SIMPLE PROOFS OF THE RIEMANN-LEBESGUE LEMMAS USING NONSTANDARD ANALYSIS

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ABSTRACT. We give simple proofs of the Riemann-Lebesgue Lemmas for Fourier series and Fourier transforms using nonstandard analysis.

Definition 0.1. *We let $\eta \in {}^{*}\mathcal{N}$ be infinite, and $\overline{V_\eta} = {}^{*}[-1,1)$, with the $*$-algebra $\mathfrak{B}_\eta$ generated by;*

$\{[\frac{i}{\eta}, \frac{i+1}{\eta}) : -\eta \leq i \leq \eta - 1\}$

and measure μ_η defined by $\mu_\eta([\frac{i}{\eta}, \frac{i+1}{\eta})) = \frac{1}{\eta}$, for $-\eta \leq i \leq \eta - 1$

We let $\{e_i : -\eta \leq i \leq \eta - 1\}$ be the standard orthonormal basis defined by;

$e_i(\frac{j}{\eta}) = \sqrt{\eta}\delta_{ij}$, *for* $-\eta \leq j \leq \eta - 1$

with respect to the inner product $<,>_\eta$, defined by;

$< f, g >_\eta = \int_{\overline{V}_\eta} f\overline{g}d\mu_\eta$, *for* $\{f, g\} \subset V(\overline{V_\eta})$

We recall the inversion theorem from [2], that for $f \in V(\overline{V_\eta})$;

$f(x) = \frac{1}{2}{}^{*}\sum_{m \in \mathfrak{Z}_\eta} (\mathcal{F}_\eta(f))(m) exp_\eta(\pi i m x)$

where $\mathfrak{Z}_\eta = {}^{*}\mathcal{Z} \cap [-\eta, \eta)$

$exp_\eta(\pi imx) = {}^{*}exp(\pi im\frac{[x\eta]}{\eta})$, *for* $-\eta \leq m \leq \eta - 1$

and $(\mathcal{F}_\eta(f))(m)$

$= \int_{\overline{V}_\eta} f(x) exp_\eta(-\pi imx) d\mu_\eta(x)$

$=< f, exp_\eta(\pi imx) >_\eta$, *for* $-\eta \leq m \leq \eta - 1$

We define the nonstandard derivative on $V(\overline{V_\eta})$ *by;*

$f'(\frac{j}{\eta}) = \eta(f(\frac{j+1}{\eta}) - f(\frac{j}{\eta}))$, *for* $-\eta \leq j \leq \eta - 2$

$f'(\frac{\eta-1}{\eta}) = \eta(f(-1) - f(\frac{\eta-1}{\eta}))$

Lemma 0.2. *If* $f \in C([-1,1])$, *with corresponding* $f_\eta \in V(\overline{V_\eta})$, *then, for infinite* $m \in \mathfrak{Z}_\eta$, *we have;*

$(\mathcal{F}_\eta(f_\eta))(m) \simeq 0$

Proof. Let $\{a,b\} \subset \overline{V_\eta}$, and let $\chi_{[a,b],\eta} \in V(\overline{V_\eta})$ be defined by;

$\chi_{[a,b],\eta}(x) = 1$ if $\frac{[a\eta]}{\eta} \leq x < \frac{[b\eta]+1}{\eta}$

$\chi_{[a,b],\eta}(x) = 0$ otherwise

We claim that, for infinite m, $\mathcal{F}_\eta(\chi_{[a,b]})(m) \simeq 0$, $(*)$

We have, as in [2], that for $m \in \mathfrak{Z}_\eta$, $x \in \overline{V_\eta}$, that;

$(exp_\eta(-\pi imx))'$

$= exp_\eta(-\pi imx)\theta_\eta(m)$

where $\theta_\eta(m) = \eta(exp_\eta(-\pi i\frac{m}{\eta}) - 1)$

and;

$exp_\eta(-\pi imx) = \frac{(exp_\eta(-\pi imx))'}{\theta_\eta(m)}$

It follows that;

$(\mathcal{F}_\eta(\chi_{[a,b]}))(m)$

$= \int_{\overline{V}_\eta} \chi_{[a,b]}(x) exp_\eta(-\pi imx) d\mu_\eta(x)$

$= \int_{\overline{V}_\eta} \chi_{[a,b]}(x) \frac{(exp_\eta(-\pi imx))'}{\theta_\eta(m)} d\mu_\eta(x)$

$= \frac{1}{\theta_\eta(m)} \int_{\overline{V}_\eta} \chi_{[a,b]}(x)(exp_\eta(-\pi imx))' d\mu_\eta(x)$

$= \frac{1}{\eta\theta_\eta(m)} {}^*\sum_{i=\frac{[a\eta]}{\eta}}^{\frac{[b\eta]+1}{\eta}} \chi_{[a,b]}(\frac{i}{\eta})(exp_\eta(-\pi im\frac{i}{\eta}))'$

$= \frac{1}{\eta\theta_\eta(m)} {}^*\sum_{i=\frac{[a\eta]}{\eta}}^{\frac{[b\eta]+1}{\eta}} \eta(exp_\eta(-\pi im\frac{i+1}{\eta}) - (exp_\eta(-\pi im\frac{i}{\eta})))$

$= \frac{1}{\theta_\eta(m)}(exp_\eta(-\pi im(\frac{[b\eta]+2}{\eta})) - (exp_\eta(-\pi im(\frac{[a\eta]}{\eta}))))$

As in [2], Lemma 0.12, for $m \in \mathfrak{Z}_\eta$, that;

$2|m| \leq |\theta_\eta(m)|$

It follows that, for m infinite;

$|(\mathcal{F}_\eta(\chi_{[a,b]}))(m)| \leq \frac{1}{2|m|}|(exp_\eta(-\pi im(\frac{[b\eta]+2}{\eta})) - (exp_\eta(-\pi im(\frac{[a\eta]}{\eta}))))|$

$\leq \frac{2}{2|m|} = \frac{1}{|m|} \simeq 0$

Hence, $(*)$ is proved. As $f \in C([-1,1])$, by Darboux's Theorem, there exists a sequence of step functions $\{g_n : n \in \mathcal{N}\}$, such that;

$\int_{-1}^{1} |f - g_n| d\mu < \frac{1}{n}$

where μ denotes Lebesgue measure. We have that;

$g_{n,\eta} = (\sum_{k=1}^{m(n)-1} c_k \chi_{[b_{kn}, b_{(k+1)n}]})_\eta$

$= (\sum_{k=1}^{m(n)-1} c_k (\chi_{[b_{kn}, b_{(k+1)n}]})_\eta$

$= (\sum_{k=1}^{m(n)-1} c_k \chi_{[\frac{[b_{kn}\eta]+1}{\eta}, \frac{[b_{(k+1)n}\eta]}{\eta}]}, \; (**)$

for a partition $-1 \leq b_{1n} \leq \ldots \leq b_{m(n)n} \leq 1$ and $c_k \in \mathcal{R}$, for $1 \leq k \leq m(n) - 1$. We have that, for infinite $m \in \mathfrak{Z}_\eta$, using $(*), (**)$, that;

$\mathcal{F}_\eta(g_{n,\eta})(m)$

$= \mathcal{F}_\eta((\sum_{k=1}^{m(n)-1} c_k \chi_{[\frac{[b_{kn}\eta]+1}{\eta}, \frac{[b_{(k+1)n}\eta]}{\eta}]}))(m)$

$= \sum_{k=1}^{m(n)-1} c_k \mathcal{F}_\eta(\chi_{[\frac{[b_{kn}\eta]+1}{\eta}, \frac{[b_{(k+1)n}\eta]}{\eta}]})(m)$

$\simeq 0$

Then, for infinite m, and $n \in \mathcal{N}$;

$|\mathcal{F}_\eta(f_\eta)(m)|$

$= |\mathcal{F}_\eta(f_\eta - g_{n,\eta} + g_{n,\eta})(m)|$

$\leq |\mathcal{F}_\eta(g_{n,\eta})(m)| + |\mathcal{F}_\eta(f_\eta - g_{n,\eta})(m)|$

$\leq \frac{1}{n} + \int_{\overline{V}_\eta} |(f_\eta - g_{n,\eta})| d\mu_\eta$, $(***)$

As $f \in C[-1,1]$, we have that, f_η is S-continuous, bounded and S-integrable, and ${}^\circ f_\eta = st^* f$, where $st : \overline{V}_\eta \rightarrow [-1,1]$ is the standard part mapping. $g_{n,\eta}$ is bounded and S-integrable, $|(f_\eta - g_{n,\eta})|$ is bounded and S-integrable. It follows, using the S-integrability criteria, see [1], and the fact the standard part mapping is measurable and measure preserving, that;

${}^\circ(\int_{\overline{V}_\eta} |(f_\eta - g_{n,\eta})| d\mu_\eta)$

$= \int_{\overline{V}_\eta} |{}^\circ(f_\eta - g_{n,\eta})| dL(\mu_\eta)$

$= \int_{\overline{V}_\eta} |(st^*(f) - ({}^\circ g_{n,\eta}))| dL(\mu_\eta)$

$\leq \int_{\overline{V}_\eta} |(st^*(f) - st^* g_n)| dL(\mu_\eta)$

$+ \int_{\overline{V}_\eta} |(st^*(g_n) - ({}^\circ g_{n,\eta}))| dL(\mu_\eta)$

$\simeq \int_{-1}^{1} |f - g_n| d\mu < \frac{1}{n}$

Therefore, using $(***)$;

$|\mathcal{F}_\eta(f_\eta)(m)|$

$< \frac{1}{n} + \int_{\overline{V}_\eta} |(f_\eta - g_{n,\eta})| < \frac{1}{n} + \frac{1}{n} = \frac{2}{n}$

As this holds for all $n \in \mathcal{N}$, using countable comprehension and overflow, see [1], $(\mathcal{F}_\eta(f_\eta))(m) \simeq 0$

$\square$

Remarks 0.3. *It is straightforward to deduce the standard Riemann-Lebesgue Lemma;*

If $f \in C([-1,1])$, *then;*

$lim_{|m|\rightarrow\infty}\mathcal{F}(f)(m) = 0$

It is sufficient to show that, given $\epsilon > 0$, *there exists* $M(\epsilon)$, *such that;*

$|\mathcal{F}(f)(m)| < \epsilon$, *for all* $m \in \mathcal{Z}$, $m \geq M(\epsilon)$. *As, for all infinite* $m \in \mathfrak{Z}_\eta$*;*

$|(\mathcal{F}_\eta(f_\eta))(m)| < \epsilon$

it follows by underflow, that, for all $|m| \geq M(\epsilon)$, $m \in \mathcal{Z}$*;*

$|(\mathcal{F}_\eta(f_\eta))(m)| < \epsilon$

The result then follows from the fact, that, for finite $m \in \mathcal{Z}$*;*

${}^\circ(\mathcal{F}_{f_\eta}(m)) = \mathcal{F}(f)(m)$

as $f_\eta(x)$ *and* $exp_\eta(-\pi imx)$ *are S-continuous and S-integrable on* $\overline{V_\eta}$. *The proof above follows the structure of the standard result.*

Definition 0.4. *We recall the definitions from* [2]. *We let* $\eta \in {}^*\mathcal{N}$ *be infinite and odd, we let;*

$\overline{\mathcal{R}_\eta} = {}^*\bigcup_{-\frac{(\eta-1)}{2}\leq i\leq\frac{(\eta-1)}{2}}[\frac{i}{\sqrt{\eta}}, \frac{i+1}{\sqrt{\eta}})$

so that $\overline{\mathcal{R}_\eta} = {}^*[-\frac{(\eta-1)}{2\sqrt{\eta}}, \frac{(\eta+1)}{2\sqrt{\eta}})$. *We let* $\mathcal{D}_\eta$ *denote the associated* $*$*-finite algebra, generated by the intervals* $[\frac{i}{\sqrt{\eta}}, \frac{i+1}{\sqrt{\eta}})$, *for* $-\frac{(\eta-1)}{2} \leq i \leq \frac{(\eta-1)}{2}$, *and* μ_η *the associated counting measure defined by* $\mu_\eta([\frac{i}{\sqrt{\eta}}, \frac{i+1}{\sqrt{\eta}})) = \frac{1}{\sqrt{\eta}}$. *We let* $(\overline{\mathcal{R}_\eta}, L(\mathcal{D}_\eta), L(\mu_\eta))$ *denote the associated Loeb space.*

We let $(\mathcal{R} \cup \{+\infty, -\infty\}, \mathfrak{D}, \mu)$ *denote the extended real line, with the completion* $\mathfrak{D}$ *of the extension of the Borel field, and* μ *the extension of Lebesgue measure, with* $\mu(+\infty) = \mu(-\infty) = \infty$. *We let* $(\mathcal{R}_{\geq 0} \cup \{+\infty\}, \mathfrak{C}, \lambda)$ *denote the extended real half line, with the completion* $\mathfrak{C}$ *of the extended Borel field, and* λ *the extension of Lebesgue* *measure, with* $\lambda(+\infty)=\infty$*, see [?], Chapter 6.*

Given a measurable $f_\eta:\overline{\mathcal{R}_\eta}\rightarrow {}^*\mathcal{C}$*, we define the nonstandard Fourier transform* $\mathcal{F}(f_\eta):\overline{\mathcal{R}_\eta}\rightarrow {}^*\mathcal{C}$ *by;*

$\mathcal{F}(f_\eta)(y)=\int_{\overline{\mathcal{R}_\eta}}f_\eta(x)exp_\eta(-2\pi iyx)d\mu_\eta(x)$

With this definition, we have, see [5] that;

$f_\eta(x)=\int_{\overline{\mathcal{R}_\eta}}\hat{f}_\eta(y)exp_\eta(2\pi ixy)d\mu_\eta(y)\ (*)$

We define the nonstandard derivative on $V(\overline{\mathcal{R}}_\eta)$ *by;*

$f'(\frac{j}{\sqrt{\eta}})=\sqrt{\eta}(f(\frac{j+1}{\sqrt{\eta}})-f(\frac{j}{\sqrt{\eta}}))$*, for* $-\frac{(\eta-1)}{2}\leq j\leq\frac{\eta-3}{2}$

$f'(\frac{\eta-1}{2\sqrt{\eta}})=\sqrt{\eta}(f(-\frac{(\eta-1)}{2\sqrt{\eta}})-f(\frac{\eta-1}{\sqrt{\eta}}))$

Lemma 0.5. *If* $f\in S(\mathcal{R})$*, with corresponding* $f_\eta\in V(\overline{V_\eta})$*, then, for infinite* $y\in\overline{\mathcal{R}_\eta}$*, we have;*

$(\mathcal{F}_\eta(f_\eta))(y)\simeq 0$

Proof. Let $n\in\mathcal{N}$, we first prove that;

$|\int_{(|x|>\frac{[n\sqrt{\eta}]}{\sqrt{\eta}})\cap\overline{\mathcal{R}_\eta}}f_\eta exp_\eta(-2\pi ixy)d\mu_\eta(x)|\leq\frac{E}{(n-1)}$, $(*)$

where $E\in\mathcal{R}$.

We have that;

$|\int_{(|x|\geq\frac{[n\sqrt{\eta}]}{\sqrt{\eta}})\cap\overline{\mathcal{R}_\eta}}f_\eta exp_\eta(-2\pi ixy)d\mu_\eta(x)|$

$\leq\frac{1}{\sqrt{\eta}}{}^*\sum_{|k|=[n\sqrt{\eta}]}^{\frac{\eta-1}{2}}|f_\eta(\frac{k}{\sqrt{\eta}})|$

$=\frac{1}{\sqrt{\eta}}{}^*\sum_{k=[n\sqrt{\eta}]}^{\frac{\eta-1}{2}}|f^*|(\frac{k}{\sqrt{\eta}})$

As $f\in S(\mathcal{R})$, we have that, $|f^*|(x)\leq\frac{C}{|x|^2}$, for $|x|\geq 1$, $C\in\mathcal{R}$. It follows that;

$|f^{*}|(\frac{k}{\sqrt{\eta}})\leq\frac{C}{|\frac{k}{\sqrt{\eta}}|^{2}}=\frac{C\eta}{|k|^{2}}$, for $|k|\geq[n\sqrt{\eta}]$

Then;

$\frac{1}{\sqrt{\eta}}{}^{*}\sum_{k=[n\sqrt{\eta}]}^{\frac{\eta-1}{2}}|f^{*}|(\frac{k}{\sqrt{\eta}})$

$\leq\frac{1}{\sqrt{\eta}}{}^{*}\sum_{k=[n\sqrt{\eta}]}^{\frac{\eta-1}{2}}\frac{C\eta}{|k|^{2}}$

$=\frac{2\eta C}{\sqrt{\eta}}{}^{*}\sum_{k=[n\sqrt{\eta}]}^{\frac{\eta-1}{2}}\frac{1}{k^{2}}$

$\leq\frac{2\eta C}{\sqrt{\eta}}\int_{[n\sqrt{\eta}]-1}^{\frac{\eta-1}{2}}\frac{dx}{x^{2}}$ (by transfer)

$=2C\sqrt{\eta}[\frac{-1}{x}]_{[n\sqrt{\eta}]-1}^{\frac{\eta-1}{2}}$

$=2C\sqrt{\eta}(\frac{1}{[n\sqrt{\eta}]-1}-\frac{2}{\eta-1})$

$\leq 2C\sqrt{\eta}\frac{1}{n\sqrt{\eta}-2}$

$\leq 2C\sqrt{\eta}\frac{1}{(n-1)\sqrt{\eta}}$

$=\frac{2C}{(n-1)}$

which gives the result $(*)$, taking $E=2C$. Let $\overline{V}_{\eta,n}=(-\frac{[n\sqrt{\eta}]}{\sqrt{\eta}},\frac{[n\sqrt{\eta}]}{\sqrt{\eta}})\cap\overline{R}_{\eta}$.

Let $\{a,b\}\subset\overline{V_{\eta,n}}$, and let $\chi_{[a,b],\eta}\in V(\overline{V_{\eta,n}})$ be defined by;

$\chi_{[a,b],\eta}(x)=1$ if $\frac{[a\sqrt{\eta}]}{\sqrt{\eta}}\leq x<\frac{[b\sqrt{\eta}]+1}{\sqrt{\eta}}$

$\chi_{[a,b],\eta}(x)=0$ otherwise

We claim that, for infinite y, $\mathcal{F}_{\eta}(\chi_{[a,b]})(y)\simeq 0$, $(**)$

We have, as in [5], that for $y\in\overline{R}_{\eta}$, $x\in\overline{V_{\eta},n}$, that;

$(exp_{\eta}(-2\pi iyx))'=\chi_{\eta}(y)exp_{\eta}(-2\pi iyx)$

where;

$\chi_{\eta}(y)=\sqrt{\eta}(exp_{\eta}(\frac{-2\pi iy}{\sqrt{\eta}})-1)$

and;

$exp_\eta(-\pi iyx)=\frac{(exp_\eta(-\pi iyx))'}{\chi_\eta(y)}$

It follows that;

$=\int_{\overline{V}_{\eta,n}}\chi_{[a,b]}(x)exp_\eta(-2\pi iyx)d\mu_\eta(x)$

$=\int_{\overline{V}_{\eta,n}}\chi_{[a,b]}(x)\frac{(exp_\eta(-2\pi iyx))'}{\chi_\eta(y)}d\mu_\eta(x)$

$=\frac{1}{\chi_\eta(y)}\int_{\overline{V}_{\eta,n}}\chi_{[a,b]}(x)(exp_\eta(-2\pi iyx))'d\mu_\eta(x)$

$=\frac{1}{\sqrt{\eta}\chi_\eta(y)}{}^*\sum_{i=\frac{[a\sqrt{\eta}]}{\sqrt{\eta}}}^{\frac{[b\sqrt{\eta}]+1}{\sqrt{\eta}}}\chi_{[a,b]}(\frac{i}{\sqrt{\eta}})(exp_\eta(-2\pi iy\frac{i}{\eta}))'$

$=\frac{1}{\sqrt{\eta}\chi_\eta(y)}{}^*\sum_{i=\frac{[a\sqrt{\eta}]}{\sqrt{\eta}}}^{\frac{[b\sqrt{\eta}]+1}{\sqrt{\eta}}}\sqrt{\eta}(exp_\eta(-2\pi iy\frac{i+1}{\sqrt{\eta}})-(exp_\eta(-2\pi iy\frac{i}{\sqrt{\eta}})))$

$=\frac{1}{\chi_\eta(y)}(exp_\eta(-2\pi iy(\frac{[b\sqrt{\eta}]+2}{\sqrt{\eta}}))-(exp_\eta(-2\pi iy(\frac{[a\sqrt{\eta}]}{\sqrt{\eta}}))))$

As in [5], Lemma 0.20, for $y\in\overline{\mathcal{R}_\eta}$, we have that;

$4|y|\leq|\chi_\eta(y)|$

It follows that, for y infinite;

$|\int_{\overline{V}_{\eta,n}}\chi_{[a,b]}(x)exp_\eta(-2\pi iyx)d\mu_\eta(x)|$

$\leq\frac{1}{4|y|}|(exp_\eta(-2\pi iy(\frac{[b\sqrt{\eta}]+2}{\sqrt{\eta}}))-(exp_\eta(-2\pi iy(\frac{[a\sqrt{\eta}]}{\sqrt{\eta}}))))|$

$\leq\frac{2}{4|y|}=\frac{1}{2|y|}\simeq 0$

Hence, $(**)$ is proved. As $f\in C([-n,n])$, by Darboux's Theorem, there exists a sequence of step functions $\{g_r:r\in\mathcal{N}\}$, such that;

$\int_{-n}^n|f-g_r|d\mu<\frac{1}{r}$

where μ denotes Lebesgue measure. We have that;

$g_{r,\eta}=(\sum_{k=1}^{m(r)-1}c_k\chi_{[b_{kr},b_{(k+1)r}]})_\eta$

$=(\sum_{k=1}^{m(r)-1}c_k(\chi_{[b_{kr},b_{(k+1)r}]})_\eta$

$= (\sum_{k=1}^{m(r)-1} c_k \chi_{[\frac{[b_{kr}\sqrt{\eta}]+1}{\sqrt{\eta}}, \frac{[b_{(k+1)r}\sqrt{\eta}]}{\sqrt{\eta}}]}, (***)$

for a partition $-n \leq b_{1r} \leq \ldots \leq b_{m(r)r} \leq n$ and $c_k \in \mathcal{R}$, for $1 \leq k \leq m(r)-1$. We have that, for infinite $y \in \overline{\mathcal{R}_\eta}$, using $(**), (***)$, that;

$\int_{\overline{V}_{\eta,n}} g_{r,\eta}(x) exp_\eta(-2\pi i y x) d\mu_\eta(x)$

$= \int_{\overline{V}_{\eta,n}} ((\sum_{k=1}^{m(r)-1} c_k \chi_{[\frac{[b_{kr}\sqrt{\eta}]+1}{\sqrt{\eta}}, \frac{[b_{(k+1)r}\sqrt{\eta}]}{\sqrt{\eta}}]})) exp_\eta(-2\pi i y x) d\mu_\eta(x)$

$= \sum_{k=1}^{m(r)-1} c_k \int_{\overline{V}_{\eta,n}} (\chi_{[\frac{[b_{kr}\sqrt{\eta}]+1}{\sqrt{\eta}}, \frac{[b_{(k+1)r}\sqrt{\eta}]}{\sqrt{\eta}}]}) exp_\eta(-2\pi i y x) d\mu_\eta(x)$

$\simeq 0$

Then, for infinite y, and $r \in \mathcal{N}$;

$| \int_{\overline{V}_{\eta,n}} f_\eta exp_\eta(-2\pi i y x) d\mu_\eta(x)|$

$= | \int_{\overline{V}_{\eta,n}} (f_\eta - g_{r,\eta} + g_{r,\eta}) exp_\eta(-2\pi i y x) d\mu_\eta(x)|$

$\leq | \int_{\overline{V}_{\eta,n}} g_{r,\eta} exp_\eta(-2\pi i y x) d\mu_\eta(x)| + | \int_{\overline{V}_{\eta,n}} (f_\eta - g_{r,\eta}) exp_\eta(-2\pi i y x) d\mu_\eta(x)|$

$\leq \frac{1}{n} + \int_{\overline{V}_{\eta,n}} |(f_\eta - g_{r,\eta})| d\mu_\eta, \; (****)$

As $f \in C[-n,n]$, we have that, f_η is S-continuous, bounded and S-integrable, and ${}^\circ f_\eta = st^* f$, where $st : \overline{V}_{\eta,n} \rightarrow [-n,n]$ is the standard part mapping. $g_{r,\eta}$ is bounded and S-integrable, $|(f_\eta - g_{r,\eta})|$ is bounded and S-integrable. It follows, using the S-integrability criteria, see [1], and the fact the standard part mapping is measurable and measure preserving, that;

${}^\circ(\int_{\overline{V}_{\eta,n}} |(f_\eta - g_{r,\eta})| d\mu_\eta)$

$= \int_{\overline{V}_{\eta,n}} |{}^\circ(f_\eta - g_{r,\eta})| dL(\mu_\eta)$

$= \int_{\overline{V}_{\eta,n}} |(st^*(f) - ({}^\circ g_{r,\eta}))| dL(\mu_\eta)$

$\leq \int_{\overline{V}_{\eta,n}} |(st^*(f) - st^* g_r)| dL(\mu_\eta)$

$+ \int_{\overline{V}_{\eta,n}} |(st^*(g_r) - ({}^\circ g_{r,\eta}))| dL(\mu_\eta)$

$\simeq \int_{-1}^{1}|f-g_r|d\mu<\frac{1}{r}$

Therefore, using $(****)$;

$|\int_{\overline{V}_{\eta,n}} f_\eta exp_\eta(-2\pi iyx)d\mu_\eta(x)|$

$<\frac{1}{n}+\int_{\overline{V}_\eta}|(f_\eta-g_{r,\eta})|d\mu_\eta(x)<\frac{1}{n}+\frac{2}{r}$

Taking $r\geq n$, we obtain that, for infinite y, that;

$|\mathcal{F}_\eta(f_\eta)(y)$

$\leq|\int_{\overline{V}_{\eta,n}} f_\eta exp_\eta(-2\pi iyx)d\mu_\eta(x)|+|\int_{(|x|>\frac{[n\sqrt{\eta}]}{\sqrt{\eta}})\cap\overline{\mathcal{R}_\eta}} f_\eta exp_\eta(-2\pi ixy)d\mu_\eta(x)|$

$<\frac{3}{n}+\frac{2C}{(n-1)}$

As this holds for all $n\in\mathcal{N}$, using countable comprehension and overflow, see [1], $(\mathcal{F}_\eta(f_\eta))(y)\simeq 0$

□

Remarks 0.6. *We let $\mathcal{S}(\mathcal{R})$ denote the Schwartz space. If $h\in\mathcal{S}(\mathcal{R})$, we define its Fourier transform by;*

$\mathcal{F}(h)(y)=\int_{-\infty}^{\infty}h(x)e^{-2\pi iyx}dx$

for $y\in\mathcal{R}$.

It is straightforward to deduce the standard Riemann-Lebesgue Lemma;

If $f\in S(\mathcal{R})$, then;

$lim_{|y|\rightarrow\infty}\mathcal{F}(f)(y)=0$

It is sufficient to show that, given $\epsilon>0$, there exists $M(\epsilon)$, such that;

$|\mathcal{F}(f)(y)|<\epsilon$, for all $y\in\mathcal{R}$, $y\geq M(\epsilon)$.

As, for all infinite $y\in\overline{\mathcal{R}}_\eta$;

$|(\mathcal{F}_\eta(f_\eta))(y)|<\epsilon$

it follows by underflow, that, for all $|y| \geq M(\epsilon)$, $y \in \overline{\mathcal{R}_\eta}$*;*

$$|(\mathcal{F}_\eta(f_\eta))(y)| < \epsilon$$

The result then follows from the fact, that, for finite $y \in \mathcal{R}$*;*

$$^\circ(\mathcal{F}_{f_\eta}(y)) = \mathcal{F}(f)(^\circ y) = \mathcal{F}(f)(y)$$

as $f_\eta exp_\eta(-\pi imx)$ *is* S*-continuous and* S*-integrable on* $\overline{V_\eta}$*, see* [4].

Lemma 0.7. *If* f *is Riemann integrable on* $[-1,1]$ *and bounded, with corresponding* $f_\eta \in V(\overline{V_\eta})$*, then* f_η *is* S*-integrable on* $\overline{V_\eta}$ *and there exist* $\{W_n : n \in \mathcal{N}\} \subset \mathfrak{B}_\eta$*, such that* $\mu_\eta(W_n) < \frac{1}{n}$ *and* $^\circ(f_\eta) = st^*(f)$ *on* $\overline{V_\eta} \setminus W_n$*. Moreover, for* m *infinite with* $m \in \mathfrak{Z}_\eta$*, we have that;*

$$(\mathcal{F}_\eta(f_\eta))(m) \simeq 0$$

Proof. Without loss of generality, assume f is real valued. As f_n is uniformly bounded, it follows that f_η is bounded, by transfer, and, in particular, S-integrable. By Lebesgue's Theorem, there exists $D \subset [-1,1]$ with $\mu(D) = 0$, such that f is continuous on $\{[-1,1] \setminus D\}$. Let $C = st^*(D)$, then $L(\mu_\eta)(C) = 0$, let $E = st^*([-1,1] \setminus D)$, then E and C are disjoint and $L(\mu_\eta)(E) = 2$. Let $x \in E$, with corresponding $^\circ x \in [-1,1] \setminus D$, then we claim that $f_\eta(x) \simeq f(^\circ x) = st^*(f)(x)$, $(*)$. As f is continuous at $^\circ x$, we have, for $n \in \mathcal{N}$, that there exists intervals $[^\circ x - \frac{1}{n}, ^\circ x + \frac{1}{n}]$, with $a(n) \in \mathcal{N}$ increasing, such that $|f(y) - f(^\circ x)| < \frac{1}{a(n)}$, for all $y \in [^\circ x - \frac{1}{n}, ^\circ x + \frac{1}{n}]$. It follows, by transfer, as $|x - ^\circ x| < \frac{1}{n}$, for all $n \in \mathcal{N}$, that $|f_\eta(x) - f(^\circ x)| < \frac{1}{a(\eta)} \simeq 0$, hence $(*)$ is shown. It follows that $^\circ(f_\eta) = st^*(f)$ on E. Using Theorem 3.4 of [**?**], there exist $\{W_n : n \in \mathcal{N}\} \subset \mathfrak{B}_\eta$, with $W_n \supset C$, such that $\mu_\eta(W_n) < \frac{1}{n}$. Clearly $^\circ(f_\eta) = st^*(f)$ on $\overline{V_\eta} \setminus W_n$ for $n \in \mathcal{N}$. This gives the first claim. Then, for infinite m, and $n \in \mathcal{N}$. Using the fact that f is Lebesgue integrable, we can find a sequence $\{g_n : n \in \mathcal{N}\}$ as in Lemma 0.2. Then;

$$|\mathcal{F}_\eta(f_\eta)(m)|$$

$$= |\mathcal{F}_\eta(f_\eta - g_{n,\eta} + g_{n,\eta})(m)|$$

$$\leq |\mathcal{F}_\eta(g_{n,\eta})(m)| + |\mathcal{F}_\eta(f_\eta - g_{n,\eta})(m)|$$

$\leq \frac{1}{n}+\int_{\overline{V}_\eta}|(f_\eta-g_{n,\eta})|d\mu_\eta$, $(***)$

We have that, f_η is bounded and S-integrable, and ${}^\circ f_\eta=st^*f$, on $\overline{V_\eta}\setminus W_n$ where $st:\overline{V}_\eta\rightarrow[-1,1]$ is the standard part mapping. $g_{n,\eta}$ is bounded and S-integrable, $|(f_\eta-g_{n,\eta})|$ is bounded and S-integrable. It follows, using the S-integrability criteria, see [1], and the fact the standard part mapping is measurable and measure preserving, that;

${}^\circ(\int_{\overline{V}_\eta}|(f_\eta-g_{n,\eta})|d\mu_\eta)$

$=\int_{\overline{V}_\eta}|({}^\circ f_\eta-({}^\circ g_{n,\eta}))|dL(\mu_\eta)$

$=\int_{\overline{V}_\eta\setminus W_n}|{}^\circ(f_\eta-g_{n,\eta})|dL(\mu_\eta)+\int_{W_n}|{}^\circ(f_\eta-g_{n,\eta})|dL(\mu_\eta)$

$\leq\int_{\overline{V}_\eta\setminus W_n}|(st^*(f)-({}^\circ g_{n,\eta}))|dL(\mu_\eta)+R\mu_\eta(V_n)$

$\leq\int_{\overline{V}_\eta\setminus W_n}|(st^*(f)-st^*g_n)|dL(\mu_\eta)+\frac{R}{n}$

$+\int_{\overline{V}_\eta\setminus W_n}|(st^*(g_n)-({}^\circ g_{n,\eta}))|dL(\mu_\eta)$

$\simeq\int_{-1}^{1}|f-g_n|d\mu+\frac{R}{n}<\frac{R+2}{n}$

Therefore, using $(***)$;

$|\mathcal{F}_\eta(f_\eta)(m)|$

$<\frac{1}{n}+\int_{\overline{V}_\eta}|(f_\eta-g_{n,\eta})|<\frac{1}{n}+\frac{R+1}{n}=\frac{R+2}{n}$

As this holds for all $n\in\mathcal{N}$, using countable comprehension and overflow, see [1], $(\mathcal{F}_\eta(f_\eta))(m)\simeq 0$

□

Remarks 0.8. *If $h\in C([-1,1])$, we define its Fourier coefficient by;*

$\mathcal{F}(h)(m)=\int_{-1}^{1}h(x)e^{-\pi imx}dx$

for $m\in\mathcal{Z}$.

As above, it is straightforward to deduce the standard Riemann-Lebesgue Lemma, in the form;

If f is Riemann integrable on $[-1,1]$ and bounded, then;

$lim_{|m|\rightarrow\infty}\mathcal{F}(f)(m)=0$

It is sufficient to show that, given $\epsilon>0$, there exists $M(\epsilon)$, such that;

$|\mathcal{F}(f)(m)|<\epsilon$*, for all* $m\in\mathcal{Z}$*,* $m\geq M(\epsilon)$*.*

As, for all infinite $m\in\mathcal{Z}_\eta$;

$|(\mathcal{F}_\eta(f_\eta))(m)|<\epsilon$

it follows by underflow, that, for all $|m|\geq M(\epsilon)$, $m\in\mathcal{Z}_\eta$;

$|(\mathcal{F}_\eta(f_\eta))(m)|<\epsilon$

The result then follows from the fact, that, for finite $m\in\mathcal{Z}$;

${}^\circ(\mathcal{F}_{f_\eta}(m))=\mathcal{F}(f)(m)$

as $f_\eta exp_\eta(-\pi imx)$ is S-continuous and S-integrable on $\overline{V_\eta}$, see [4].

We now give a different, but more geometric proof of Lemma 0.5, but we require an extra assumption;

Lemma 0.9. *If $f\in S(\mathcal{R})$, with $Re(f)$ and $Im(f)$ analytic, with corresponding $f_\eta\in V(\overline{V_\eta})$, η prime, then, there exists $N\in\overline{R_\eta}$ infinite, $N=\frac{\sqrt{\eta}}{4}$, such that for infinite $y\in\overline{\mathcal{R}_\eta}$, with $|y|\leq N$, we have;*

$(\mathcal{F}_\eta(f_\eta))(y)\simeq 0$

Proof. The first part is the same as Lemma 0.5, for $n\in\mathcal{N}$;

$|\int_{(|x|>\frac{[n\sqrt{\eta}]}{\sqrt{\eta}})\cap\overline{\mathcal{R}_\eta}}f_\eta exp_\eta(-2\pi ixy)d\mu_\eta(x)|\leq\frac{E}{(n-1)}$, $(*)$

where $E\in\mathcal{R}$.

We claim that there exists $N\in\overline{\mathcal{R}_\eta}$ infinite, such that for y infinite, with $|y|\leq N$, and $n\in\mathcal{N}$;

$\int_{\overline{V}_{\eta,n}} f_\eta exp_\eta(-2\pi iyx)d\mu_\eta(x) \simeq 0$, $(*)$

where $\overline{V}_{\eta,n} = (-\frac{[n\sqrt{\eta}]}{\sqrt{\eta}}, \frac{[n\sqrt{\eta}]}{\sqrt{\eta}}) \cap \overline{R}_\eta$.

As in [5], we have that;

$\int_{\overline{V}_{\eta,n}} f_\eta exp_\eta(-2\pi iyx)d\mu_\eta(x)$

$= \int_{\overline{V}_{\eta,n}} Re(f_\eta)cos_\eta(2\pi xy)d\mu_\eta(x)$

$+i\int_{\overline{V}_{\eta,n}} Im(f_\eta)cos_\eta(2\pi xy)d\mu_\eta(x)$

$-i\int_{\overline{V}_{\eta,n}} Re(f_\eta)sin_\eta(2\pi xy)d\mu_\eta(x)$

$+\int_{\overline{V}_{\eta,n}} Im(f_\eta)sin_\eta(2\pi xy)d\mu_\eta(x)$, $(\sharp)$

It is sufficient to prove that $\int_{\overline{V}_{\eta,n}} Re(f_\eta)cos_\eta(2\pi xy)d\mu_\eta(x) \simeq 0$, the remaining cases are left to the reader;

We have that;

$\int_{\overline{V}_{\eta,n}} Re(f_\eta)cos_\eta(2\pi xy)d\mu_\eta(x)$

$= \frac{1}{\sqrt{\eta}} {}^*\sum_{|l|<[n\sqrt{\eta}]} Re(f^*)(\frac{l}{\sqrt{\eta}})\theta_k(\frac{l}{\sqrt{\eta}})$, where $y = \frac{k}{\sqrt{\eta}}$

where $\theta_k(\frac{l}{\sqrt{\eta}}) = cos_\eta(\frac{2\pi lk}{\eta})$, $\theta_k(x) = cos^*(\frac{2\pi kx}{\sqrt{\eta}})$

We compute an upper bound, for given $k \in {}^*\mathcal{Z}$, with $-\frac{(\eta-1)}{2} \leq k \leq \frac{(\eta-1)}{2}$ $n \in \mathcal{N}$, of;

$\frac{1}{\sqrt{\eta}} {}^*\sum_{|l|<n\sqrt{\eta}} Re(f^*)(\frac{l}{\sqrt{\eta}})\theta_k(\frac{l}{\sqrt{\eta}}))$

by transfer of the result for;

$\frac{1}{m} {}^*\sum_{|l|<nm} Re(f)(\frac{l}{m})\theta_r(\frac{l}{m})$

where $m \in \mathcal{R}_{>0}$, $r \in \mathcal{Z}$, $-\frac{(m^2-1)}{2} \leq r \leq \frac{(m^2-1)}{2}$ $n \in \mathcal{N}$, and $\theta_r(\frac{l}{m}) = cos_{m^2}(\frac{2\pi lr}{m^2})$

For $r \in \mathcal{Z}$, $-\frac{(m^2-1)}{2} \leq r \leq \frac{(m^2-1)}{2}$, $r \neq 0$, $m \in \mathcal{R}_{>0}$, $x \in \mathcal{R}$, we let;

$\theta_r(x) = cos(\frac{2\pi rx}{m})$

For $x \in \mathcal{R}$, we have that;

$\theta_r(x) = cos(\frac{2\pi rx}{m}) = 0$

iff $\frac{2\pi rx}{m} = \frac{\pi}{2} + t\pi$, $(t \in \mathcal{Z})$

iff $x = \frac{\frac{\pi m}{2} + \pi tm}{2\pi r}$

iff $x = (\frac{1}{4r} + \frac{t}{2r})m$, $(\sharp\sharp)$

$\{r, t\} \subset \mathcal{Z}$, $m \in \mathcal{R}_{>0}$

With the assumption that $|\frac{r}{m}| \leq \frac{m}{4}$, we have that $|r| \leq \frac{m^2}{4}$, $\frac{1}{|r|} \geq \frac{4}{m^2}$, $\frac{m}{2|r|} \geq \frac{2}{m} > \frac{1}{m}$, where $\frac{m}{2|r|} = z_2 - z_1$, $\theta_r(z_1) = 0$ and $z_2 = \mu z(z > z_1 : \theta_r(z_2) = 0)$. It follows, by transfer, with m corresponding to $\sqrt{\eta}$ and r corresponding to k, that for $|\frac{k}{\sqrt{\eta}}| \leq \frac{\sqrt{\eta}}{4}$, $\frac{\sqrt{\eta}}{2|k|} \geq \frac{2}{\sqrt{\eta}} > \frac{1}{\sqrt{\eta}}$, where $\frac{\sqrt{\eta}}{2|k|} = z_2 - z_1$, $\theta_k(z_1) = 0$ and $z_2 = \mu z(z > z_1 : \theta_k(z_2) = 0)$. We let $N = \frac{\sqrt{\eta}}{4} < \frac{\sqrt{\eta}}{2}$. Suppose y is infinite, with $|y| \leq N$ and $y = \frac{k}{\sqrt{\eta}}$. Using the above calculation, we have, by transfer, that;

$\frac{\sqrt{\eta}}{2|k|} = \frac{\sqrt{\eta}}{2|y\sqrt{\eta}|} = \frac{1}{2|y|} = z_2 - z_1 \simeq 0$

With the assumption that m^2 is prime, we have that $m^2(1+2t)$ is odd, so that, for $r \neq 0$ $\frac{m^2(1+2t)}{4r} \notin \mathcal{Z}$, so if $\theta_r(x_0) = 0$, then $mx_0 \notin \mathcal{Z}$, $(\dagger)$.

As in [5], using the assumption that $Re(f)$ is analytic, we have that $Re(f)$ has finitely many zeroes at $\{x_1, \ldots, x_{a(n)}\}$, with $-n \leq x_1 \leq \ldots \leq x_i \leq \ldots \leq x_{a(n)} \leq n$, and finitely many maxima and minima, $\{x_{i,1}, \ldots, x_{i,j}, \ldots, x_{i,b(i)}\}$, with $x_i \leq x_{i,1} \leq x_{i,j} \leq \ldots \leq x_{i,b(i)} \leq x_{i+1}$, $0 \leq i \leq a(n)$, $1 \leq j \leq b(i) - 1$. As in [5], let $x_{i,0} = x_i$, for $0 \leq i \leq a(n) + 1$, then it follows that $Re(f)|_{[x_{i,j}, x_{i,j+1}]}$ is monotone for $0 \leq i \leq a(n)$, $0 \leq j \leq b(i) - 1$ and $Re(f)|_{[x_{i,b(i)}, x_{i+1,0}]}$ is monotone for $0 \leq i \leq a(n)$.

Again, as in [5], we have that, for sufficiently large m, with $m \notin \mathcal{Z}$, $mx_i \notin \mathcal{Z}$, for $1 \leq i \leq a(n) - 1$ and $mx_{i,j} \notin \mathcal{Z}$, for $1 \leq i \leq a(n) - 1$, $1 \leq j \leq b(i) - 1$, $(\dagger\dagger)$.

Let $\{r_{ijs}:1\leq s\leq e(i,j)\}$ enumerate the zeroes of θ_r on $[x_{i,j},x_{i,j+1}]$, $r_{ij0}=x_{i,j}$, $r_{ij(e(i,j)+1))}=x_{i,j+1}$, for $0\leq i\leq a(n)$, $0\leq j\leq b(i)-1$, and let $\{r_{is}:1\leq s\leq e(i)\}$ enumerate the zeroes of θ_r on $[x_{i,b(i)},x_{i+1,0}]$, $r_{i0}=x_{i,b(i)}$, $r_{i(e(i)+1)}=x_{i+1}$, $0\leq i\leq a(n)$. Then, using $(\dagger),(\dagger\dagger)$, we have that;

$\frac{1}{m}{}^*\sum_{|l|<nm}Re(f)(\frac{l}{m})\theta_r(\frac{l}{m})$

$=\frac{1}{m}{}^*\sum_{i=0}^{a(n)}{}^*\sum_{j=0}^{b(i)-1}{}^*\sum_{s=0}^{e(i,j)}{}^*\sum_{l=[mr_{ijs}]+1}^{[mr_{ij(s+1)}]-1}Re(f)(\frac{l}{m})\theta_r(\frac{l}{m})$

$+\frac{1}{m}{}^*\sum_{i=0}^{a(n)}{}^*\sum_{s=0}^{e(i)}{}^*\sum_{l=[mr_{is}]+1}^{[mr_{i(s+1)}]-1}Re(f)(\frac{l}{m})\theta_r(\frac{l}{m})$

We compute $\frac{1}{m}|{}^*\sum_{s=1}^{e(i,j)-1}{}^*\sum_{l=[mx_{ijs}]+1}^{[mx_{ij(s+1)}]-1}Re(f)(\frac{l}{m})\theta_r(\frac{l}{m})|$.

Let $\theta_{i,j}(s)=\frac{1}{m}{}^*\sum_{l=[mr_{ijs}]+1}^{[mr_{ij(s+1)}-1]}Re(f)(\frac{l}{m})\theta_r(\frac{l}{m})$

Considering Case 1 in [5], we have that;

$0\leq|{}^*\sum_{s=1}^{e(i,j)-1}\theta_{i,j}(s)|\leq l_{i,j}$, $(*)$

where;

$l_{i,j}=\frac{1}{m}{}^*\sum_{l=[mr_{ij1}]}^{[mr_{ij2}]}|Re(f)|(\frac{l}{m})\theta_r(\frac{l}{m})$

$\leq\frac{1}{m}{}^*\sum_{l=[mr_{ij1}]+1}^{[mr_{ij2}]-1}D$, where $|Re(f)|\leq D$, and $0\leq\theta_r|_{[r_{ij1},r_{ij2}]}\leq 1$

$\leq\frac{D}{m}(([mr_{ij2}]-1)-([mr_{ij1}]+1)+1)$

$=\frac{D}{m}([mr_{ij2}]-[mr_{ij1}]-1)$

$\leq\frac{D}{m}((mr_{ij2}+1)-(mr_{ij1}-1)-1)$

$=D((r_{ij2}-r_{ij1})+\frac{1}{m})$

so that, using $(*),(\sharp\sharp)$;

$0\leq\frac{1}{m}|{}^*\sum_{s=1}^{e(i,j)-1}{}^*\sum_{l=[mx_{ijs}]+1}^{[mx_{ij(s+1)}]-1}Re(f)(\frac{l}{m})\theta_r(\frac{l}{m})|\leq l_{i,j}$

$\leq D((r_{ij2}-r_{ij1})+\frac{1}{m})$

$= D(\frac{m}{2|r|} + \frac{1}{m})$

Similarly, for Case 2 in [5]. In the other cases, reversing the sequences, we obtain;

$\frac{1}{m}|^* \sum_{s=1}^{e(i,j)-1} {}^* \sum_{l=[mx_{ijs}]+1}^{[mx_{ij(s+1)}]-1} Re(f)(\frac{l}{m})\theta_r(\frac{l}{m})|$

$\leq D((r_{ije(i,j)} - r_{ij(e(i,j)-1)}) + \frac{1}{m})$

$= D(\frac{m}{2|r|} + \frac{1}{m})$

Considering all 4 cases, we obtain the same bound;

$\frac{1}{m}|^* \sum_{s=1}^{e(i)-1} {}^* \sum_{l=[mr_{is}]+1}^{[mr_{i(s+1)}]-1} Re(f)(\frac{l}{m})\theta_r(\frac{l}{m})| \leq D(\frac{m}{2|r|} + \frac{1}{m})$

Similarly;

$max(A_{i,j}, B_{i,j}, C_i, D_i)$

$\leq D(\frac{m}{2|r|} + \frac{1}{m})$

where;

$A_{i,j} = \frac{1}{m}|^* \sum_{l=[mr_{ij0}]+1}^{[mr_{ij1}]-1} Re(f)(\frac{l}{m})\theta_r(\frac{l}{m})|$

$B_{i,j} = \frac{1}{m}|^* \sum_{l=[mr_{ije(i,j)}]+1}^{[mr_{ij(e(i,j)+1)}]-1} Re(f)(\frac{l}{m})\theta_r(\frac{l}{m})|$

$C_i = \frac{1}{m}|^* \sum_{l=[mr_{i0}]+1}^{[mr_{i1}]-1} Re(f)(\frac{l}{m})\theta_r(\frac{l}{m})|$

$D_i = \frac{1}{m}|^* \sum_{l=[mr_{ie(i)}]+1}^{[mr_{i(e(i)+1)}]-1} Re(f)(\frac{l}{m})\theta_r(\frac{l}{m})|)$

As in [5], we have that;

$\frac{1}{m}|^* \sum_{|l|<nm} Re(f)(\frac{l}{m})\theta_r(\frac{l}{m})|$

$\leq (w(n) + a(n) + 1)3D(\frac{m}{2|r|} + \frac{1}{m})$

where $w(n) = Card(Re(f)'|_{[-n,n]} = 0)$, $a(n) = Card(Re(f)|_{[-n,n]} = 0)$

It follows, by transfer, with the assumption that y is infinite, $y=\frac{k}{\sqrt{\eta}}$ and $y\leq N$, using ($\sharp\sharp$);

$|\int_{(\frac{-n\sqrt{\eta}}{\sqrt{\eta}},\frac{n\sqrt{\eta}}{\sqrt{\eta}})}Re(f_\eta)cos_\eta(2\pi xy)d\mu_\eta(x)|$

$=\frac{1}{\sqrt{\eta}}|^*\sum_{|l|<n\sqrt{\eta}}Re(f^*)(\frac{l}{\sqrt{\eta}})\theta_k(\frac{l}{\sqrt{\eta}}))|$

$\leq(w(n)+a(n)+1)3D(\frac{\sqrt{\eta}}{2|k|}+\frac{1}{\sqrt{\eta}})\simeq 0$, as $w(n),a(n)$ are finite and $\frac{\sqrt{\eta}}{2|k|}=\frac{1}{2|y|}\simeq 0$.

and, similarly, as in [5], considering the additional terms from ($\sharp$), we have that;

$|\int_{(|x|<\frac{[n\sqrt{\eta}]}{\sqrt{\eta}})\cap\overline{\mathcal{R}_\eta}}f_\eta exp_\eta(-2\pi ixy)d\mu_\eta(x)|$

$\leq(w(n)+a(n)+w'(n)+a'(n)+2)6D(\frac{\sqrt{\eta}}{2|k|}+\frac{1}{\sqrt{\eta}})\simeq 0$

again, as $w(n),a(n),w'(n),a'(n)$ are finite and $\frac{\sqrt{\eta}}{2|k|}=\frac{1}{2|y|}\simeq 0$.

It follows that, for infinite $y\in\overline{\mathcal{R}_\eta}$, with $|y|\leq N$, that;

$|\int_{\overline{\mathcal{R}_\eta}}f_\eta exp_\eta(-2\pi ixy)d\mu_\eta(x)|$

$\leq|\int_{(|x|<\frac{[n\sqrt{\eta}]}{\sqrt{\eta}})\cap\overline{\mathcal{R}_\eta}}f_\eta exp_\eta(-2\pi ixy)d\mu_\eta(x)|+|\int_{(|x|\geq\frac{[n\sqrt{\eta}]}{\sqrt{\eta}})\cap\overline{\mathcal{R}_\eta}}f_\eta exp_\eta(-2\pi ixy)d\mu_\eta(x)|$

$\leq\delta+\frac{E}{n-1}<\frac{E+1}{n-1}$

where $\delta\simeq 0$. As this holds for all $n\in\mathcal{N}$, we obtain that;

$\int_{\overline{\mathcal{R}_\eta}}f_\eta exp_\eta(-2\pi ixy)d\mu_\eta(x)\simeq 0$

□

Remarks 0.10. *It is straightforward to deduce the standard Riemann-Lebesgue Lemma;*

If $f\in S(\mathcal{R})$, with $Re(f)$ and $Im(f)$ analytic, then;

$lim_{|y|\rightarrow\infty}\mathcal{F}(f)(y)=0$

Just follow the proof in Remark 0.6, the addition restraint that $|y|\leq N$ doesn't effect the application of underflow.

Lemma 0.11. *If $f \in C[-1,1]$, with $Re(f)$ and $Im(f)$ analytic, and corresponding $f_\eta \in V(\overline{V_\eta})$, η prime, then, there exists $N \in \mathcal{Z}_\eta$ infinite, $N = \eta - 1$, such that for infinite $m \in \mathcal{Z}_\eta$, with $|m| \leq N$, we have;*

$(\mathcal{F}_\eta(f_\eta))(m) \simeq 0$

Proof. We claim that there exists $N \in \mathcal{Z}_\eta$ infinite, such that for m infinite, with $|m| \leq N$;

$\int_{\overline{V}_\eta} f_\eta(x) exp_\eta(-\pi imx) d\mu_\eta(x) \simeq 0$, $(*)$

As above, we have that;

$\int_{\overline{V}_\eta} f_\eta exp_\eta(-\pi imx) d\mu_\eta(x)$

$= \int_{\overline{V}_\eta} Re(f_\eta) cos_\eta(\pi mx) d\mu_\eta(x)$

$+ i \int_{\overline{V}_\eta} Im(f_\eta) cos_\eta(\pi mx) d\mu_\eta(x)$

$- i \int_{\overline{V}_\eta} Re(f_\eta) sin_\eta(\pi mx) d\mu_\eta(x)$

$+ \int_{\overline{V}_\eta} Im(f_\eta) sin_\eta(\pi mx) d\mu_\eta(x)$, $(\sharp)$

It is sufficient to prove that $\int_{\overline{V}_\eta} Re(f_\eta) cos_\eta(\pi mx) d\mu_\eta(x) \simeq 0$, the remaining cases are left to the reader;

As $Re(f_\eta)$ and $cos^*(\pi m \frac{[x\eta]}{\eta})$ are bounded, $\frac{1}{\eta} Re(f_\eta)(-1) cos^*(-\pi m) \simeq 0$, so we have that;

$\int_{\overline{V}_\eta} Re(f_\eta) cos_\eta(\pi mx) d\mu_\eta(x)$

$\simeq \frac{1}{\eta} {}^*\sum_{|l|<\eta-1} Re(f^*)(\frac{l}{\eta}) cos_\eta(\frac{\pi ml}{\eta})$

$= \frac{1}{\eta} {}^*\sum_{|l|<\eta-1} Re(f^*)(\frac{l}{\eta}) \theta_m(\frac{l}{\eta})$

where $\theta_m(\frac{l}{\eta}) = cos_\eta(\frac{\pi ml}{\eta})$, $\theta_m(x) = cos^*(\pi mx)$

We compute an upper bound, given $m \in {}^*\mathcal{Z}$, $-\eta \leq m \leq \eta - 1$, for;

$\frac{1}{\eta} {}^*\sum_{|l|<\eta-1} Re(f^*)(\frac{l}{\eta}) \theta_m(\frac{l}{\eta})$

by transfer of the result for;

$\frac{1}{n}*\sum_{|l|<n-1}Re(f)(\frac{l}{n})\theta_r(\frac{l}{n})$

where $n\in\mathcal{N}_{>0}$ is prime, $r\in\mathcal{Z}$, $-n\leq r\leq n-1$, and $\theta_r(\frac{l}{n})=cos(\frac{\pi rl}{n})$

For $r\in\mathcal{Z}$, $-n\leq r\leq n-1$, $r\neq 0$, n prime, $x\in[-1,1]$, we let;

$\theta_r(x)=cos(\pi rx)$

For $x\in[-1,1]$, we have that;

$\theta_r(x)=cos(\pi rx)=0$

iff $\pi rx=\frac{\pi}{2}+t\pi$, $(t\in\mathcal{Z})$

iff $x=\frac{\frac{\pi}{2}+\pi t}{\pi r}$

iff $x=(\frac{1}{2r}+\frac{t}{r})$, $(\sharp\sharp)$

$\{r,t\}\subset\mathcal{Z}$, $-n\leq r\leq n-1$, $-r-\frac{1}{2}\leq t\leq r-\frac{1}{2}$, for $r>0$, $r-\frac{1}{2}\leq t\leq -r-\frac{1}{2}$, for $r<0$

With the assumption that $|r|\leq n-1$, we have that, $\frac{1}{|r|}\geq\frac{1}{n-1}>\frac{1}{n}$, where $\frac{1}{|r|}=z_2-z_1$, $\theta_r(z_1)=0$ and $z_2=\mu z(z>z_1:\theta_r(z_2)=0)$. It follows, by transfer, with n corresponding to η and r corresponding to m, that for $|m|\leq\eta-1$, $\frac{1}{|m|}\geq\frac{1}{\eta-1}>\frac{1}{\eta}$, where $\frac{1}{|m|}=z_2-z_1$, $\theta_m(z_1)=0$ and $z_2=\mu z(z>z_1:\theta_m(z_2)=0)$. We let $N=\eta-1$. Suppose m is infinite, with $|m|\leq\eta-1$. Clearly, we have that;

$\frac{1}{|m|}=z_2-z_1\simeq 0$

With the assumption that n is prime, we have that $n(1+2t)$ is odd, so that, for $r\neq 0$ $\frac{n(1+2t)}{2r}\notin\mathcal{Z}$, so if $\theta_r(x_0)=0$, then $nx_0\notin\mathcal{Z}$, $(\dagger)$.

As in Lemma 0.9, using the assumption that $Re(f)$ is analytic, we have that $Re(f)$ has finitely many zeroes at $\{x_1,\ldots,x_{a(1)}\}$, with $-1\leq x_1\leq\ldots\leq x_i\leq\ldots\leq x_{a(n)}\leq 1$, and finitely many maxima and minima, $\{x_{i,1},\ldots,x_{i,j},\ldots,x_{i,b(i)}\}$, with $x_i\leq x_{i,1}\leq x_{i,j}\leq\ldots\leq x_{i,b(i)}\leq x_{i+1}$, $0\leq i\leq a(1)$, $1\leq j\leq b(i)-1$. As in Lemma 0.9 , let $x_{i,0}=x_i$, for $0\leq i\leq a(1)+1$, then it follows that $Re(f)|_{[x_{i,j},x_{i,j+1}]}$ is monotone for $0 \leq i \leq a(1)$, $0 \leq j \leq b(i)-1$ and $Re(f)|_{[x_{i,b(i)},x_{i+1,0}]}$ is monotone for $0 \leq i \leq a(1)$.

If $w \in \mathcal{Q}$, then, for any $d \in \mathcal{R} \setminus \mathcal{Q}$, we have that $w + d \in \mathcal{R} \setminus \mathcal{Q}$. Given a finite set $\{w_1, \ldots, w_i, \ldots, w_m\} \subset \mathcal{R} \setminus \mathcal{Q}$, let $U_i = \{u \in \mathcal{R} \setminus \mathcal{Q} : w_i + u \in \mathcal{R} \setminus \mathcal{Q}\}$, then U_i is dense in $\mathcal{R}$, hence, the finite intersection $\bigcap_{1 \leq i \leq m} U_i$ is dense in $\mathcal{R}$, in particular nonempty. It follows that there exists $d \in \mathcal{R}$, with $x_i + d \in \mathcal{R} \setminus \mathcal{Q}$, $x_{i,j} + d \in \mathcal{R} \setminus \mathcal{Q}$, for $1 \leq i \leq a(1)-1$, $1 \leq i \leq a(1)-1$, $1 \leq j \leq b(i)-1$. By considering $g(x) = f(x-\epsilon)$, for sufficiently small ϵ, with $\epsilon \in \mathcal{R} \setminus \mathcal{Q}$, we can assume that $\{x_1, \ldots, x_{a(1)}\} \subset \mathcal{R} \setminus \mathcal{Q}$, for $1 \leq i \leq a(n)-1$, and $\{x_{i,1}, \ldots, x_{i,j}, \ldots, x_{i,b(i)}\} \subset \mathcal{R} \setminus \mathcal{Q}$, for $1 \leq i \leq a(n)-1$, $1 \leq j \leq b(i)-1$. In particular, it follows that for n prime, $nx_i \notin \mathcal{Z}$, for $1 \leq i \leq a(n)-1$ and $nx_{i,j} \notin \mathcal{Z}$, for $1 \leq i \leq a(n)-1$, $1 \leq j \leq b(i)-1$, $(\dagger\dagger)$.

Let $\{r_{ijs} : 1 \leq s \leq e(i,j)\}$ enumerate the zeroes of θ_r on $[x_{i,j}, x_{i,j+1}]$, $r_{ij0} = x_{i,j}$, $r_{ij(e(i,j)+1))} = x_{i,j+1}$, for $0 \leq i \leq a(1)$, $0 \leq j \leq b(i)-1$, and let $\{r_{is} : 1 \leq s \leq e(i)\}$ enumerate the zeroes of θ_r on $[x_{i,b(i)}, x_{i+1,0}]$, $r_{i0} = x_{i,b(i)}$, $r_{i(e(i)+1)} = x_{i+1}$, $0 \leq i \leq a(1)$. Then, using $(\dagger), (\dagger\dagger)$, we have that;

$\frac{1}{n}{}^* \sum_{|l|<n-1} Re(f)(\frac{l}{n})\theta_r(\frac{l}{n})$

$= \frac{1}{n}{}^* \sum_{i=0}^{a(1)} {}^* \sum_{j=0}^{b(i)-1} {}^* \sum_{s=0}^{e(i,j)} {}^* \sum_{l=[nr_{ijs}]+1}^{[nr_{ij(s+1)}]-1} Re(f)(\frac{l}{n})\theta_r(\frac{l}{n})$

$+ \frac{1}{n}{}^* \sum_{i=0}^{a(1)} {}^* \sum_{s=0}^{e(i)} {}^* \sum_{l=[nr_{is}]+1}^{[nr_{i(s+1)}]-1} Re(f)(\frac{l}{n})\theta_r(\frac{l}{n})$

We compute $\frac{1}{n}|^* \sum_{s=1}^{e(i,j)-1} {}^* \sum_{l=[nx_{ijs}]+1}^{[nx_{ij(s+1)}]-1} Re(f)(\frac{l}{n})\theta_r(\frac{l}{n})|$.

Let $\theta_{i,j}(s) = \frac{1}{n}{}^* \sum_{l=[nr_{ijs}]+1}^{[nr_{ij(s+1)}-1]} Re(f)(\frac{l}{n})\theta_r(\frac{l}{n})$

Considering Case 1 as above, we have that;

$0 \leq |^* \sum_{s=1}^{e(i,j)-1} \theta_{i,j}(s)| \leq l_{i,j}$, $(*)$

where;

$l_{i,j} = \frac{1}{n}{}^* \sum_{l=[nr_{ij1}]}^{[nr_{ij2}]} |Re(f)|(\frac{l}{n})\theta_r(\frac{l}{n})$

$\leq \frac{1}{n}{}^* \sum_{l=[nr_{ij1}]+1}^{[nr_{ij2}]-1} D$, where $|Re(f)| \leq D$, and $0 \leq \theta_r|_{[r_{ij1}, r_{ij2}]} \leq 1$

$\leq \frac{D}{n}(([nr_{ij2}]-1)-([nr_{ij1}]+1)+1)$

$= \frac{D}{n}([nr_{ij2}]-[nr_{ij1}]-1)$

$\leq \frac{D}{n}((nr_{ij2}+1)-(nr_{ij1}-1)-1)$

$= D((r_{ij2}-r_{ij1})+\frac{1}{n})$

so that, using $(*), (\sharp\sharp)$;

$0 \leq \frac{1}{n}|^{*}\sum_{s=1}^{e(i,j)-1}{}^{*}\sum_{l=[nx_{ijs}]+1}^{[nx_{ij(s+1)}]-1} Re(f)(\frac{l}{n})\theta_r(\frac{l}{n})| \leq l_{i,j}$

$\leq D((r_{ij2}-r_{ij1})+\frac{1}{n})$

$= D(\frac{1}{|r|}+\frac{1}{n})$

Similarly, for Case 2 in [5]. In the other cases, reversing the sequences, we obtain;

$\frac{1}{n}|^{*}\sum_{s=1}^{e(i,j)-1}{}^{*}\sum_{l=[nx_{ijs}]+1}^{[nx_{ij(s+1)}]-1} Re(f)(\frac{l}{n})\theta_r(\frac{l}{n})|$

$\leq D((r_{ije(i,j)}-r_{ij(e(i,j)-1)})+\frac{1}{n})$

$= D(\frac{1}{|r|}+\frac{1}{n})$

Considering all 4 cases, we obtain the same bound;

$\frac{1}{n}|^{*}\sum_{s=1}^{e(i)-1}{}^{*}\sum_{l=[nr_{is}]+1}^{[nr_{i(s+1)}]-1} Re(f)(\frac{l}{n})\theta_r(\frac{l}{n})| \leq D(\frac{1}{|r|}+\frac{1}{n})$

Similarly;

$max(A_{i,j}, B_{i,j}, C_i, D_i)$

$\leq D(\frac{1}{|r|}+\frac{1}{n})$

where;

$A_{i,j} = \frac{1}{n}|^{*}\sum_{l=[nr_{ij0}]+1}^{[nr_{ij1}]-1} Re(f)(\frac{l}{n})\theta_r(\frac{l}{n})|$

$B_{i,j} = \frac{1}{n}|^{*}\sum_{l=[nr_{ije(i,j)}]+1}^{[nr_{ij(e(i,j)+1)}]-1} Re(f)(\frac{l}{n})\theta_r(\frac{l}{n})|$

$C_i = \frac{1}{n}|^*\sum_{l=[nr_{i0}]+1}^{[nr_{i1}]-1} Re(f)(\frac{l}{n})\theta_r(\frac{l}{n})|$

$D_i = \frac{1}{n}|^*\sum_{l=[nr_{ie(i)}]+1}^{[nr_{i(e(i)+1)}]-1} Re(f)(\frac{l}{n})\theta_r(\frac{l}{n})|)$

As in [5], we have that;

$\frac{1}{n}|^*\sum_{|l|<n-1} Re(f)(\frac{l}{n})\theta_r(\frac{l}{n})|$

$\leq (w(1)+a(1)+1)3D(\frac{1}{|r|}+\frac{1}{n})$

where $w(1) = Card(Re(f)'|_{[-1,1]} = 0)$, $a(1) = Card(Re(f)|_{[-1,1]} = 0)$

It follows, by transfer, with the assumption that m is infinite and $m \leq N$, using (♯♯);

$|\int_{\overline{V}_\eta} Re(f_\eta)cos_\eta(\pi mx)d\mu_\eta(x)|$

$\simeq \frac{1}{\eta}|^*\sum_{|l|<\eta-1} Re(f^*)(\frac{l}{\eta})\theta_m(\frac{l}{\eta})|$

$\leq (w(1)+a(1)+1)3D(\frac{1}{|m|}+\frac{1}{\eta}) \simeq 0$, as $w(1), a(1)$ are finite and $\frac{1}{|m|} \simeq 0$, $\frac{1}{\eta} \simeq 0$.

and, similarly, as in [5], considering the additional terms from (♯), we have that;

$|\int_{\overline{V}_\eta} f_\eta exp_\eta(-\pi imx)d\mu_\eta(x)|$

$\simeq \frac{1}{\eta}|^*\sum_{|l|<\eta-1} Re(f^*)(\frac{l}{\eta})\theta_m(\frac{l}{\eta})|$

$\leq (w(1)+a(1)+w'(1)+a'(1)+2)6D(\frac{1}{|r|}+\frac{1}{n}) \simeq 0$

again, as $w(1), a(1), w'(1), a'(1)$ are finite and $\frac{1}{|r|} \simeq 0$, $\frac{1}{\eta} \simeq 0$

It follows that, for infinite $m \in \mathcal{Z}_\eta$, with $|m| \leq N$, that;

$\int_{\overline{V}_\eta} f_\eta exp_\eta(-\pi imx)d\mu_\eta(x) \simeq 0$

□

Remarks 0.12. *Again, it is straightforward to deduce the standard Riemann-Lebesgue Lemma;*

If $f\in C[-1,1]$, with $Re(f)$ and $Im(f)$ analytic, then;

$lim_{|m|\rightarrow\infty}\mathcal{F}(f)(m)=0$

Just follow the proof in Remark 0.6, the addition restraint that $|m|\leq N$ doesn't effect the application of underflow.

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