting that $[a,b]$ is a finite interval and $\{g_j:0\leq j\leq n-1\}\subset C^\infty([a,b])$, by continuity, there exists a constant D, with $max(|g_j(y)|:0\leq j\leq n-1,y\in[a,b])\leq D$, so, as in the proof of Lemma 0.4, we can use the bound $C=\sum_{0\leq j\leq n-1}L_jD$, for $m>1$. The proof of $(*)$ follows uniformly in y, as in the proof of 0.4, for sufficiently large m, again using the fact that the data $\{g_j(y):0\leq j\leq n-1,y\in[a,b]\}$ is bounded. The next claim is just the FTC again. For the second part, when we calculate $\frac{\partial^i h}{\partial y^i}$, for $i\in\mathcal{N}$, we are just differentiating the coefficients which are linear in the data $\{g_j(y):0\leq j\leq n-1\}$, so we obtain a function which fits the data $\{g_j^{(i)}(y):0\leq j\leq n-1\}$ and $(i)',(ii)'$ follow. Noting that, for $i\in\mathcal{N}$,

$\{g_j^{(i)} : 0 \leq j \leq n-1\} \subset C^\infty([a,b])$, again by continuity, there exist constants D_i, with $max(|g_j^{(i)}(y)| : 0 \leq j \leq n-1, y \in [a,b]) \leq D_i$, so, again, as in the proof of Lemma 0.4, we can use the bound $C_i = \sum_{0\leq j\leq n-1} L_j D_i$, for $m > 1$. The proof of $(**)$ follows uniformly in y, for each $i \in \mathcal{N}$, as in the proof of Lemma 0.4, for sufficiently large m, again using the fact that the data $\{g_j^{(i)}(y) : 0 \leq j \leq n-1, y \in [a,b]\}$ is bounded. The last claim is again just the FTC. □

Lemma 0.26. *If $[a,b] \subset \mathcal{R}$, $[c,d] \subset \mathcal{R}$, with a,b,c,d finite, $n \geq 3$, and $\{g_j : 0 \leq j \leq n-1\} \subset C^\infty([a,b] \times [c,d])$, then, if $m \in \mathcal{R}_{>0}$ is sufficiently large, there exists $h \in C^\infty([m, m+\frac{1}{m}] \times [a,b] \times [c,d])$, with the property that;*

$\frac{\partial^{(j)} h}{\partial x^j}|_{(m,y,z)} = g_j(y,z)$, $(y,z) \in [a,b] \times [c,d]$, (i)

$\frac{\partial h^j}{\partial x^j}(m+\frac{1}{m}, y, z) = 0$, $(y,z) \in [a,b] \times [c,d]$, (ii)

$|h|_{[m,m+\frac{1}{m}]\times[a,b]\times[c,d]}| \leq C$

for some $C \in \mathcal{R}_{>0}$, independent of m sufficiently large, and, if $\frac{\partial^n h}{\partial x^n}(m,y,z) > 0$, $\frac{\partial^n h}{\partial x^n}(x,y,z) > 0$, for $x \in [m, m+\frac{1}{m}]$, and if $\frac{\partial^n h}{\partial x^n}(m,y,z) < 0$, $\frac{\partial^n h}{\partial x^n}(x,y,z) < 0$, for $x \in [m, m+\frac{1}{m}]$, $(*)$. In particularly;

$\int_m^{m+\frac{1}{m}} |\frac{\partial^n h}{\partial x^n}|_{(x,y,z)}|dx = |g_{n-1}(y,z)|$

Moreover, for $(i,k) \subset \mathcal{N}^2$, $0 \leq j \leq n-1$, $\frac{\partial^{i+k} h}{\partial y^i \partial z^k}$, has the property that;

$\frac{\partial^{i+j+k} h}{\partial x^j \partial y^i \partial z^k}(m,y,z) = \frac{\partial^{i+k} g_j}{\partial y^i \partial z^k}(y,z)$, $(y,z) \in [a,b] \times [c,d]$, $(i)'$

$\frac{\partial^{i+j+k} h}{\partial x^j \partial y^i \partial z^k}(m+\frac{1}{m}, y, z) = 0$, $(y,z) \in [a,b] \times [c,d]$, $(ii)'$

$|\frac{\partial^{i+k} h}{\partial y^i \partial z^k}|_{[m,m+\frac{1}{m}]\times[a,b]\times[c,d]}| \leq C_{i,k}$

for some $C_{i,k} \in \mathcal{R}_{>0}$, independent of m sufficiently large, and, if $\frac{\partial^{i+k+n} h}{\partial y^i \partial z^k \partial x^n}(m,y,z) > 0$, $\frac{\partial^{i+k+n} h}{\partial y^i \partial z^k \partial x^n}(x,y,z) > 0$, for $x \in [m, m+\frac{1}{m}]$, and if $\frac{\partial^{i+k+n} h}{\partial y^i \partial z^k \partial x^n}(m,y) < 0$, $\frac{\partial^{i+k+n} h}{\partial y^i \partial z^k \partial x^n}(x,y,z) < 0$, for $x \in [m, m+\frac{1}{m}]$, $(**)$. In particularly;

$\int_m^{m+\frac{1}{m}} |\frac{\partial^{i+k+n} h}{\partial y^i \partial z^k \partial x^n}|_{(x,y,z)}|dx = |\frac{\partial^{i+k} g_{n-1}}{\partial y^i \partial z^k}(y,z)|$

Proof. For the construction of h in the first part, just use the proof of Lemma 0.24, replacing the constant coefficients $\{a_j : 0 \leq j \leq n-1\} \subset \mathcal{R}$ with the data $\{g_j(y,z) : 0 \leq j \leq n-1\}$. The properties $(i),(ii)$ are then clear. Noting that $[a,b]\times[c,d]$ is compact and $\{g_j : 0 \leq j \leq n-1\} \subset C^\infty([a,b]\times[c,d])$, by continuity, there exists a constant D, with $max(|g_j(y,z)| : 0 \leq j \leq n-1, (y,z) \in [a,b]\times[c,d]) \leq D$, so, as in the proof of Lemma 0.24, we can use the bound $C = \sum_{0\leq j\leq n-1} L_j D$, for $m > 1$. The proof of $(*)$ follows uniformly in y, as in the proof of 0.24, for sufficiently large m, again using the fact that the data $\{g_j(y,z) : 0 \leq j \leq n-1, (y,z) \in [a,b]\}$ is bounded. The next claim is just the FTC again. For the second part, when we calculate $\frac{\partial^{i+k}h}{\partial y^i \partial z^k}$, for $(i,j \in \mathcal{N}^2$, we are just differentiating the coefficients which are linear in the data $\{g_j(y,z) : 0 \leq j \leq n-1\}$, so we obtain a function which fits the data $\{\frac{\partial^{i+k}g_j}{\partial y^i \partial z^k}(y,z) : 0 \leq j \leq n-1\}$ and $(i)',(ii)'$ follow. Noting that, for $(i,k) \in \mathcal{N}^2$, $\{\frac{\partial^{i+k}g_j}{\partial y^i \partial z^k} : 0 \leq j \leq n-1\} \subset C^\infty([a,b]\times[c,d])$, again by continuity, there exist constants $D_{i,k}$, with $max(|\frac{\partial^{i+k}g_j}{\partial y^i \partial z^k}(y,z)| : 0 \leq j \leq n-1, y \in [a,b]\times[c,d]) \leq D_{i,k}$, so, again, as in the proof of Lemma 0.24, we can use the bound $C_{i,k} = \sum_{0\leq j\leq n-1} L_j D_{i,k}$, for $m > 1$. The proof of $(**)$ follows uniformly in (y,z), for each $(i,k) \in \mathcal{N}^2$, as in the proof of Lemma 0.24, for sufficiently large m, again using the fact that the data $\{\frac{\partial^{i+k}g_j}{\partial Y^i \partial z^k}(y) : 0 \leq j \leq n-1, (y,z) \in [a,b]\times[c,d]\}$ is bounded. The last claim is again just the FTC. □

Lemma 0.27. *For $f \in C^{27}(\mathcal{R}^2)$ with $\frac{\partial^{i_1+i_2}f}{\partial x^{i_1}\partial y^{i_2}}$ bounded by some constant $F \in \mathcal{R}_{>0}$, for $0 \leq i_1 + i_2 \leq 27$. Then for sufficiently large m, there exists an inflexionary approximation sequence $\{f_m : m \in \mathcal{N}\}$, with the property that;*

$$max(\int_{\mathcal{R}^2} |\frac{\partial f_m}{\partial x^{14}}|dxdy, \int_{\mathcal{R}^2} |\frac{\partial f_m}{\partial y^{14}}|dxdy) \leq Gm^2$$

for some $G \in \mathcal{R}_{>0}$, for sufficiently large m.

Proof. Define $f_m = f$ on C_m, so that (ii) of Definition 0.21 is satisfied. Using two applications of Lemma 0.25 with $n = 14$, changing to a vertical rather than horizontal orientation, and the fact that, for $0 \leq i \leq 13$, $|x| \leq m$, $\frac{\partial^i f}{\partial y^i}|_{(x,m)}$ and $\frac{\partial^i f}{\partial y^i}|_{(x,-m)}$ define smooth functions on $[-m,m]$, we can extend f_m to $R = \{(x,y) : |x| \leq m, m \leq |y| \leq m+\frac{1}{m}\}$, such that $f_m|R_1$ satisfies conditions $(iv),(v)$ of Definition 0.21, where $R_1 = \{(x,y) : |x| \leq m, 0 \leq |y| \leq m+\frac{1}{m}\}$. Again, using two applications of Lemma 0.25 with $n = 14$, and the original horizontal orientation, and the fact that, for $0 \leq i \leq 13$, $0 \leq |y| \leq m + \frac{1}{m}$, $\frac{\partial^i f_m}{\partial x^i}|_{(m,y)}$ and $\frac{\partial^i f}{\partial x^i}|_{(-m,y)}$ define smooth functions on $[-m - \frac{1}{m}, m + \frac{1}{m}]$, we can extend f_m to $S = \{(x,y) : m \leq |x| \leq m + \frac{1}{m}, 0 \leq |y| \leq m + \frac{1}{m}\}$, such that $f_m|C_{m+\frac{1}{m}}$ satisfies conditions $(vi), (vii)$ of Definition 0.21. Conditions $(i), (iii)$ are then clear. We then have, using (iii), that;

$\int_{\mathcal{R}^2} |\frac{\partial f_m}{\partial x^{14}}|dxdy = \int_{C_{m+\frac{1}{m}}} |\frac{\partial f_m}{\partial x^{14}}|dxdy$

$= \int_{|x|\leq m, |y|\leq m} |\frac{\partial f_m}{\partial x^{14}}|dxdy + \int_{|x|\leq m, m\leq |y|\leq m+\frac{1}{m}} |\frac{\partial f_m}{\partial x^{14}}|dxdy + \int_{m\leq |x|\leq m+\frac{1}{m}, |y|\leq m} |\frac{\partial f_m}{\partial x^{14}}|dxdy$

$+ \int_{m\leq |x|\leq m+\frac{1}{m}, m\leq |y|\leq m+\frac{1}{m}} |\frac{\partial f_m}{\partial x^{14}}|dxdy$

$\int_{\mathcal{R}^2} |\frac{\partial f_m}{\partial y^{14}}|dxdy = \int_{C_{m+\frac{1}{m}}} |\frac{\partial f_m}{\partial y^{14}}|dxdy$

$= \int_{|x|\leq m, |y|\leq m} |\frac{\partial f_m}{\partial y^{14}}|dxdy + \int_{|x|\leq m, m\leq |y|\leq m+\frac{1}{m}} |\frac{\partial f_m}{\partial y^{14}}|dxdy + \int_{m\leq |x|\leq m+\frac{1}{m}, |y|\leq m} |\frac{\partial f_m}{\partial y^{14}}|dxdy$

$+ \int_{m\leq |x|\leq m+\frac{1}{m}, m\leq |y|\leq m+\frac{1}{m}} |\frac{\partial f_m}{\partial y^{14}}|dxdy \; (*)$

We then have the following cases, using the second clause in Lemma 0.25 repeatedly with the appropriate orientations;

Case 1;

$\int_{|x|\leq m, |y|\leq m} |\frac{\partial^{14} f_m}{\partial x^{14}}|dxdy$

$= \int_{|x|\leq m, |y|\leq m} |\frac{\partial^{14} f}{\partial x^{14}}|dxdy \leq Fm^2$

$\int_{|x|\leq m, |y|\leq m} |\frac{\partial^{14} f_m}{\partial y^{14}}|dxdy$

$= \int_{|x|\leq m, |y|\leq m} |\frac{\partial^{14} f}{\partial y^{14}}|dxdy \leq Fm^2$

Case 2;

$\int_{|x|\leq m, m\leq |y|\leq m+\frac{1}{m}} |\frac{\partial^{14} f_m}{\partial x^{14}}|dxdy$

$= \int_{|x|\leq m} (\int_{|y|\leq m+\frac{1}{m}} |\frac{\partial^{14} f_m}{\partial x^{14}}|dy)dx$

$\leq \frac{2}{m} \int_{|x|\leq m} C_{14} dx$

$\leq 2m\frac{2}{m} C_{14}$

$= 4C_{14}$

Case 3;

$\int_{m\leq |x|\leq m+\frac{1}{m},|y|\leq m} |\frac{\partial^{14} f_m}{\partial x^{14}}|dxdy$

$= \int_{|y|\leq m}(\int_{m\leq |x|\leq m+\frac{1}{m}} |\frac{\partial^{14} f_m}{\partial x^{14}}|dx)dy$

$= \int_{|y|\leq m}(|\frac{\partial^{13} f}{\partial x^{13}}|_{(m,y)} + |\frac{\partial^{13} f}{\partial x^{13}}|_{(-m,y)})dy$

$\leq 4mF$

Case 4.

$\int_{m\leq |x|\leq m+\frac{1}{m},m\leq |y|\leq m+\frac{1}{m}} |\frac{\partial^{14} f_m}{\partial x^{14}}|dxdy$

$= \int_{m\leq |y|\leq m+\frac{1}{m}}(\int_{m\leq |x|\leq m+\frac{1}{m}} |\frac{\partial^{14} f_m}{\partial x^{14}}|dx)dy$

$= \int_{m\leq |y|\leq m+\frac{1}{m}}(|\frac{\partial^{13} f_m}{\partial x^{13}}|_{(m,y)} + |\frac{\partial^{13} f_m}{\partial x^{13}}|_{(-m,y)}dy$

$\leq \int_{m\leq y\leq m+\frac{1}{m}} C_{13,1}dy + \int_{-m-\frac{1}{m}\leq -m} C_{13,2}dy$

$\leq \frac{max(C_{13,1},C_{13,2})}{m}$ (the constants $\{C_{13,1}, C_{13,2}\}$ coming from the two applications of Lemma 0.25 at the two boundaries)

Case 5;

$\int_{|x|\leq m,m\leq |y|\leq m+\frac{1}{m}} |\frac{\partial^{14} f_m}{\partial y^{14}}|dxdy$

$= \int_{|x|\leq m}(\int_{m\leq |y|\leq m+\frac{1}{m}} |\frac{\partial^{14} f_m}{\partial y^{14}}|dy)dx$

$= \int_{|x|\leq m}(|\frac{\partial f}{\partial y^{13}}|_{(x,m)} + |\frac{\partial f(x,y)}{\partial y^{13}}|_{(x,-m)}dx)$

$\leq 4mF$

Case 6;

$\int_{|y|\leq m,m\leq |x|\leq m+\frac{1}{m}} |\frac{\partial^{14} f_m}{\partial y^{14}}|dxdy$

$= \int_{|y|\leq m}(\int_{m\leq |x|\leq m+\frac{1}{m}} |\frac{\partial^{14} f_m}{\partial y^{14}}|dx)dy$

$\leq \frac{1}{m}\int_{|y|\leq m}(|\sum_{i=0}^{13}D_i|\frac{\partial^i\partial^{14}f}{\partial y^{14}\partial x^i}|(m,y)+|\sum_{i=0}^{13}D_i|\frac{\partial^i\partial^{14}f}{\partial y^{14}\partial x^i}|(-m,y))dy$

$\leq \frac{2}{m}(2m)F(\sum_{i=0}^{13}D_i)$

$= 4F(\sum_{i=0}^{13}D_i)$

Case 7.

$\int_{m\leq|x|\leq m+\frac{1}{m},m\leq|y|\leq m+\frac{1}{m}}|\frac{\partial^{14}f_m}{\partial y^{14}}|dxdy$

$= \int_{m\leq|y|\leq m+\frac{1}{m}}(\int_{m\leq|x|\leq m+\frac{1}{m}}|\frac{\partial^{14}f_m}{\partial y^{14}}|dx)dy$

$\leq \frac{1}{m}\int_{m\leq|y|\leq m+\frac{1}{m}}(\sum_{i=0}^{13}L_{i,14}|\frac{\partial^{i+14}f_m}{\partial x^i\partial y^{14}}|_{(m,y)}+L_{i,14}|\frac{\partial^{i+14}f_m}{\partial x^i\partial y^{14}}|_{(-m,y)})dy$

$= \frac{1}{m}\sum_{i=0}^{13}L_{i,14}(|\frac{\partial^{i+13}f}{\partial x^i\partial y^{13}}|_{(m,m)}|+|\frac{\partial^{i+13}f}{\partial x^i\partial y^{13}}|_{(m,-m)}|+|\frac{\partial^{i+13}f}{\partial x^i\partial y^{13}}|_{(-m,m)}|+|\frac{\partial^{i+13}f}{\partial x^i\partial y^{13}}|_{(-m,-m)}|)$

$\leq \frac{4F(\sum_{i=0}^{13}L_{i,14})}{m}$ (the constants $L_{i,14}$,$0\leq i\leq 13$ coming from the proof of Lemma 0.25)

Combining the seven cases and $(*)$, we obtain, for sufficiently large m, that;

$\int_{\mathcal{R}^2}|\frac{\partial f_m}{\partial x^{14}}|dxdy \leq Fm^2+4C_{14}+4mF+\frac{max(C_{13,1},C_{13,2})}{m}\leq Gm^2$

$\int_{\mathcal{R}^2}|\frac{\partial f_m}{\partial y^{14}}|dxdy \leq Fm^2+4mF+4F(\sum_{i=0}^{13}D_i)+\frac{4F(\sum_{i=0}^{13}L_{i,14})}{m}\leq Gm^2$

□

Lemma 0.28. *For $f\in C^{40}(\mathcal{R}^3)$ with $\frac{\partial^{i_1+i_2+i_3}f}{\partial x^{i_1}\partial y^{i_2}\partial z^{i_3}}$ bounded by some constant $F\in\mathcal{R}_{>0}$, for $0\leq i_1+i_2+i_3\leq 40$. Then for sufficiently large m, there exists an inflexionary approximation sequence $\{f_m:m\in\mathcal{N}\}$, with the property that;*

$max(\int_{\mathcal{R}^3}|\frac{\partial f_m}{\partial x^{14}}|dxdydz,\int_{\mathcal{R}^3}|\frac{\partial f_m}{\partial y^{14}}|dxdydz,\int_{\mathcal{R}^3}|\frac{\partial f_m}{\partial z^{14}}|dxdydz)\leq Gm^3$

for some $G\in\mathcal{R}_{>0}$, for sufficiently large m.

Proof. Define $f_m=f$ on W_m, so that (ii) of Definition 0.22 is satisfied. Using two applications of Lemma 0.26 with $n=14$, with a horizontal orientation, and the fact that, for $0\leq i\leq 13$, $0\leq|y|\leq m$, $0\leq|z|\leq m$ $\frac{\partial^i f}{\partial x^i}|_{(m,y,z)}$ and $\frac{\partial^i f}{\partial x^i}|_{(-m,y,z)}$ define smooth functions on $[-m,m]^2$, we can extend f_m to $A_1=\{(x,y,z):m\leq |x|\leq m+\frac{1}{m},0\leq |y|\leq m,0\leq |z|\leq m\}$, such that $f_m|A_2$ satisfies conditions $(iv),(v)$ of Definition 0.22, where $A_2=\{(x,y,z):0\leq |x|\leq m+\frac{1}{m},0\leq |y|\leq m,0\leq |z|\leq m\}$. Again, using two applications of Lemma 0.26 with $n=14$ again, this time with a vertical orientation, and the fact that, for $0\leq i\leq 13$, $0\leq |x|\leq m+\frac{1}{m}$, $0\leq |z|\leq m$, $\frac{\partial^i f_m}{\partial y^i}|_{(x,m,z)}$ and $\frac{\partial^i f_m}{\partial y^i}|_{(x,-m,z)}$ define smooth functions on $[-m-\frac{1}{m},m+\frac{1}{m}]\times[-m,m]$, we can extend f_m to $A_3=\{(x,y,z):0\leq |x|\leq m+\frac{1}{m},m\leq |y|\leq m+\frac{1}{m},0\leq |z|\leq m\}$, such that $f_m|A_4$ satisfies conditions $(vi),(vii)$ of Definition 0.22, where $A_4=\{(x,y,z):0\leq |x|\leq m+\frac{1}{m},0\leq |y|\leq m+\frac{1}{m},0\leq |z|\leq m\}$. Again, using two applications of Lemma 0.26 with $n=14$ again, this time with a lateral orientation, and the fact that, for $0\leq i\leq 13$, $0\leq |x|\leq m+\frac{1}{m}$, $0\leq |y|\leq m+\frac{1}{m}$, $\frac{\partial^i f_m}{\partial z^i}|_{(x,y,m)}$ and $\frac{\partial^i f_m}{\partial z^i}|_{(x,y,-m)}$ define smooth functions on $[-m-\frac{1}{m},m+\frac{1}{m}]^2$, we can extend f_m to $W_{m+\frac{1}{m}}$ such that $f_m|_{W_{m+\frac{1}{m}}}$ satisfies conditions $(viii),(ix)$ of Definition 0.22.

Conditions $(i),(iii)$ are then clear. We then have, using (iii), that;

(a). $\int_{\mathcal{R}^3}|\frac{\partial f_m}{\partial x^{14}}|dxdydz=\int_{W_{m+\frac{1}{m}}}|\frac{\partial f_m}{\partial x^{14}}|dxdydz$

$=\int_{|x|\leq m,|y|\leq m,|z|\leq m}|\frac{\partial f_m}{\partial x^{14}}|dxdydz+\int_{m\leq |x|\leq m+\frac{1}{m},|y|\leq m,|z|\leq m}|\frac{\partial f_m}{\partial x^{14}}|dxdydz$

$+\int_{|x|\leq m,m\leq |y|\leq m+\frac{1}{m},|z|\leq m}|\frac{\partial f_m}{\partial x^{14}}|dxdydz+\int_{m\leq |x|\leq m+\frac{1}{m},m\leq |y|\leq m+\frac{1}{m},|z|\leq m}|\frac{\partial f_m}{\partial x^{14}}|dxdydz$

$+\int_{|x|\leq m,|y|\leq m,m\leq |z|\leq m+\frac{1}{m}}|\frac{\partial f_m}{\partial x^{14}}|dxdydz+\int_{m\leq |x|\leq m+\frac{1}{m},|y|\leq m,m\leq |z|\leq m+\frac{1}{m}}|\frac{\partial f_m}{\partial x^{14}}|dxdydz$

$+\int_{|x|\leq m,m\leq |y|\leq m+\frac{1}{m},m\leq |z|\leq m+\frac{1}{m}}|\frac{\partial f_m}{\partial x^{14}}|dxdydz+\int_{m\leq |x|\leq m+\frac{1}{m},m\leq |y|\leq m+\frac{1}{m},m\leq |z|\leq m+\frac{1}{m}}|\frac{\partial f_m}{\partial x^{14}}|dxdydz$

(b). $\int_{\mathcal{R}^3}|\frac{\partial f_m}{\partial y^{14}}|dxdydz=\int_{W_{m+\frac{1}{m}}}|\frac{\partial f_m}{\partial y^{14}}|dxdydz$

$=\int_{|x|\leq m,|y|\leq m,|z|\leq m}|\frac{\partial f_m}{\partial y^{14}}|dxdydz+\int_{m\leq |x|\leq m+\frac{1}{m},|y|\leq m,|z|\leq m}|\frac{\partial f_m}{\partial y^{14}}|dxdydz$

$+\int_{|x|\leq m,m\leq |y|\leq m+\frac{1}{m},|z|\leq m}|\frac{\partial f_m}{\partial y^{14}}|dxdydz+\int_{m\leq |x|\leq m+\frac{1}{m},m\leq |y|\leq m+\frac{1}{m},|z|\leq m}|\frac{\partial f_m}{\partial y^{14}}|dxdydz$

$+\int_{|x|\leq m,|y|\leq m,m\leq |z|\leq m+\frac{1}{m}}|\frac{\partial f_m}{\partial y^{14}}|dxdydz+\int_{m\leq |x|\leq m+\frac{1}{m},|y|\leq m,m\leq |z|\leq m+\frac{1}{m}}|\frac{\partial f_m}{\partial y^{14}}|dxdydz$

$+\int_{|x|\leq m,m\leq |y|\leq m+\frac{1}{m},m\leq |z|\leq m+\frac{1}{m}}|\frac{\partial f_m}{\partial y^{14}}|dxdydz+\int_{m\leq |x|\leq m+\frac{1}{m},m\leq |y|\leq m+\frac{1}{m},m\leq |z|\leq m+\frac{1}{m}}|\frac{\partial f_m}{\partial y^{14}}|dxdydz$

(c). $\int_{\mathcal{R}^3}|\frac{\partial f_m}{\partial z^{14}}|dxdydz=\int_{W_{m+\frac{1}{m}}}|\frac{\partial f_m}{\partial z^{14}}|dxdydz$

$=\int_{|x|\leq m,|y|\leq m,|z|\leq m}|\frac{\partial f_m}{\partial z^{14}}|dxdydz+\int_{m\leq |x|\leq m+\frac{1}{m},|y|\leq m,|z|\leq m}|\frac{\partial f_m}{\partial z^{14}}|dxdydz$

$+\int_{|x|\leq m,m\leq|y|\leq m+\frac{1}{m},|z|\leq m}|\frac{\partial f_m}{\partial z^{14}}|dxdydz+\int_{m\leq|x|\leq m+\frac{1}{m},m\leq|y|\leq m+\frac{1}{m},|z|\leq m}|\frac{\partial f_m}{\partial z^{14}}|dxdydz$

$+\int_{|x|\leq m,|y|\leq m,m\leq|z|\leq m+\frac{1}{m}}|\frac{\partial f_m}{\partial z^{14}}|dxdydz+\int_{m\leq|x|\leq m+\frac{1}{m},|y|\leq m,m\leq|z|\leq m+\frac{1}{m}}|\frac{\partial f_m}{\partial z^{14}}|dxdydz$

$+\int_{|x|\leq m,m\leq|y|\leq m+\frac{1}{m},m\leq|z|\leq m+\frac{1}{m}}|\frac{\partial f_m}{\partial z^{14}}|dxdydz+\int_{m\leq|x|\leq m+\frac{1}{m},m\leq|y|\leq m+\frac{1}{m},m\leq|z|\leq m+\frac{1}{m}}|\frac{\partial f_m}{\partial z^{14}}|dxdydz$

$(*)$

We then have the following cases, using the second clause in Lemma 0.26 repeatedly with the appropriate orientations;

Case 1;

$\int_{|x|\leq m,|y|\leq m,|z|\leq m}|\frac{\partial^{14}f_m}{\partial x^{14}}|dxdydz$

$=\int_{|x|\leq m,|y|\leq m,|z|\leq m}|\frac{\partial^{14}f}{\partial x^{14}}|dxdydz\leq Fm^3$

$\int_{|x|\leq m,|y|\leq m,|z|\leq m}|\frac{\partial^{14}f_m}{\partial y^{14}}|dxdydz$

$=\int_{|x|\leq m,|y|\leq m,|z|\leq m}|\frac{\partial^{14}f}{\partial y^{14}}|dxdydz\leq Fm^3$

$\int_{|x|\leq m,|y|\leq m,|z|\leq m}|\frac{\partial^{14}f_m}{\partial z^{14}}|dxdydz$

$=\int_{|x|\leq m,|y|\leq m,|z|\leq m}|\frac{\partial^{14}f}{\partial z^{14}}|dxdydz\leq Fm^3$

Case 2;

$\int_{m\leq|x|\leq m+\frac{1}{m},|y|\leq m,|z|\leq m}|\frac{\partial^{14}f_m}{\partial x^{14}}|dxdydz$

$=\int_{|y|\leq m,|z|\leq m}(\int_{m\leq|x|\leq m+\frac{1}{m}}|\frac{\partial^{14}f_m}{\partial x^{14}}|dx)dydz$

$=\int_{|y|\leq m}(|\frac{\partial^{13}f}{\partial x^{13}}|_{(m,y,z)}+|\frac{\partial^{13}f}{\partial x^{13}}|_{(-m,y,z)})dydz$

$\leq 2(2m)^2F$

$=8m^2F$

Case 3;

$\int_{m\leq|x|\leq m+\frac{1}{m},|y|\leq m,|z|\leq m}|\frac{\partial^{14}f_m}{\partial y^{14}}|dxdydz$

$= \int_{|y|\leq m,|z|\leq m}(\int_{m\leq|x|\leq m+\frac{1}{m}}|\frac{\partial^{14}f_m}{\partial y^{14}}|dx)dydz$

$\leq \frac{1}{m}\int_{|y|\leq m,|z|\leq m}(|\sum_{i=0}^{13}D_i|\frac{\partial^i\partial^{14}f}{\partial y^{14}\partial x^i}|(m,y,z)+|\sum_{i=0}^{13}D_i|\frac{\partial^i\partial^{14}f}{\partial y^{14}\partial x^i}|(-m,y,z))dydz$

$\leq \frac{2}{m}(2m)^2F(\sum_{i=0}^{13}D_i)$

$= 8mF(\sum_{i=0}^{13}D_i)$

$\int_{m\leq|x|\leq m+\frac{1}{m},|y|\leq m,|z|\leq m}|\frac{\partial^{14}f_m}{\partial z^{14}}|dxdydz$

$= \int_{|y|\leq m,|z|\leq m}(\int_{m\leq|x|\leq m+\frac{1}{m}}|\frac{\partial^{14}f_m}{\partial z^{14}}|dx)dydz$

$\leq \frac{1}{m}\int_{|y|\leq m,|z|\leq m}(|\sum_{i=0}^{13}D_i|\frac{\partial^i\partial^{14}f}{\partial z^{14}\partial x^i}|(m,y,z)+|\sum_{i=0}^{13}D_i|\frac{\partial^i\partial^{14}f}{\partial z^{14}\partial x^i}|(-m,y,z))dydz$

$\leq \frac{2}{m}(2m)^2F(\sum_{i=0}^{13}D_i)$

$= 8mF(\sum_{i=0}^{13}D_i)$

Case 4.

$\int_{|x|\leq m,m\leq|y|\leq m+\frac{1}{m},|z|\leq m}|\frac{\partial^{14}f_m}{\partial x^{14}}|dxdydz$

$= \int_{|x|\leq m,|z|\leq m}(\int_{|y|\leq m+\frac{1}{m}}|\frac{\partial^{14}f_m}{\partial x^{14}}|dy)dxdz$

$\leq \frac{2}{m}\int_{|x|\leq m,|z|\leq m}C_{14}dx$

$= (2m)^2\frac{2}{m}C_{14,0}$

$= 8mC_{14,0}$

$\int_{|x|\leq m,m\leq|y|\leq m+\frac{1}{m},|z|\leq m}|\frac{\partial^{14}f_m}{\partial z^{14}}|dxdydz$

$= \int_{|x|\leq m,|z|\leq m}(\int_{|y|\leq m+\frac{1}{m}}|\frac{\partial^{14}f_m}{\partial z^{14}}|dy)dxdz$

$\leq \frac{2}{m}\int_{|x|\leq m,|z|\leq m}C_{0,14}dx$

$= (2m)^2\frac{2}{m}C_{0,14}$

$= 8mC_{0,14}$

Case 5.

$\int_{|x|\leq m, m\leq|y|\leq m+\frac{1}{m}, |z|\leq m} |\frac{\partial^{14} f_m}{\partial y^{14}}| dxdydz$

$= \int_{|x|\leq m, |z|\leq m} (\int_{m\leq|y|\leq m+\frac{1}{m}} |\frac{\partial^{14} f_m}{\partial y^{14}}| dy) dxdz$

$= \int_{|x|\leq m, |z|\leq m} (|\frac{\partial f}{\partial y^{13}}|_{(x,m,z)} + |\frac{\partial f}{\partial y^{13}}|_{(x,-m,z)}) dxdz)$

$\leq 2(2m)^2 F$

$= 8m^2 F$

Case 6.

$\int_{m\leq|x|\leq m+\frac{1}{m}, m\leq|y|\leq m+\frac{1}{m}, |z|\leq m} |\frac{\partial^{14} f_m}{\partial x^{14}}| dxdydz$

$= \int_{m\leq|x|\leq m+\frac{1}{m}, |z|\leq m} (\int_{m\leq|y|\leq m+\frac{1}{m}} |\frac{\partial^{14} f_m}{\partial x^{14}}| dy) dxdz$

$\leq \frac{1}{m} \int_{m\leq|x|\leq m+\frac{1}{m}, |z|\leq m} (\sum_{i=0}^{13} L_{i,14} |\frac{\partial^{i+14}\partial^{14} f_m}{\partial y^i \partial x^{14}}|_{(x,m,z)} + L_{i,14} |\frac{\partial^{i+14}\partial^{14} f_m}{\partial y^i \partial x^{14}}|_{(x,-m,z)}) dxdz$

$= \frac{1}{m} \int_{|z|\leq m} (\sum_{i=0}^{13} L_{i,14} (|\frac{\partial^{i+13}\partial^{14} f}{\partial y^i \partial x^{13}}|_{(m,m,z)}| + |\frac{\partial^{i+13}\partial^{14} f}{\partial y^i \partial x^{13}}|_{(m,-m,z)}| + |\frac{\partial^{i+13}\partial^{14} f}{\partial y^i \partial x^{13}}|_{(-m,m,z)}|$

$+ |\frac{\partial^{i+13}\partial^{14} f}{\partial y^i \partial x^{13}}|_{(-m,-m,z)}|)) dz$

$\leq (2m) \frac{4F(\sum_{i=0}^{13} L_{i,14})}{m}$

$= 8F(\sum_{i=0}^{13} L_{i,14})$

(the constants $L_{i,14}, 0 \leq i \leq 13$ coming from the proof of Lemma 0.25)

Case 7.

$\int_{m\leq|x|\leq m+\frac{1}{m}, m\leq|y|\leq m+\frac{1}{m}, |z|\leq m} |\frac{\partial^{14} f_m}{\partial y^{14}}| dxdydz$

$= \int_{m\leq|x|\leq m+\frac{1}{m}, |z|\leq m} (\int_{m\leq|y|\leq m+\frac{1}{m}} |\frac{\partial^{14} f_m}{\partial y^{14}}| dy) dxdz$

$= \int_{m\leq|x|\leq m+\frac{1}{m}, |z|\leq m} (|\frac{\partial^{13} f_m}{\partial y^{13}}|_{(x,m,z)} + |\frac{\partial^{13} f_m}{\partial y^{13}}|_{(x,-m,z)}) dxdz$

$\leq \int_{m\leq x\leq m+\frac{1}{m}, |z|\leq m} C_{13,1} dxdz + \int_{-m-\frac{1}{m}\leq -m, |z|\leq m} C_{13,2} dxdz$

$\leq (2m)\frac{max(C_{13,1},C_{13,2})}{m}$

$= 2max(C_{13,1}, C_{13,2})$

(the constants $\{C_{13,1}, C_{13,2}\}$ coming from the two applications of Lemma 0.25 at the two boundaries)

Case 8.

$\int_{m\leq|x|\leq m+\frac{1}{m}, m\leq|y|\leq m+\frac{1}{m}, |z|\leq m} |\frac{\partial^{14} f_m}{\partial z^{14}}|dxdydz$

$= \int_{m\leq|x|\leq m+\frac{1}{m}, |z|\leq m}(\int_{m\leq|y|\leq m+\frac{1}{m}} |\frac{\partial^{14} f_m}{\partial z^{14}}|dy)dxdz$

$\leq \frac{1}{m}\int_{m\leq|x|\leq m+\frac{1}{m}, |z|\leq m}(\sum_{i=0}^{13} L_{i,14}|\frac{\partial^{i+14} f_m}{\partial y^i \partial z^{14}}|_{(x,m,z)} + L_{i,14}|\frac{\partial^{i+14} f_m}{\partial y^i \partial z^{14}}|_{(x,-m,z)})dxdz$

$\leq \frac{1}{m^2}\int_{|z|\leq m}(\sum_{i=0}^{13}\sum_{j=0}^{13} L_{i,14}L_{j,i,14}(|\frac{\partial^{i+j+14} f}{\partial x^j \partial y^i \partial z^{14}}|_{(m,m,z)}| + |\frac{\partial^{i+j+14} f}{\partial x^j \partial y^i \partial z^{14}}|_{(m,-m,z)}|$

$+|\frac{\partial^{i+j+14} f}{\partial x^j \partial y^i \partial z^{14}}|_{(-m,m,z)}| + |\frac{\partial^{i+j+14} f}{\partial x^j \partial y^i \partial z^{14}}|_{(-m,-m,z)}|))dz$

$\leq (2m)\frac{4F(\sum_{i=0}^{13}\sum_{j=0}^{13} L_{i,14}L_{j,i,14})}{m^2}$

$= \frac{8F}{m}(\sum_{i=0}^{13}\sum_{j=0}^{13} L_{i,14}L_{j,i,14})$

(the constants $L_{i,14}, L_{j,i,14}$, $0 \leq i \leq 13, 0 \leq j \leq 13$ coming from two applications of the proof of Lemma 0.26)

Case 9.

$\int_{|x|\leq m, |y|\leq m, m\leq|z|\leq m+\frac{1}{m}} |\frac{\partial f_m}{\partial x^{14}}|dxdydz$

$= \int_{|x|\leq m, |y|\leq m}(\int_{m\leq|z|\leq m+\frac{1}{m}} |\frac{\partial f_m}{\partial x^{14}}|dz)dxdy$

$\leq \frac{2}{m}\int_{|x|\leq m, |y|\leq m}(E_{14,0})$

$= (2m)^2\frac{2}{m}E_{14,0}$

$= 8mE_{14,0}$

$\int_{|x|\leq m, |y|\leq m, m\leq|z|\leq m+\frac{1}{m}} |\frac{\partial f_m}{\partial y^{14}}|dxdydz$

$= \int_{|x|\leq m,|y|\leq m}(\int_{m\leq|z|\leq m+\frac{1}{m}}|\frac{\partial f_m}{\partial y^{14}}|dz)dxdy$

$\leq \frac{2}{m}\int_{|x|\leq m,|y|\leq m}(E_{0,14})$

$= (2m)^2\frac{2}{m}E_{0,14}$

$= 8mE_{0,14}$

(the constants $E_{0,14}, E_{14,0}$ coming from an application of Lemma 0.26 with a different orientation)

Case 10.

$\int_{|x|\leq m,|y|\leq m,m\leq|z|\leq m+\frac{1}{m}}|\frac{\partial f_m}{\partial z^{14}}|dxdydz$

$= \int_{|x|\leq m,|y|\leq m}(\int_{m\leq|z|\leq m+\frac{1}{m}}|\frac{\partial f_m}{\partial z^{14}}|dz)dxdy$

$= \int_{|x|\leq m,|y|\leq m}(|\frac{\partial f}{\partial z^{13}}|(x,y,m)+|\frac{\partial f}{\partial z^{13}}|(x,y,m))dxdy$

$\leq 2(2m)^2F$

$= 8m^2F$

Case 11.

$\int_{m\leq|x|\leq m+\frac{1}{m},|y|\leq m,m\leq|z|\leq m+\frac{1}{m}}|\frac{\partial^{14} f_m}{\partial x^{14}}|dxdydz$

$= \int_{m\leq|x|\leq m+\frac{1}{m},|y|\leq m}(\int_{m\leq|z|\leq m+\frac{1}{m}}|\frac{\partial^{14} f_m}{\partial x^{14}}|dz)dxdy$

$\leq \int_{m\leq|x|\leq m+\frac{1}{m},|y|\leq m}(\sum_{i=0}^{13}L_{i,14}|\frac{\partial^{i+14} f_m}{\partial z^i\partial x^{14}}|(x,y,m)+L_{i,14}|\frac{\partial^{i+14} f_m}{\partial z^i\partial x^{14}}|(x,y,-m))dxdy$

$= \int_{|y|\leq m}(\int_{m\leq|x|\leq m+\frac{1}{m}}(\sum_{i=0}^{13}L_{i,14}(|\frac{\partial^{i+14} f_m}{\partial z^i\partial x^{14}}|(x,y,m)+L_{i,14}(|\frac{\partial^{i+14} f_m}{\partial z^i\partial x^{14}}|(x,y,-m))dx)dy$

$= \int_{|y|\leq m}(\sum_{i=0}^{13}L_{i,14}|\frac{\partial^{i+13} f}{\partial z^i\partial x^{13}}|(m,y,m)+\sum_{i=0}^{13}L_{i,14}|\frac{\partial^{i+13} f}{\partial z^i\partial x^{13}}|(-m,y,m)$

$+\sum_{i=0}^{13}L_{i,14}|\frac{\partial^{i+13} f}{\partial z^i\partial x^{13}}|(m,y,m)+\sum_{i=0}^{13}L_{i,14}|\frac{\partial^{i+13} f}{\partial z^i\partial x^{13}}|(-m,y,-m))dy$

$\leq (2m)(4F)(\sum_{i=0}^{13}L_{i,14})$

$= 8mF(\sum_{i=0}^{13}L_{i,14})$

Case 12.

$\int_{m\leq|x|\leq m+\frac{1}{m},|y|\leq m,m\leq|z|\leq m+\frac{1}{m}}|\frac{\partial^{14}f_m}{\partial y^{14}}|dxdydz$

$=\int_{m\leq|x|\leq m+\frac{1}{m},|y|\leq m}(\int_{m\leq|z|\leq m+\frac{1}{m}}|\frac{\partial^{14}f_m}{\partial y^{14}}|dz)dxdy$

$\leq\int_{m\leq|x|\leq m+\frac{1}{m},|y|\leq m}(\sum_{i=0}^{13}L_{i,14}|\frac{\partial^{i+14}f_m}{\partial z^i\partial y^{14}}|(x,y,m)+L_{i,14}|\frac{\partial^{i+14}f_m}{\partial y^i\partial x^{14}}|(x,y,-m))dxdy$

$=\int_{|y|\leq m}(\int_{m\leq|x|\leq m+\frac{1}{m}}(\sum_{i=0}^{13}L_{i,14}(|\frac{\partial^{i+14}f_m}{\partial z^i\partial y^{14}}|(x,y,m)+L_{i,14}(|\frac{\partial^{i+14}f_m}{\partial z^i\partial y^{14}}|(x,y,-m))dx)dy$

$=\int_{|y|\leq m}(\sum_{i=0}^{13}\sum_{j=0}^{13}L_{i,14}L_{i,j,14}|\frac{\partial^{i+j+14}f}{\partial x^j\partial z^i\partial y^{14}}|(m,y,m)$

$+\sum_{i=0}^{13}\sum_{j=0}^{13}L_{i,14}L_{i,j,14}|\frac{\partial^{i+j+14}f}{\partial x^j\partial z^i\partial y^{14}}|(-m,y,m)$

$+\sum_{i=0}^{13}\sum_{j=0}^{13}L_{i,14}L_{i,j,14}|\frac{\partial^{i+j+14}f}{\partial x^j\partial z^i\partial y^{14}}|(m,y,-m)$

$+\sum_{i=0}^{13}\sum_{j=0}^{13}L_{i,14}L_{i,j,14}|\frac{\partial^{i+j+14}f}{\partial x^j\partial z^i\partial y^{14}}|(-m,y,-m))dy$

$\leq(2m)(4F)(\sum_{i=0}^{13}\sum_{j=0}^{13}L_{i,14}L_{i,j,14})$

$=8mF(\sum_{i=0}^{13}\sum_{j=0}^{13}L_{i,14}L_{i,j,14})$

Case 13.

$\int_{m\leq|x|\leq m+\frac{1}{m},|y|\leq m,m\leq|z|\leq m+\frac{1}{m}}|\frac{\partial^{14}f_m}{\partial z^{14}}|dxdydz$

$=\int_{m\leq|x|\leq m+\frac{1}{m},|y|\leq m}(\int_{m\leq|z|\leq m+\frac{1}{m}}|\frac{\partial^{14}f_m}{\partial z^{14}}|dz)dxdy$

$=\int_{m\leq|x|\leq m+\frac{1}{m},|y|\leq m}(|\frac{\partial^{13}f_m}{\partial z^{13}}|(x,y,m)+|\frac{\partial^{13}f_m}{\partial z^{13}}|(x,y,-m))dxdy$

$=\int_{|y|\leq m}(\int_{m\leq|x|\leq m+\frac{1}{m}}(|\frac{\partial^{13}f_m}{\partial z^{13}}|(x,y,m)+|\frac{\partial^{13}f_m}{\partial z^{13}}|(x,y,-m))dx)dy$

$\leq\int_{|y|\leq m}(\sum_{i=0}^{13}L_{i,13}|\frac{\partial^{i+13}f}{\partial x^i\partial z^{13}}|(m,y,m)+\sum_{i=0}^{13}L_{i,13}|\frac{\partial^{i+13}f}{\partial x^i\partial z^{13}}|(-m,y,m)$

$+\sum_{i=0}^{13}L_{i,13}|\frac{\partial^{i+13}f}{\partial x^i\partial z^{13}}|(m,y,-m)+\sum_{i=0}^{13}L_{i,13}|\frac{\partial^{i+13}f}{\partial x^i\partial z^{13}}|(-m,y,-m))$

$\leq(2m)(4F)(\sum_{i=0}^{13}L_{i,13})$

$=8mF(\sum_{i=0}^{13}L_{i,13})$

Cases 14-16 are similar to cases 11-13, interchanging the orders of integration, with case 14 corresponding to case 12, case 15 corresponding to case 11 and case 16 corresponding to case 13, so that;

Case 14.

$\int_{|x|\leq m, m\leq|y|\leq m+\frac{1}{m}, m\leq|z|\leq m+\frac{1}{m}}|\frac{\partial^{14} f_m}{\partial x^{14}}|dxdydz$

$\leq 8mF(\sum_{i=0}^{13}\sum_{j=0}^{13}L_{i,14}L_{i,j,14})$

Case 15.

$\int_{|x|\leq m, m\leq|y|\leq m+\frac{1}{m}, m\leq|z|\leq m+\frac{1}{m}}|\frac{\partial^{14} f_m}{\partial y^{14}}|dxdydz$

$\leq 8mF(\sum_{i=0}^{13}L_{i,14})$

Case 16.

$\int_{|x|\leq m, m\leq|y|\leq m+\frac{1}{m}, m\leq|z|\leq m+\frac{1}{m}}|\frac{\partial^{14} f_m}{\partial z^{14}}|dxdydz$

$\leq 8mF(\sum_{i=0}^{13}L_{i,13})$

Case 17.

$\int_{m\leq|x|\leq m+\frac{1}{m}, m\leq|y|\leq m+\frac{1}{m}, m\leq|z|\leq m+\frac{1}{m}}|\frac{\partial^{14} f_m}{\partial x^{14}}|dxdydz$

$= \int_{m\leq|x|\leq m+\frac{1}{m}, m\leq|y|\leq m+\frac{1}{m}}(\int_{m\leq|z|\leq m+\frac{1}{m}}|\frac{\partial^{14} f_m}{\partial x^{14}}|dz)dxdy$

$\leq \frac{1}{m}\int_{m\leq|x|\leq m+\frac{1}{m}, m\leq|y|\leq m+\frac{1}{m}}(\sum_{i=0}^{13}L_{i,14}|\frac{\partial^{i+14} f_m}{\partial z^i\partial x^{14}}|(x,y,m)+\sum_{i=0}^{13}L_{i,14}|\frac{\partial^{i+14} f_m}{\partial z^i\partial x^{14}}|(x,y,-m))dxdy$

$= \frac{1}{m}\int_{m\leq|x|\leq m+\frac{1}{m}}(\int_{m\leq|y|\leq m+\frac{1}{m}}(\sum_{i=0}^{13}L_{i,14}(|\frac{\partial^{i+14} f_m}{\partial z^i\partial x^{14}}|(x,y,m)$

$+\sum_{i=0}^{13}L_{i,14}(|\frac{\partial^{i+14} f_m}{\partial z^i\partial x^{14}}|(x,y,-m))dy)dx$

$\leq \frac{1}{m^2}\int_{m\leq|x|\leq m+\frac{1}{m}}(\sum_{j=0}^{13}\sum_{i=0}^{13}L_{i,14}L_{i,j,14}|\frac{\partial^{i+j+14} f_m}{\partial y^j\partial z^i\partial x^{14}}|(x,m,m)$

$+\sum_{j=0}^{13}\sum_{i=0}^{13}L_{i,14}L_{i,j,14}|\frac{\partial^{i+j+14} f_m}{\partial y^j\partial z^i\partial x^{14}}|(x,-m,m)$

$+\sum_{j=0}^{13}\sum_{i=0}^{13}L_{i,14}L_{i,j,14}|\frac{\partial^{i+j+14} f_m}{\partial y^j\partial z^i\partial x^{14}}|(x,m,-m)$

$+\sum_{j=0}^{13}\sum_{i=0}^{13}L_{i,14}L_{i,j,14}|\frac{\partial^{i+j+14} f_m}{\partial y^j\partial z^i\partial x^{14}}|(x,-m,-m))dx$

$= \frac{1}{m^2}(\sum_{j=0}^{13}\sum_{i=0}^{13} L_{i,14}L_{i,j,14}|\frac{\partial^{i+j+13}f}{\partial y^j\partial z^i\partial x^{13}}|(m,m,m)+\sum_{j=0}^{13}\sum_{i=0}^{13} L_{i,14}L_{i,j,14}|\frac{\partial^{i+j+13}f}{\partial y^j\partial z^i\partial x^{13}}|(-m,m,m)$

$+\sum_{j=0}^{13}\sum_{i=0}^{13} L_{i,14}L_{i,j,14}|\frac{\partial^{i+j+13}f}{\partial y^j\partial z^i\partial x^{13}}|(m,-m,m)+\sum_{j=0}^{13}\sum_{i=0}^{13} L_{i,14}L_{i,j,14}|\frac{\partial^{i+j+13}f}{\partial y^j\partial z^i\partial x^{13}}|(-m,-m,m)$

$+\sum_{j=0}^{13}\sum_{i=0}^{13} L_{i,14}L_{i,j,14}|\frac{\partial^{i+j+13}f}{\partial y^j\partial z^i\partial x^{13}}|(m,m,-m)+\sum_{j=0}^{13}\sum_{i=0}^{13} L_{i,14}L_{i,j,14}|\frac{\partial^{i+j+13}f}{\partial y^j\partial z^i\partial x^{13}}|(-m,m,-m)$

$+\sum_{j=0}^{13}\sum_{i=0}^{13} L_{i,14}L_{i,j,14}|\frac{\partial^{i+j+13}f}{\partial y^j\partial z^i\partial x^{13}}|(m,-m,-m)$

$+\sum_{j=0}^{13}\sum_{i=0}^{13} L_{i,14}L_{i,j,14}|\frac{\partial^{i+j+13}f}{\partial y^j\partial z^i\partial x^{13}}|(-m,-m,-m))$

$\leq \frac{8F}{m^2}(\sum_{j=0}^{13}\sum_{i=0}^{13} L_{i,14}L_{i,j,14})$

Case 18.

$\int_{m\leq|x|\leq m+\frac{1}{m},m\leq|y|\leq m+\frac{1}{m},m\leq|z|\leq m+\frac{1}{m}}|\frac{\partial^{14}f_m}{\partial y^{14}}|dxdydz$

$=\int_{m\leq|x|\leq m+\frac{1}{m},m\leq|y|\leq m+\frac{1}{m}}(\int_{m\leq|z|\leq m+\frac{1}{m}}|\frac{\partial^{14}f_m}{\partial y^{14}}|dz)dxdy$

$\leq \frac{1}{m}\int_{m\leq|x|\leq m+\frac{1}{m},m\leq|y|\leq m+\frac{1}{m}}(\sum_{i=0}^{13} L_{i,14}|\frac{\partial^{i+14}f_m}{\partial z^i\partial y^{14}}|(x,y,m)+\sum_{i=0}^{13} L_{i,14}|\frac{\partial^{i+14}f_m}{\partial z^i\partial y^{14}}|(x,y,-m))dxdy$

$=\frac{1}{m}\int_{|x|\leq m+\frac{1}{m}}(\int_{m\leq|y|\leq m+\frac{1}{m}}(\sum_{i=0}^{13} L_{i,14}(|\frac{\partial^{i+14}f_m}{\partial z^i\partial y^{14}}|(x,y,m)+\sum_{i=0}^{13} L_{i,14}(|\frac{\partial^{i+14}f_m}{\partial z^i\partial y^{14}}|(x,y,-m))dy)dx$

$=\frac{1}{m}\int_{m\leq|x|\leq m+\frac{1}{m}}(\sum_{i=0}^{13} L_{i,14}|\frac{\partial^{i+13}f_m}{\partial z^i\partial y^{13}}|(x,m,m)$

$+\sum_{i=0}^{13} L_{i,14}|\frac{\partial^{i+13}f_m}{\partial z^i\partial y^{13}}|(x,-m,m)$

$+\sum_{i=0}^{13} L_{i,14}|\frac{\partial^{i+13}f_m}{\partial z^i\partial y^{13}}|(x,m,-m)$

$+\sum_{i=0}^{13} L_{i,14}|\frac{\partial^{i+13}f_m}{\partial z^i\partial y^{13}}|(x,-m,-m))dx$

$\leq \frac{1}{m^2}(\sum_{j=0}^{13}\sum_{i=0}^{13} L_{i,14}L_{i,j,13}|\frac{\partial^{i+j+13}f}{\partial x^j\partial z^i\partial y^{13}}|(m,m,m)+\sum_{j=0}^{13}\sum_{i=0}^{13} L_{i,14}L_{i,j,13}|\frac{\partial^{i+j+13}f}{\partial x^j\partial z^i\partial y^{13}}|(-m,m,m)$

$+\sum_{j=0}^{13}\sum_{i=0}^{13} L_{i,14}L_{i,j,13}|\frac{\partial^{i+j+13}f}{\partial x^j\partial z^i\partial y^{13}}|(m,-m,m)+\sum_{j=0}^{13}\sum_{i=0}^{13} L_{i,14}L_{i,j,13}|\frac{\partial^{i+j+13}f}{\partial x^j\partial z^i\partial y^{13}}|(-m,-m,m)$

$+\sum_{j=0}^{13}\sum_{i=0}^{13} L_{i,14}L_{i,j,13}|\frac{\partial^{i+j+13}f}{\partial x^j\partial z^i\partial y^{13}}|(m,m,-m)+\sum_{j=0}^{13}\sum_{i=0}^{13} L_{i,14}L_{i,j,13}|\frac{\partial^{i+j+13}f}{\partial x^j\partial z^i\partial y^{13}}|(-m,m,-m)$

$+\sum_{j=0}^{13}\sum_{i=0}^{13} L_{i,14}L_{i,j,13}|\frac{\partial^{i+j+13}f}{\partial x^j\partial z^i\partial y^{13}}|(m,-m,-m)$

$+\sum_{j=0}^{13}\sum_{i=0}^{13} L_{i,14}L_{i,j,13}|\frac{\partial^{i+j+13}f}{\partial x^j\partial z^i\partial y^{13}}|(-m,-m,-m))$

$\leq \frac{8F}{m^2}(\sum_{j=0}^{13}\sum_{i=0}^{13} L_{i,14}L_{i,j,13})$

Case 19.

$\int_{m\leq|x|\leq m+\frac{1}{m}, m\leq|y|\leq m+\frac{1}{m}, m\leq|z|\leq m+\frac{1}{m}} |\frac{\partial^{14} f_m}{\partial z^{14}}|dxdydz$

$= \int_{m\leq|x|\leq m+\frac{1}{m}, m\leq|y|\leq m+\frac{1}{m}}(\int_{m\leq|z|\leq m+\frac{1}{m}} |\frac{\partial^{14} f_m}{\partial z^{14}}|dz)dxdy$

$= \int_{m\leq|x|\leq m+\frac{1}{m}, m\leq|y|\leq m+\frac{1}{m}}(|\frac{\partial^{13} f_m}{\partial z^{13}}|(x,y,m) + |\frac{\partial^{13} f_m}{\partial z^{13}}|(x,y,-m))dxdy$

$= \int_{m\leq|x|\leq m+\frac{1}{m}}(\int_{m\leq|y|\leq m+\frac{1}{m}}((|\frac{\partial^{13} f_m}{\partial z^{13}}|(x,y,m)+|\frac{\partial^{13} f_m}{\partial z^{13}}|(x,y,-m))dy)dx$

$\leq \frac{1}{m}\int_{|x|\leq m+\frac{1}{m}}(\sum_{i=0}^{13} L_{i,13}|\frac{\partial^{i+13} f_m}{\partial y^i \partial z^{13}}|(x,m,m)$

$+\sum_{i=0}^{13} L_{i,13}|\frac{\partial^{i+13} f_m}{\partial y^i \partial z^{13}}|(x,-m,m)$

$+\sum_{i=0}^{13} L_{i,13}|\frac{\partial^{i+13} f_m}{\partial y^i \partial z^{13}}|(x,m,-m)$

$+\sum_{i=0}^{13} L_{i,13}|\frac{\partial^{i+13} f_m}{\partial y^i \partial z^{13}}|(x,-m,-m))dx$

$\leq \frac{1}{m^2}(\sum_{j=0}^{13}\sum_{i=0}^{13} L_{i,13}L_{i,j,13}|\frac{\partial^{i+j+13} f}{\partial x^j \partial y^i \partial z^{13}}|(m,m,m)+\sum_{j=0}^{13}\sum_{i=0}^{13} L_{i,13}L_{i,j,13}|\frac{\partial^{i+j+13} f}{\partial x^j \partial y^i \partial z^{13}}|(-m,m,m)$

$+\sum_{j=0}^{13}\sum_{i=0}^{13} L_{i,13}L_{i,j,13}|\frac{\partial^{i+j+13} f}{\partial x^j \partial y^i \partial z^{13}}|(m,-m,m)+\sum_{j=0}^{13}\sum_{i=0}^{13} L_{i,13}L_{i,j,13}|\frac{\partial^{i+j+13} f}{\partial x^j \partial y^i \partial z^{13}}|(-m,-m,m)$

$+\sum_{j=0}^{13}\sum_{i=0}^{13} L_{i,13}L_{i,j,13}|\frac{\partial^{i+j+13} f}{\partial x^j \partial y^i \partial z^{13}}|(m,m,-m)+\sum_{j=0}^{13}\sum_{i=0}^{13} L_{i,13}L_{i,j,13}|\frac{\partial^{i+j+13} f}{\partial x^j \partial y^i \partial z^{13}}|(-m,m,-m)$

$+\sum_{j=0}^{13}\sum_{i=0}^{13} L_{i,13}L_{i,j,13}|\frac{\partial^{i+j+13} f}{\partial x^j \partial y^i \partial z^{13}}|(m,-m,-m)$

$+\sum_{j=0}^{13}\sum_{i=0}^{13} L_{i,13}L_{i,j,13}|\frac{\partial^{i+j+13} f}{\partial x^j \partial y^i \partial z^{13}}|(-m,-m,-m))$

$\leq \frac{8F}{m^2}(\sum_{j=0}^{13}\sum_{i=0}^{13} L_{i,13}L_{i,j,13})$

It is then clear from $(*)$, summing the bounds from the individual cases 1-19, as at the end of the proof of Lemma 0.27, that there exists a constant $G \in \mathcal{R}_{>0}$ with;

$max(\int_{\mathcal{R}^3} |\frac{\partial f_m}{\partial x^{14}}|dxdydz, \int_{\mathcal{R}^3} |\frac{\partial f_m}{\partial y^{14}}|dxdydz, \int_{\mathcal{R}^3} |\frac{\partial f_m}{\partial z^{14}}|dxdydz) \leq Gm^3$

for sufficiently large m.

□

Lemma 0.29. *Let $\{f_m : m \in \mathcal{N}\}$ be an inflexionary sequence, then for $\overline{k} \neq 0$, sufficiently large m, we have that there exists $D \in \mathcal{R}_{>0}$, with;*

$|\mathcal{F}(f_m)(\overline{k})| \leq \frac{Dm^3}{|\overline{k}|^{14}}$

Moreover, for sufficiently large m, $\mathcal{F}(f_m) \in L^1(\mathcal{R}^3)$.

Proof. For $(k_1, k_2, k_3) \in \mathcal{R}^3$, using repeated integration by parts, and the fact that $f_m \in L1(\mathcal{R}^3)$, we have, for $m \in \mathcal{N}$;

$\mathcal{F}(\frac{\partial^{14} f_m}{\partial x^{14}} + \frac{\partial^{14} g}{\partial y^{14}} + \frac{\partial^{14} g}{\partial z^{14}})(\overline{k})$

$= \frac{1}{(2\pi)^{\frac{3}{2}}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (\frac{\partial^{14} f_m}{\partial x^{14}} + \frac{\partial^{14} f_m}{\partial y^{14}} + \frac{\partial^{14} f_m}{\partial z^{14}}) e^{-ik_1 x} e^{-ik_2 y} e^{-ik_3 z} dxdydz$

$= ((ik_1)^{14} + (ik_2)^{14} + (ik_3)^{14}) \frac{1}{(2\pi)^{\frac{3}{2}}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_m(x, y, z) e^{-ik_1 x} e^{-ik_2 y} e^{-ik_3 z} dxdydz$

$= (-k_1^{14} - k_2^{14} - k_3^{14}) \mathcal{F}(f_m)(\overline{k})$

so that, for $\overline{k} \neq \overline{0}$;

$|\mathcal{F}(f_m)(\overline{k})| \leq \frac{|\mathcal{F}(\frac{\partial^{14} f_m}{\partial x^{14}} + \frac{\partial^{14} g}{\partial y^{14}} + \frac{\partial^{14} g}{\partial z^{14}})(\overline{k})|}{(k_1^{14} + k_2^{14} + k_3^{14})}$ $(\dagger)$

We have, using the result of Lemma 0.28, for sufficiently large m, that;

$|\mathcal{F}(\frac{\partial^{14} f_m}{\partial x^{14}} + \frac{\partial^{14} g}{\partial y^{14}} + \frac{\partial^{14} g}{\partial z^{14}})(\overline{k})|$

$\frac{1}{(2\pi)^{\frac{3}{2}}} | \int_{\mathcal{R}^3} (\frac{\partial^{14} f_m}{\partial x^{14}} + \frac{\partial^{14} f_m}{\partial y^{14}} + \frac{\partial^{14} f_m}{\partial z^{14}}) e^{-ik_1 x} e^{-ik_2 y} e^{-ik_3 z} dxdydz|$

$\leq \frac{1}{(2\pi)^{\frac{3}{2}}} \int_{\mathcal{R}^3} (|\frac{\partial f_m}{\partial x^{14}}| + |\frac{\partial f_m}{\partial y^{14}}| + |\frac{\partial f_m}{\partial z^{14}}|) dxdydz$

$\leq \frac{3G}{(2\pi)^{\frac{3}{2}}} m^3$ $(\dagger\dagger)$

so that, combining $(\dagger)$ and $(\dagger\dagger)$, we have, for $\overline{k} \neq \overline{0}$, sufficiently large m;

$|\mathcal{F}(f_m)(\overline{k})| \leq \frac{3G}{(2\pi)^{\frac{3}{2}}} \frac{m^3}{(k_1^{14} + k_2^{14} + k_3^{14})}$ $(*)$

Using polar coordinates $k_1 = rsin(\theta)cos(\phi)$, $k_2 = rsin(\theta)sin(\phi)$, $k_3 = rcos(\theta)$, $0 \leq \theta \leq \pi$, $-\pi < \phi \leq \pi$, we have that;

$\frac{1}{(k_1^{14}+k_2^{14}+k_3^{14})} = \frac{1}{r^{14}}\frac{1}{\alpha(\theta,\phi)}$

where $\alpha(\theta,\phi) = sin^{14}(\theta)(cos^{14}(\phi) + sin^{14}(\phi)) + cos^{14}(\theta)$

We have that, in the range $0 \leq \theta \leq \pi$, $-\pi \leq \phi \leq \pi$, with $\theta \neq \frac{\pi}{2}$, $|\phi| \neq \frac{\pi}{2}$;

$\alpha(\theta,\phi) = 0$

iff $tan^{14}(\theta)(1 + tan^{14}(\phi)) + \frac{1}{cos^{14}(\phi)} = 0$

iff $tan^{14}(\theta)(1 + tan^{14}(\phi)) = -\frac{1}{cos^{14}(\phi)}$

which has no solution, as the two sides of the equation have opposite signs.

and, with $\theta = \frac{\pi}{2}$, , $|\phi| \neq \frac{\pi}{2}$

$\alpha(\theta,\phi) = 0$

iff $cos^{14}(\phi) + sin^{14}(\phi) = 0$

iff $tan^{14}(\phi) = -1$

which has no solution, as the two sides of the equation have opposite signs.

and, with $\theta \neq \frac{\pi}{2}$, , $|\phi| = \frac{\pi}{2}$

$\alpha(\theta,\phi) = 0$

iff $cos^{14}(\theta) + sin^{14}(\theta) = 0$

iff $tan^{14}(\theta) = -1$

which has no solution, as the two sides of the equation have opposite signs.

and, with $\theta = \frac{\pi}{2}$, , $|\phi| = \frac{\pi}{2}$

$\alpha(\theta,\phi)=0$

iff $1=0$

which is not the case. It follows that $\alpha(\theta,\phi)=0$ has no solution in the range $0\leq\theta\leq\pi$, $-\pi\leq\phi\leq\pi$. By continuity, compactness of $[0\,\pi]\times[-\pi,\pi]$ and the fact that $\alpha(\frac{\pi}{2},\frac{\pi}{2})=1$, restricting the interval $[-\pi,\pi]$, there exists $\epsilon>0$, with $\alpha(\theta,\phi)\geq\epsilon$, for $0\leq\theta\leq\pi$, $-\pi<\phi\leq\pi$. In particularly;

$\frac{1}{(k_1^{14}+k_2^{14}+k_3^{14})}\leq\frac{1}{\epsilon r^{14}}$

$=\frac{1}{\epsilon|\overline{k}|^{14}}$

so that, from $(*)$;

$|\mathcal{F}(f_m)(\overline{k})|\leq\frac{3G}{(2\pi)^{\frac{3}{2}}}\frac{m^3}{\epsilon|\overline{k}|^{14}}$

$=\frac{Dm^3}{|\overline{k}|^{14}}$

where $D=\frac{3G}{\epsilon(2\pi)^{\frac{3}{2}}}$

For the final claim, we have, for $1\leq i\leq 3$, $m\in\mathcal{N}$, as f_m is supported on $W_{m+\frac{1}{m}}$ and continuous, that $x_if_m\in L^1(\mathcal{R}^3)$ and, differentiating under the integral sign;

$|\frac{\partial\mathcal{F}(f_m)(\overline{k})}{\partial k^i}|=|\frac{\partial}{\partial k^i}(\frac{1}{(2\pi)^{\frac{3}{2}}}\int_{\mathcal{R}^3}f_m(\overline{x})e^{-i\overline{k}.\overline{x}}d\overline{x})|$

$=|\frac{-i}{(2\pi)^{\frac{3}{2}}}\int_{\mathcal{R}^3}x_if_m(\overline{x})e^{-i\overline{k}.\overline{x}}d\overline{x})|$

$\leq\frac{1}{(2\pi)^{\frac{3}{2}}}\int_{\mathcal{R}^3}|x_if_m(\overline{x})|d\overline{x}$

$=\frac{1}{(2\pi)^{\frac{3}{2}}}||x_if_m(\overline{x})||_1$

so that $\frac{\partial\mathcal{F}(f_m)(\overline{k})}{\partial k^i}$ is bounded, and, in particularly, $\mathcal{F}(f_m)$ is continuous, for $m\in\mathcal{N}$. It follows, using the first result, and polar coordinates, that, for $n>1$, sufficiently large m;

$|\int_{\mathcal{R}^3}\mathcal{F}(f_m)(\overline{k})d\overline{k}|\leq\int_{B(\overline{0},n)}|\mathcal{F}(f_m)(\overline{k})|d\overline{k}+\int_{\mathcal{R}^3\setminus B(\overline{0},n)}|\mathcal{F}(f_m)(\overline{k})|d\overline{k}$

$\leq \frac{4C_n\pi^3}{3} + \int_{\mathcal{R}^3 \setminus B(\overline{0},n)} \frac{Dm^3}{|\overline{k}|^{14}}$

$\leq \frac{4C_n\pi^3}{3} + \int_0^\pi \int_{-\pi}^\pi \int_n^\infty \frac{Dm^3}{r^{14}} |r^2 sin(\theta)| dr d\theta d\phi$

$\leq \frac{4C_n\pi^3}{3} + 2D\pi^2 m^3 \int_n^\infty \frac{dr}{r^{12}}$

$\leq \frac{4C_n\pi^3}{3} + 2D\pi^2 m^3 [\frac{-1}{11r^{11}}]_n^\infty$

$= \frac{4C_n\pi^3}{3} + \frac{2D\pi^2 m^3}{11n^{11}}$

where $C_n = ||\mathcal{F}(f_m)|_{B(\overline{0},n)}||_\infty$, so that $\mathcal{F}(f_m) \in L^1(\mathcal{R}^3)$.

□

Lemma 0.30. *Let $f \in C^{40}(\mathcal{R}^3)$, with $\frac{\partial^{i_1+i_2+i_3}}{\partial x^{i_1} \partial y^{i_2} \partial z^{i_3}}$ bounded for $0 \leq i_1 + i_2 + i_3 \leq 40$, f analytic for $|\overline{x}| > r$, where $r \in \mathcal{R}_{>0}$, and f of very moderate decrease. Then;*

$f(\overline{x}) = \mathcal{F}^{-1}(\mathcal{F}(f))(\overline{x})$, $(\overline{x} \in \mathcal{R}^3)$

where, for $g \in L^1(\mathcal{R}^3)$;

$\mathcal{F}^{-1}(g)(\overline{x}) = \frac{1}{(2\pi)^{\frac{3}{2}}} \int_{\mathcal{R}^3} g(\overline{k}) e^{i\overline{k}.\overline{x}} d\overline{k}$

Proof. By Lemma 0.20, we have that $\mathcal{F}(f) \in L^1(\mathcal{R})$. Let $\{f_m : m \in \mathcal{N}\}$ be the approximating sequence, given by Lemma 0.27, then, for sufficiently large m, $f_m \in L^1(\mathcal{R})$ and $\mathcal{F}(f_m) \in L^1(\mathcal{R})$ by Lemma 0.29. It follows, see [4] or the method of [10], that for such m, $f_m = \mathcal{F}^{-1}(\mathcal{F}(f_m))$, $(***)$, By the proof of Lemma 0.19, we have that, for $\overline{k}$ with $min(|k_1|, |k_2|, |k_3|) > \epsilon > 0$, $|\mathcal{F}(f)(k) - \mathcal{F}(f_m)(k)| \leq \frac{E_\epsilon}{m}$, (B). By the proof of Lemma 0.20, we have that $\mathcal{F}(f) - \mathcal{F}(f_m) \in L^{\frac{4}{3}}(\mathcal{R}^3)$, with $||\mathcal{F}(f) - \mathcal{F}(f_m)||_{L^{\frac{4}{3}}(\mathcal{R}^3)} \rightarrow 0$ as $m \rightarrow \infty$. In particularly, there exists a constant $H \in \mathcal{R}_{>0}$ with $||\mathcal{F}(f) - \mathcal{F}(f_m)||_{L^{\frac{4}{3}}(\mathcal{R}^3)} \leq H$, for sufficiently large m. We then have, using the Holder's inequality that, for $\epsilon > 0$, m sufficiently large;

$||\mathcal{F}(f) - \mathcal{F}(f_m)||_{L^1(W_\epsilon)}$

$= ||(\mathcal{F}(f) - \mathcal{F}(f_m))|_{W_\epsilon} 1_{W_\epsilon}||_{L^1(W_\epsilon)}$

$\leq ||(\mathcal{F}(f)-\mathcal{F}(f_m))|_{W_\epsilon}||_{L^{\frac{4}{3}}(W_\epsilon)}||1_{W_\epsilon}||_{L^4(W_\epsilon)}$

$\leq H||1_{W_\epsilon}||_{L^4(W_\epsilon)}$

$= 8H\epsilon^3$

Letting $W_{i,\epsilon}=\{\overline{k}\in\mathcal{R}^3:|k_i|<\epsilon\}$, $1\leq i\leq 3$, and $V_\epsilon=\bigcup_{1\leq i\leq 3}W_{i,\epsilon}$, we have that;

$\mathcal{R}^3\setminus V_\epsilon=\{\overline{k}\in\mathcal{R}^3:min(|k_1|,|k_2|,|k_3|)>\epsilon\}$

Using the notation of Lemma 0.20, we have that $W_{1,\epsilon}=W_\epsilon\cup V_{1,\epsilon}\cup V_{12,\epsilon}\cup V_{13,\epsilon}$, with ϵ replacing the parameters $\{E_1,E_2,E_3\}$. Using the method of Lemma 0.20, we can show that;

$\theta_m(x,y)=\int_{|k_3|\geq\epsilon}\mathcal{F}(f-f_m)(x,y,k_3)dk_3$

is non oscillatory and of very moderate decrease, with;

$\int_{V_{12,\epsilon}}\mathcal{F}(f-f_m)(k_1,k_2,k_3)dk_1dk_2dk_3$

$\int_{|k_1|<\epsilon,|k_2|<\epsilon,|k_3|\geq\epsilon}\mathcal{F}(f-f_m)(k_1,k_2,k_3)dk_1dk_2dk_3$

$=\int_{|k_1<\epsilon,|k_2|<\epsilon}\mathcal{F}(\theta_m)dk_1dk_2$

where $\mathcal{F}$ is the fourier transform for non oscillatory functions of very moderate decrease in 2 variables. As $\theta_m\in L^3(\mathcal{R}^2)$, $\mathcal{F}(\theta_m)\in L^{\frac{3}{2}}(\mathcal{R})$ by the Haussdorff-Young inequality, so that, by Holder's inequality;

$|\int_{V_{12,\epsilon}}\mathcal{F}(f-f_m)(k_1,k_2,k_3)dk_1dk_2dk_3|$

$\leq ||\mathcal{F}(\theta_m)||_{L^1(|k_1<\epsilon,|k_2|<\epsilon)}$

$\leq 4\epsilon^2||\mathcal{F}(\theta_m)||_{L^{\frac{3}{2}}(|k_1<\epsilon,|k_2|<\epsilon)}$

$\leq 4\epsilon^2||\theta_m||_{L^3(\mathcal{R}^2)}$

$\leq 4C_{12}D_{12}\epsilon^2$

$=E_{12}\epsilon^2$

where $C_{12} \in \mathcal{R}_{>0}$ is a uniform bound for $||\theta_m||_{L^3(\mathcal{R}^2)}$, D_{12} is the functional bound in the Haussdorff-Young inequality.

Similarly, we can show that;

$||\mathcal{F}(f) - \mathcal{F}(f_m)||_{L^1(V_{13,\epsilon})} \leq E_{13}\epsilon^2$

$||\mathcal{F}(f) - \mathcal{F}(f_m)||_{L^1(V_{1,\epsilon})} \leq E_1\epsilon$

so that;

$||\mathcal{F}(f) - \mathcal{F}(f_m)||_{L^1(W_{1,\epsilon})} \leq F_1\epsilon$, $(0 < \epsilon < 1)$

and, similarly;

$||\mathcal{F}(f) - \mathcal{F}(f_m)||_{L^1(W_{i,\epsilon})} \leq F_i\epsilon$, $(0 < \epsilon < 1)$

$||\mathcal{F}(f) - \mathcal{F}(f_m)||_{L^1(V_\epsilon)} \leq (F_1 + F_2 + F_3)\epsilon = F\epsilon$ $(0 < \epsilon < 1)$ (A)

Using the fact from Lemma 0.20, that $\mathcal{F}(f) \in L^1(\mathcal{R})$, for $\delta > 0$ arbitrary, we have that;

$\int_{\mathcal{R}^3 \setminus B(\overline{0},n)} |\mathcal{F}(f)(\overline{k}|d\overline{k} < \delta$

for $n \in \mathcal{N}$, sufficiently large, $n \geq n_0$. Choosing $m \in \mathcal{N}$, with $m = [n^{\frac{10}{3}}]$, and using $(A), (B)$, Lemma 0.29we have, for $\overline{x} \in \mathcal{R}^3$, that;

$|\mathcal{F}^{-1}(\mathcal{F}(f))(\overline{x}) - \mathcal{F}^{-1}(\mathcal{F}(f_m))(\overline{x})| = |\mathcal{F}^{-1}(\mathcal{F}(f)(\overline{k}) - \mathcal{F}(f_m)(\overline{k}))|$

$= \frac{1}{(2\pi)^{\frac{3}{2}}}|\int_{B(\overline{0},n)}(\mathcal{F}(f)(\overline{k}) - \mathcal{F}(f_m)(\overline{k}))e^{i\overline{k}\centerdot\overline{x}}d\overline{k}$

$+ \int_{\mathcal{R}^3 \setminus B(\overline{0},n)}(\mathcal{F}(f)(\overline{k}) - \mathcal{F}(f_m)(\overline{k}))e^{i\overline{k}\centerdot\overline{x}}d\overline{k}|$

$\leq \frac{1}{(2\pi)^{\frac{3}{2}}}(\int_{B(\overline{0},n)} |\mathcal{F}(f)(\overline{k}) - \mathcal{F}(f_m)(\overline{k})|d\overline{k}$

$+ \int_{\mathcal{R}^3 \setminus B(\overline{0},n)} |\mathcal{F}(f)(\overline{k})|d\overline{k} + \int_{\mathcal{R}^3 \setminus B(\overline{0},n)} |\mathcal{F}(f_m)(\overline{k})|d\overline{k})$

$\leq \frac{1}{(2\pi)^{\frac{3}{2}}}(\int_{V_\epsilon \cap B(\overline{0},n)} |\mathcal{F}(f)(\overline{k}) - \mathcal{F}(f_m)(\overline{k})|d\overline{k} + \frac{4\pi n^3 E_\epsilon}{3m} + \delta + \int_{\mathcal{R}^3 \setminus B(\overline{0},n)} \frac{Dm^3}{|k|^{14}}d\overline{k})$

$\leq \frac{1}{(2\pi)^{\frac{3}{2}}}(\int_{V_\epsilon} |\mathcal{F}(f)(\overline{k}) - \mathcal{F}(f_m)(\overline{k})|d\overline{k} + \frac{4\pi n^3 E_\epsilon}{3m} + \delta + \int_{\mathcal{R}^3 \setminus B(\overline{0},n)} \frac{Dm^3}{|k|^{14}}d\overline{k})$

$\leq \frac{1}{(2\pi)^{\frac{3}{2}}}(F\epsilon+\frac{4\pi n^3 E_\epsilon}{3(n^{\frac{10}{3}}-1)}+\delta+\int_{\mathcal{R}^3\setminus B(\overline{0},n)}\frac{Dn^{10}}{|k|^{14}}d\overline{k})$

$\leq \frac{1}{(2\pi)^{\frac{3}{2}}}(F\epsilon+\frac{4\pi E_\epsilon}{3n^{\frac{1}{3}}}+\delta+2\pi^2\int_{r>n}\frac{Dn^{10}}{r^{14}}dr)$

$= \frac{1}{(2\pi)^{\frac{3}{2}}}(F\epsilon+\frac{4\pi E_\epsilon}{3n^{\frac{1}{3}}}+\delta+2D\pi^2 n^{10}[\frac{-1}{13r^{13}}]_n^\infty)$

$= \frac{1}{(2\pi)^{\frac{3}{2}}}(F\epsilon+\frac{4\pi E_\epsilon}{3n^{\frac{1}{3}}}+\delta+\frac{2D\pi^2}{13n^3})$

$< \frac{2\delta+F\epsilon}{(2\pi)^{\frac{3}{2}}}$

for sufficiently large $n \geq n_0$, so that, as $\epsilon > 0$ and $\delta > 0$ were arbitrary, for $\overline{x} \in \mathcal{R}^3$;

$lim_{m\rightarrow\infty}\mathcal{F}^{-1}(\mathcal{F}(f_m))(\overline{x}) = \mathcal{F}^{-1}\mathcal{F}(f)(\overline{x})$, $(****)$

and, by Definition 0.22, $(***)$, $(****)$;

$f(\overline{x}) = lim_{m\rightarrow\infty}f_m(\overline{x}) = lim_{m\rightarrow\infty}\mathcal{F}^{-1}(\mathcal{F}(f_m))(\overline{x}) = \mathcal{F}^{-1}\mathcal{F}(f)(\overline{x})$

□

References

[1] An Introduction to Electrodynamics, Third Edition, Pearson Education, (2008)

[2] On convergence and growth of partial sums of Fourier series, L. Carlsen, Acta Mathematica, 116 (1): 135157, (1966).

[3] Partial Differential Equations, L. C. Evans, Graduate Studies in Mathematics, Volume 19, AMS, (1998).

[4] Fourier Analysis and its Applications, Folland, G. B., Wadsworth, (1992).

[5] Fourier Analysis, T. Korner, CUP, (1988)

[6] Maths Stack Exchange; https://math.stackexchange.com/questions/291877/bounding-the-solution-of-a-wave-equation-in-3-dimensions

[7] Simple Proofs of the Riemann-Lebesgue Lemma using Nonstandard Analysis, Tristram de Piro, available at http://www.curvalinea.net (51), (2019)

[8] Some Arguments for the Wave Equation in Quantum Theory 2, Tristram de Piro, Open Journal of Mathematical Sciences, available at

http://www.curvalinea.net (59), (2022)

[9] Some Arguments for the Wave Equation in Quantum Theory 3, Tristram de Piro, submitted to the Open Journal of Mathematical Sciences, available at http://www.curvalinea.net (63), (2022)

[10] A Simple Proof of the Fourier Inversion Theorem Using Nonstandard Analysis, Tristram de Piro, available at http://www.curvalinea.net (19), (2013)

[11] Some Arguments for the Wave Equation in Quantum Theory, Tristram de Piro, Open Journal of Mathematical Sciences, available at http://www.curvalinea.net (58), (2021)

[12] The Continuity Equation and Particle Motion, Tristram de Piro, in progress, (2022)

[13] College Calculus with Analytic Geometry, Protter, Murray H.; Morrey, Charles B. Jr, (2nd ed.), Reading: Addison-Wesley, (1970)

[14] Fourier Analysis, An Introduction, E. Stein and R. Sakarchi, Princeton Lectures in Mathematics, (2003).

[15] Introfuction to Fourier Analysis on Euclidean Spaces, E. Stein and G. Weiss, Princeton University Press, (1971).

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