A NONSTANDARD APPROACH TO EQUIDISTRIBUTION IN ERGODIC THEORY

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Abstract. We develop a nonstandard approach in ergodic theory, considering the doubling map sh_d on the unit interval, which is known to be chaotic in the sense of [2]. We show that there exists a typical element for sh_d, using arguments from [6], motivated indirectly by the early work of J. Ville on probabilistic properties of binary sequences. In particular, we obtain new methods for finding equidistributed and weakly equidistributed sequences, in the sense of [3].

Definition 0.1. *We adapt slightly the notation in Definition 0.1 and Definition 0.2 of* [4]. *As in Definition 0.1, we let* $\eta = 2^\nu$, *for* $\nu \in {}^*\mathcal{N} \setminus \mathcal{N}$, $(\overline{\Omega_\eta}, \mathcal{C}_\eta, \mu_\eta)$ *the corresponding* $*$*-finite measure space, and* $(\overline{\Omega_\eta}, L(\mathcal{C}_\eta), L(\mu_\eta))$ *the associated Loeb space. We denote by* $\leq$ *the usual ordering on* $\overline{\Omega_\eta}$, *induced by* ${}^*[0,1)$. *We let* Ω_ν *be as in Definition 0.2, but* C_ν *the set of sequences of length* ν, *consisting of* 0*'s and* 1*'s. For* ν *even, we let* $B_\nu = \{\bar{t} \in C_\nu : \sum_{j=1}^{\nu} \bar{t}(j) = \frac{\nu}{2}\}$. *Observe that, given* $\{x, y\} \subset \overline{\Omega_\eta}$, *we have that* $x \leq y$ *iff* $\frac{[\eta x]}{\eta} \leq \frac{[\eta y]}{\eta}$ *iff* $[\eta x] \leq [\eta y]$ *in* Ω_ν. *We let* $\theta_\nu : \Omega_\nu \rightarrow C_\nu$ *associate* $i \in \Omega_\nu$ *with its binary representation. We define the right shift map* $sh_{\nu,r} : C_\nu \rightarrow C_\nu$, *by setting;*

$sh_{\nu,r}(\bar{t})(j) = \bar{t}(j-1)$, *for* $2 \leq j \leq \nu$

$sh_{\nu,r}(\bar{t})(1) = \bar{t}(\nu)$

Letting $sh'_{\nu,r} = (\theta_\nu^{-1} \circ sh_{\nu,r} \circ \theta_\nu) : \Omega_\nu \rightarrow \Omega_\nu$, *we have that;*

$sh'_{\nu,r}(\alpha) = 2\alpha$, *if* $\alpha \in \Omega_{\nu-1}$

$sh'_{\nu,r}(\alpha) = 2(\alpha - 2^{\nu-1}) + 1$, *otherwise*

We define the associated forward shift $sh_{\nu,f} : \overline{\Omega_\eta} \rightarrow \overline{\Omega_\eta}$ *by setting;*

$sh_{\nu,f}(\frac{i}{\eta}) = \frac{sh'_{\nu,r}(i)}{\eta}$

$sh_{\nu,f}(x) = sh_{\nu,f}(\frac{[x\eta]}{\eta}) + (x - \frac{[x\eta]}{\eta})$

If $r \in {}^*\mathcal{N} \cap [1,\nu]$, we define;

$C_{\nu,r} = \{\bar{t} \in C_\nu : \bar{t}(\nu - j) = 0, 0 \leq j \leq r-1\}$

with corresponding initial segment $\Omega_{\nu-r} \subset \Omega_\nu$.

Lemma 0.2. *If $r \in {}^*\mathcal{N} \cap [1,\nu]$, we have that;*

$Im(sh_{\nu,r}|_{C_{\nu-r}}) \subset C_{\nu-r+1}$

Proof. □

Lemma 0.3. *The forward shift map $sh_{\nu,f}$ is measurable and measure preserving.*

Proof. Observe that, if $A \in \mathcal{C}_\eta$, we have $A = \bigcup_{i\in I}[\frac{i}{\eta}, \frac{i+1}{\eta})$, with $I \subset \Omega_\nu$ internal. Then;

$sh_{\nu,f}^{-1}(A) = \bigcup_{i\in I} sh_{\nu,f}^{-1}([\frac{i}{\eta}, \frac{i+1}{\eta}))$

$= \bigcup_{i\in I}([\frac{sh_{\nu,r}^{-1}(i)}{\eta}, \frac{sh_{\nu,r}^{-1}(i+1)}{\eta}))$

$= \bigcup_{i\in sh_{\nu,r}^{-1}(I)}[\frac{i}{\eta}, \frac{i+1}{\eta})$

As $sh_{\nu,r}^{-1}(I) \subset \Omega_\nu$ is internal, we obtain that $sh_{\nu,f}^{-1}(A) \in \mathcal{C}_\eta$.
Moreover;

$\mu_\eta(sh_{\nu,f}^{-1}(A)) = \frac{{}^*Card(sh_{\nu,r}^{-1}(I))}{\eta} = \frac{{}^*Card(I)}{\eta} = \mu_\eta(A)$ $(*)$

as $sh_{\nu,r}$ is a bijection. Letting $\mathfrak{C}_\eta$ denote the σ-algebra generated by $\mathcal{C}_\eta$, we have that $sh_{\nu,f}^{-1}(\mathfrak{C}_\eta)$ is also a σ-algebra generating $\mathcal{C}_\eta$, as $sh_{\nu,f}$ is $\mathcal{C}_\eta$-measurable, hence, $sh_{\nu,f}^{-1}(\mathfrak{C}_\eta) = \mathfrak{C}_\eta$, and $sh_{\nu,f}$ is $\mathfrak{C}_\eta$-measurable. A similar argument with the completion $L(\mathcal{C}_\eta)$ of $\mathfrak{C}_\eta$, shows that $sh_{\nu,f}$ is $L(\mathcal{C}_\eta)$-measurable. We have, using Theorem 3.4, see also Lemma 3.15, of [5], that, if $B \in L(\mathcal{C}_\eta)$, and $\epsilon > 0$ there exist $\{A, C\} \subset \mathcal{C}_\eta$, such that $A \subset B \subset C$, and $max(\mu_{\eta,ext}(B \setminus A), \mu_{\eta,ext}(C \setminus B)) < \epsilon$. We have that $sh_{\nu,f}^{-1}(A) \subset sh_{\nu,f}^{-1}(B) \subset sh_{\nu,f}^{-1}(C)$. Hence, using $(*)$, we obtain that

$\mu_{\eta,ext}(B)-\epsilon<\mu_{\eta,ext}(sh_{\nu,f}^{-1}(B))<\mu_{\eta,ext}(B)+\epsilon$. As ϵ was arbitrary, we have that $\mu_{\eta,ext}(sh_{\nu,f}^{-1}(B))=\mu_{\eta,ext}(B)$, as required. □

Lemma 0.4. *If $x\in\overline{\Omega_\eta}$, then, if $0\leq x<\frac{1}{2}$, $sh_{\nu,f}(\frac{[\eta x]}{\eta})=\frac{2[\eta x]}{\eta}$, and, if $\frac{1}{2}\leq x<1$, $sh_{\nu,f}(\frac{[\eta x]}{\eta})=2x(mod1)+\frac{1}{\eta}$.*

Proof. If $0\leq x<\frac{1}{2}$, then $[\eta x]\in\Omega_{\nu-1}$, and, using Definition 0.1 (make lemma?), $sh_{\nu,f}(\frac{[\eta x]}{\eta})=\frac{2[\eta x]}{\eta}$. If $\frac{1}{2}\leq x<1$, then $[\eta x]=2^{\nu-1}+y$, with $y\in\Omega_{\nu-1}$, then;

$$sh_{\nu,f}(\frac{[\eta x]}{\eta})=\frac{2y+1}{\eta}=\frac{2([\eta x]-2^{\nu-1})+1}{\eta}=2(\frac{[\eta x]}{\eta})(mod1)+\frac{1}{\eta}$$

□

Definition 0.5. *We let $([0,1),\mathcal{B},\mu)$ denote the unit interval, with endpoints identified, and the completion of the Borel field and Lebesgue measure. Recall from [1], that;*

$$st:(\overline{\Omega_\eta},L(\mathcal{C}_\eta),L(\mu_\eta))\rightarrow([0,1),\mathcal{B},\mu)$$

is measurable and measure preserving, with the convention that the endpoints are identified. We define the doubling map $sh_d:[0,1)\rightarrow[0,1)$ by;

$$sh_d(x)=2x(mod1)$$

Lemma 0.6. *The doubling map sh_d is measurable and measurable preserving. Moreover, if $x'\in[0,1)$ and $x\in\mathcal{V}_{x'}\cap\overline{\Omega_\eta}$, then $sh_d(x')={}^\circ sh_{\nu,f}(x)$.*

Proof. The first part is a standard result. If $x'\in[0,\frac{1}{2})$, and ${}^\circ x=x'$, then $0\leq x<\frac{1}{2}$, $0\leq\frac{[\eta x]}{\eta}<\frac{1}{2}$, and;

$${}^\circ sh_{\nu,f}(x)={}^\circ(2\frac{[\eta x]}{\eta}+(x-\frac{[\eta x]}{\eta}))=2x=sh_d(x')$$

If $x'\in(\frac{1}{2},1)$, and ${}^\circ x=x'$, then $\frac{1}{2}<x<1$, $\frac{1}{2}<\frac{[\eta x]}{\eta}<1$, and;

$${}^\circ sh_{\nu,f}(x)={}^\circ(2\frac{[\eta x]}{\eta}(mod1)+\frac{1}{\eta}+(x-\frac{[\eta x]}{\eta}))=2x(mod1)$$

If $x'=\frac{1}{2}$, we have, if $x=\frac{1}{2}-\delta$, $\delta\simeq 0$, $\delta>0$;

$${}^\circ sh_{\nu,f}(x)={}^\circ(2\frac{[\eta(\frac{1}{2}-\delta)]}{\eta}+(\frac{1}{2}-\delta-\frac{[\eta(\frac{1}{2}-\delta)]}{\eta}))=0=sh_d(x')$$

and, if $x=\frac{1}{2}+\delta$, $\delta\simeq 0$, $\delta\geq 0$;

$${}^{\circ}sh_{\nu,f}(x)={}^{\circ}(2\frac{[\eta(\frac{1}{2}-\delta)]}{\eta}(mod1)+\frac{1}{\eta}+(\frac{1}{2}+\delta-\frac{[\eta(\frac{1}{2}+\delta)]}{\eta}))=0=sh_d(x')$$

as required.

□

Definition 0.7. *We let $D_\infty=\{0,1\}^{\mathcal{N}}$ denote the set of infinite sequences of 0's and 1's, and let $\overline{t}_1$ be defined by $\overline{t}_1(i)=1$, for $i\in\mathcal{N}$. We say that $\overline{t}$ is permitted, if $\overline{t}=\overline{t}_1$, or $\overline{t}\neq\overline{t}_1$, and there does not exist $i_0\in\mathcal{N}$, with $\overline{t}(i_0)=0$, and $\overline{t}(i_0+i)=1$, for $i\in\mathcal{N}$. We let $E_\infty\subset D_\infty$ be the set of permitted sequences. We define $\gamma:[0,1]\rightarrow E_\infty$ by;*

$$\gamma(x)(1)=0,\ \text{if } x\neq 1,\ x\in[0,\tfrac{1}{2}),\gamma(x)(1)=1,\ \text{if } x\in[\tfrac{1}{2},1)$$

$$\gamma(x)(i+1)=0,\ \text{if } x\neq 1,\ x-\sum_{k=1}^{i}\gamma(x)(k)2^{-k}\in[0,\tfrac{1}{2^{i+1}}),\ i\in\mathcal{N}$$

$$\gamma(x)(i+1)=1,\ \text{if } x\neq 1,\ x-\sum_{k=1}^{i}\gamma(x)(k)2^{-k}\in[\tfrac{1}{2^{i+1}},\tfrac{1}{2^{i}}),\ i\in\mathcal{N}$$

$$\gamma(1)(i)=1,\ \text{if } i\in\mathcal{N}$$

We let $\gamma:[0,1)\rightarrow E_\infty$, associate $x\in[0,1)$ with its decimal binary representation. We define the left shift $sh_l:E_\infty\rightarrow E_\infty$ by $(sh_l(\overline{t}))(i)=\overline{t}(i+1)$, for $i\in\mathcal{N}$. We let $T:D_\infty\rightarrow C_\nu$ associate the standard sequence $\{\overline{w}(i):i\in\mathcal{N}\}$ with its transfer $\{\overline{w}(i):i\in{}^*\mathcal{N}\cap[1,\nu]\}$, let $rev:C_\nu\rightarrow C_\nu$ be a sequence reversal $rev(\overline{w})(i)=\overline{w}(\nu-i+1)$, and $S=(rev\circ T)$. Let $res:C_\nu\rightarrow D_\infty$ be the restriction $res(\overline{w})(i)=\overline{w}(\nu-i+1)$, for $i\in\mathcal{N}$.

Lemma 0.8. *If $x\in[0,1)$, we have that;*

$$sh_d(x)=(\gamma^{-1}\circ sh_l\circ\gamma)(x)$$

$$\frac{(\theta_\nu^{-1}\circ S\circ\gamma)(x)}{\eta}\simeq x$$

$$(\gamma\circ sh_d)(x)=(res\circ sh_{\nu,r}\circ S\circ\gamma)(x)=res\circ sh_{\nu,r}\circ\theta_\nu([\eta x])$$

Proof. Let $x=\sum_{n=1}^{\infty}a_n2^{-n}$, then, if $x\in[0,\frac{1}{2})$, $a_1=0$, $sh_d(x)=2x=\sum_{n=2}^{\infty}a_n2^{-n+1}=\sum_{n=1}^{\infty}a_{n+1}2^{-n}$, and, if $x\in[\frac{1}{2},1)$, $a_1=1$, $sh_d(x)=2x-1=\sum_{n=1}^{\infty}a_n2^{-n+1}-1=\sum_{n=2}^{\infty}a_n2^{-n+1}=\sum_{n=1}^{\infty}a_{n+1}2^{-n}$. Hence, $sh_d(x)=(\gamma^{-1}\circ sh_l\circ\gamma)(x)$. Let $s_n(x)=\sum_{i=1}^{n}a_i2^{-i}$, then $|s_n(x)-$

$x| \leq \sum_{i=n+1}^{\infty} a_i 2^{-i} \leq \frac{1}{2^n}$. Using Theorem 3.4(i) of [5], we have that $|s_\eta(x) - x| \simeq 0$, where $\eta s_\eta(x) = (\theta_\nu^{-1} \circ S \circ \gamma)(x)$. The fact that $(\gamma \circ sh_d)(x) = (res \circ sh_{\nu,r} \circ S \circ \gamma)(x)$ follows immediately from the definitions of $sh_d, \gamma, sh_{\nu,r}$ and res in Definition 0.7. □

Remarks 0.9. *The doubling map sh_d^2 has the horseshoe property in the sense that there exist disjoint closed subintervals $\{[0,\frac{1}{4}],[\frac{1}{2},\frac{3}{4}]\} \subset P([0,1))$, with $sh_d^2([0,\frac{1}{4}]) = sh_d^2([\frac{1}{2},\frac{3}{4}]) = [0,1)$, endpoint preserving. One can relax the property, in the sense that there exist disjoint subintervals $\{[0,\frac{1}{2}),[\frac{1}{2},1)\} \subset P([0,1))$, with $sh_d([0,\frac{1}{2})) = sh_d([\frac{1}{2},1)) = [0,1)$, endpoint preserving. Using this idea, or directly, using the decimal representation, one can show that sh_d^n has at least 2^n fixed points, sh_d has periodic points of every minimal period and sh_d has uncountably many non periodic points.*

Definition 0.10. *If $x \in \overline{\Omega_\eta}$, we say that x is periodic with period $r \in {}^*\mathcal{N} \cap [1,\nu]$, if r is minimal in ${}^*\mathcal{N} \cap [1,\nu]$, with the property that $sh_{\nu,f}^r(x) = x$. If x is periodic, with period r, we define the associated probability measure ρ_x on Ω_η, by;*

$$\rho_x = \frac{1}{r}({}^*\textstyle\sum_{0\leq j\leq r-1} \delta_{sh_{\nu,f}^j x})$$

and corresponding Loeb measure $L(\rho_x)$, where, for $y \in \overline{\Omega_\eta}$, δ_y is the point measure with respect to $\mathcal{C}_\eta$, supported on y. If $\{x_n : n \in \mathcal{N}\} \subset \overline{\Omega_\eta}$, we say that $lim_{w,n\to\infty} L(\rho_{x_n}) = L(\mu_\eta)$, if, for every $A \in \mathcal{C}_\eta$;

$$lim_{n\to\infty} L(\rho_{x_n})(A) = L(\mu_\eta)(A).$$

Remarks 0.11. *Observe that every element $x \in \overline{\Omega_\eta}$ has a period $1 \leq r \leq \nu$, and, if $sh_{\nu,f}^t(x) = x$, with $t \in {}^*\mathcal{N}$, $t \geq 1$, then $r|t$. This is an elementary consequence, by transfer and Definition 0.1, of the corresponding result for finite sets. Observe that, if $r \in \mathcal{N}$ is finite, and x has period r then;*

$$L(\rho_x) = \frac{1}{r}(\textstyle\sum_{0\leq j\leq r-1} L(\delta_{sh_{\nu,f}^j x}))$$

and, for $A \in \mathcal{C}_\eta$, $L(\rho_x)(A) = \rho_x(A)$. Using Theorem 3.4(ii) of [5]*, if $lim_{w,n\to\infty} L(\rho_{x_n}) = L(\mu_\eta)$, then, for* every *$A \in L(\mathcal{C}_\eta)$, $lim_{n\to\infty} L(\rho_{x_n})(A) = L(\mu_\eta)(A)$.*

Lemma 0.12. *If $\{x_n : n \in \mathcal{N}\} \subset \overline{\Omega_\eta}$, with $r_n \in \mathcal{N}$, such that the weak limit $lim_{w,n\to\infty} L(\rho_{r_n}) = L(\mu_\eta)$, then, for any $\kappa \in {}^*\mathcal{N}$ infinite, we have that, for the internal sequences $\{x_n\}_{1\leq n\leq\kappa}$, $\{r_n\}_{1\leq n\leq\kappa}$, $\{\rho_{x_n}\}_{1\leq n\leq\kappa}$, and*

$f \in V(\overline{\Omega_\eta})$, $|f| \leq C$, with $C \in \mathcal{R}$;

$$\int_{\overline{\Omega_\eta}} f d\mu_\eta \simeq \frac{1}{r_\kappa}{}^*\sum_{0\leq j\leq r_\kappa-1} f(sh^j_{\nu,f}x_\kappa) \ (\dagger)$$

and the sequence $\{sh^j_{\nu,f}x_\kappa\}_{0\leq j\leq r_\kappa-1}$ is weakly equidistributed, in the sense of [3], *($\dagger\dagger$), ([1]), (change δ notation in* [3]*).*

Proof. By the hypotheses on f, $f \in SL^1(\overline{\Omega_\eta}, \mu_\eta)$, hence, ${}^\circ f \in L^1(\overline{\Omega_\eta}, L(\mu_\eta))$, and;

$$\int_{\overline{\Omega_\eta}} f d\mu_\eta \simeq \int_{\overline{\Omega_\eta}} {}^\circ f dL(\mu_\eta), \ (***)$$

As $|f| \leq C$, we have that $f \in SL^1(\overline{\Omega_\eta}, \rho_{x_n})$, for $1 \leq n \leq \kappa$, $(*)$. Observe that for $A \in \mathcal{C}_\eta$, and $n \in \mathcal{N}$, $L(\rho_{r_n})(A) = \rho_{r_n}(A)$. Hence, $lim_{n\to\infty}\rho_{r_n}(A) = lim_{n\to\infty}L(\rho_{r_n})(A) = L(\mu_\eta)(A)$. Then, by Lemma 2.22(i) of [5], we have that $\rho_{r_\kappa}(A) \simeq L(\mu_\eta(A))$, hence $L(\rho_{r_\kappa})(A) = L(\mu_\eta)(A)$, and $L(\rho_{r_\kappa}) = L(\mu_\eta)$, $(**)$, by Theorem 3.4(ii) of [5]. By $(*)$, $(**)$, $(***)$;

$$\frac{1}{r_\kappa}{}^*\sum_{0\leq j\leq r_\kappa-1} f(sh^j_{\nu,f}x_\kappa) = \int_{\overline{\Omega_\eta}} f d\rho_{r_\kappa}$$

$$\simeq \int_{\overline{\Omega_\eta}} {}^\circ f dL(\rho_{r_\kappa})$$

$$= \int_{\overline{\Omega_\eta}} {}^\circ f dL(\mu_\eta)$$

$$\simeq \int_{\overline{\Omega_\eta}} f d\mu_\eta, \ (****)$$

In particularly, the sequence $\{sh^j_{\nu,f}x_\kappa\}_{0\leq j\leq \kappa-1}$ is weakly equidistributed, as, if $\{a,b\} \subset [0,1)$, ${}^*[a,b] \in \mathcal{C}_\eta$, and we can apply $(****)$ to $\chi_{[a,b]} \in V(\overline{\Omega_\eta})$, with $|\chi_{[a,b]}| \leq 1$. □

Definition 0.13. *If $x \in [0,1)$, we say that x is periodic with period $r \in \mathcal{N}$, if r is minimal in $\mathcal{N}$, with the property that $sh^r_d(x) = x$. Similarly to the above, if x is periodic, with period r, we define the associated probability measure ρ_x on $[0,1)$, by;*

$$\rho_x = \frac{1}{r}\left(\sum_{0\leq j\leq r-1} \delta_{sh^j_d x}\right)$$

[1] We slightly extend the sense of weakly equidistributed, to mean that $L(\rho_{x_\kappa})(a,b) = b - a$ for $\{a,b\} \subset \mathcal{R}$. Observe that ρ_{x_κ} is defined in this paper relative to $\mathcal{C}_\eta$ rather than $\mathcal{C}_{r_\kappa}$. The nonstandard analogue of Weyl's criterion may be more difficult to prove, though, when $r_\kappa \neq \eta$. The reader should check that the proof below still goes through using the definition with respect to $\mathcal{C}_{r_\kappa}$.

where, for $y \in [0,1)$, δ_y is the point measure with respect to $\mathfrak{B}$, supported on y. More generally, if $x \in [0,1)$ and $n \in \mathcal{N}$, we define the associated probability measure $\rho_{x,n}$ on $[0,1)$, by;

$\rho_{x,n} = \frac{1}{n}(\sum_{0 \leq j \leq n-1} \delta_{sh_d^j x})$

If $m \in \mathcal{N}$, $x \in [0,1)$ is periodic with respect to sh_d^m, with period r, we define the associated probability measure $\rho_{m,x}$ on $[0,1)$, by;

$\rho_{m,x} = \frac{1}{r}(\sum_{0 \leq j \leq r-1} \delta_{sh_d^{mj} x})$

and, if $x \in [0,1)$ and $n \in \mathcal{N}$, the associated probability measure $\rho_{m,x,n}$ is defined on $[0,1)$, by;

$\rho_{m,x,n} = \frac{1}{n}(\sum_{0 \leq j \leq n-1} \delta_{sh_d^{mj} x})$

We define $C([0,1)) = \{g|_{[0,1)} : g \in C([0,1])\}$. Given a sequence of complete probability measures $\{\mu_n : n \in \mathcal{N}\}$, on $([0,1),\mathfrak{B})$, we say that $lim_{w,n\rightarrow\infty}\mu_n = \mu$, if, for every $g \in C([0,1))$;

$lim_{n\rightarrow\infty} \int_{[0,1)} g d\mu_n = \int_{[0,1)} g d\mu$

If $x \in [0,1)$, and $m \in \mathcal{N}$, we say that x is typical for sh_d^m, if, $lim_{w,n\rightarrow\infty}\rho_{m,x,n} = \mu$.

Lemma 0.14. *If $\{\mu_n\}_{n\in\mathcal{N}}$ is a sequence of complete probability measures on $([0,1),\mathfrak{B})$, then $lim_{w,n\rightarrow}\mu_n = \mu$ iff for every $A \in \mathfrak{B}$, $lim_{n\rightarrow\infty}\mu_n(A) = \mu(A)$.*

Proof. Suppose that, for every $A \in \mathfrak{B}$, $lim_{n\rightarrow\infty}\mu_n(A) = \mu(A)$. Let $g \in C([0,1)$, and $\epsilon > 0$, then we can find $g_\epsilon = \sum_{k=1}^N \lambda_k \chi_{A_k}$, $A_k \in \mathfrak{B}$, $1 \leq k \leq N$, with $\lambda_k \in \mathcal{C}$, such that, for $x \in [0,1)$, $|g(x) - g_\epsilon(x)| < \epsilon$. We have, for $n \in \mathcal{N}$, with $max_{1\leq k\leq N}(\mu_n(A_k) - \mu(A_k)) \leq \frac{max_{1\leq k\leq N}(\lambda_k)\epsilon}{N}$, that;

$|\int_{[0,1)} g d\mu - \int_{[0,1)} g d\mu_n|$

$\leq |\int_{[0,1)}(g - g_\epsilon)d\mu| + |\int_{[0,1)}(g - g_\epsilon)d\mu_n| + |\int_{[0,1)} g_\epsilon d\mu - \int_{[0,1)} g_\epsilon d\mu_n|$

$\leq 2\epsilon + \sum_{k=1}^N |\lambda_k||\mu(A_k) - \mu_n(A_k)| \leq 3\epsilon$

As ϵ was arbitrary, it follows that $lim_{w,n\rightarrow}\mu_n = \mu$. Conversely, suppose that $lim_{w,n\rightarrow}\mu_n = \mu$, and let $A \in \Sigma_0$, where Σ_0 consists of finite unions of sets of the form $[a,b) \subset [0,1)$. Given $\epsilon > 0$, we can find $g_{A,\epsilon} \in C([0,1))$, such that $|\chi_A - g_{A,\epsilon}| < \epsilon$. Then, for $n \in \mathcal{N}$, with $|\int_{[0,1)} g_{A,\epsilon} d\mu - \int_{[0,1)} g_{A,\epsilon} d\mu_n| < \epsilon$, we have;

$|\int_{[0,1)} \chi_A d\mu - \int_{[0,1)} \chi_A d\mu_n|$

$\leq |\int_{[0,1)}(\chi_A - g_{A,\epsilon})d\mu| + |\int_{[0,1)}(\chi_A - g_{A,\epsilon})d\mu_n| + |\int_{[0,1)} g_{A,\epsilon} d\mu - \int_{[0,1)} g_{A,\epsilon} d\mu_n|$

$\leq 3\epsilon$

Hence, as ϵ was arbitrary;

$\mu(A) = \int_{[0,1)} \chi_A d\mu = lim_{n\rightarrow\infty} \chi_A d\mu_n = lim_{n\rightarrow\infty} \mu_n(A)$

It follows that $lim_{n\rightarrow\infty}\mu_n = \mu$ on Σ_0, hence, by Caratheodery's Lemma, $lim_{n\rightarrow\infty}\mu_n = \mu$ as required.

□

Lemma 0.15. *Let $\{x_n\}_{n\in\mathcal{N}}$ be a sequence of periodic elements in $[0,1)$, such that $lim_{w,n\rightarrow\infty}\rho_{x_n} = \mu$, then there exists $x \in [0,1)$, which is typical for sh_d.*

Proof. By Definition 0.13, using the fact that $\mu(\{1\}) = 0$, we can replace $[0,1)$ by $[0,1]$. We can now follow through the proof of Lemma 1.11 in [6], replacing the left shift σ on $[0,1]^{\mathcal{N}}$ by the left shift sh_l on D_∞. □

Lemma 0.16. *Suppose that, for every $f \in C([0,1))$ and $\epsilon > 0$, there exists a periodic element $x_\epsilon \in [0,1)$, such that;*

$|\int_{[0,1]} f d\mu - \int_{[0,1]} f d\rho_{x_\epsilon}| < \epsilon$

then there exists a sequence of periodic elements $\{x_n\}_{n\in\mathcal{N}}$ in $[0,1)$, such that $lim_{w,n\rightarrow\infty}\rho_{x_n} = \mu$

Proof. The proof is similar to Lemma 1.12 in [6], using the Stone-Weierstrass Theorem for $[0,1]$, instead of $[0,1]^n$, for $n \in \mathcal{N}$. We obtain a countable base for $\Omega_\mu \subset \mathcal{M}$, where $\mathcal{M}$ is the set of regular real valued measures on $[0,1]$, and Ω_μ is the set of open sets containing the measure μ. □

Definition 0.17. *Given $n \in \mathcal{N}$, we define $\overline{j}_{1,n} \in 2^n$, by $\overline{j}_{1,n}(i) = 1$, for $1 \leq i \leq n$. If $\overline{j} \in 2^n$, we let $x_{\overline{j}} = \sum_{k=1}^{n} \overline{j}(k)2^{-k}$, and;*

$B_{\overline{j},n} = \{x \in [0,1] : 0 \leq x - x_{\overline{j}} < \frac{1}{2^n}$, *if* $\overline{j} \neq \overline{j}_{1,n}$

$B_{\overline{j}_{1,n},n} = \{x \in [0,1] : 0 \leq x - x_{\overline{j}_{1,n}} \leq \frac{1}{2^n}$

Lemma 0.18. *Let* $\epsilon > 0$, $g \in C([0,1])$, *and a periodic element* $x \in [0,1)$ *be given, then there exists* $n \in \mathcal{N}$, *and* $\delta > 0$, *such that if* $|\rho_x(B_{\overline{j},n}) - \mu(B_{\overline{j},n})| < \delta$, *for all* $\overline{j} \in 2^n$, *then;*

$$| \int_{[0,1)} g d\rho_x - \int_{[0,1)} g d\mu| < \epsilon$$

Proof. The proof is a simple adaptation of Lemma 1.14 in [6]. □

Lemma 0.19. *Given* $n \in \mathcal{N}$, *and* $\delta > 0$, *there exists a periodic element* $x_\delta \in [0,1)$, *such that;*

$$|\rho_{x_\delta}(B_{\overline{j},n}) - \mu(B_{\overline{j},n})| < \delta$$

for every $\overline{j} \in 2^n$.

Proof. Again, the proof uses Theorem 1.15 of [6]. After defining κ, using $(\mu, [0,1])$, the conditions $(i) - (iii)$ hold for κ. In the graph theory argument, with $\Sigma = \{0,1\}$, observe that, in the longest sequence $(\xi^1, \ldots, \xi^{r-1})$ of elements from Σ^n with the properties (1) and (2), and $\xi^r = \xi^0$, if β is not permitted, then $\beta(i) = 1$, for $0 \leq i \leq n-r-2$, and, therefore $\xi_i^j = 1$, for $1 \leq j \leq r-1$, $0 \leq i \leq n-1$. Such a sequence is stationary, $\xi^j = \xi^1$, for $1 \leq j \leq r-1$, and can clearly be extended to a longer sequence, by taking a new sequence;

$$\pi^1 = \xi^1,\ \pi_{n-k}^{1+k} = 0,\ \pi_j^{1+k} = 1, \text{ for } j \neq n-k, \text{ for } 1 \leq k \leq\leq n$$

$$\pi^{1+k} = \xi^1, \text{ for } n+1 \leq k \leq n+r-1$$

satisfying $(1), (2)$, $\pi^1 = \pi^{n+r}$.

It follows that β is permitted, and corresponds to $x_\delta \in [0,1)$. □

Theorem 0.20. *There exists* $x \in [0,1)$, *which is typical for* sh_d, *the sequence* $\{sh_d^{j-1}(x)\}_{j\in\mathcal{N}}$ *is equidistributed. Moreover, for all* $\nu \in {}^*\mathcal{N}$ *infinite, if* $\overline{t} \in \{0,1\}^{*\mathcal{N}}$ *transfers* $\gamma(x)$, $y = \frac{\theta_\nu^{-1}(S(\overline{t}))}{\eta}$, *then* $\{{}^*sh_d^{j-1}(x)\}_{1\leq j\leq\nu}$ *and* $\{sh_{\nu,f}^{j-1}(y)\}_{1\leq j\leq\nu}$ *are weakly equidistributed in the sense of* [3].

Proof. The first part follows by combining Lemmas 0.15, 0.16, 0.18 and 0.19. The second part is a simple adaptation of Lemma 0.14. The third claim follows from Remarks 0.3 of [3]. For the final part, let $m \in \mathcal{N}$, $a = \frac{k}{2^m}$, with associated decimal binary expansion given by $(a_1, \ldots, a_m)$, $b = \frac{k+1}{2^m}$, where $0 \leq k \leq 2^m - 1$. By the second part and Lemma 0.8, we have that;

$lim_{n\rightarrow\infty}\frac{1}{n}Card(\{j \in \mathcal{N} \cap [1,n] : sh_d^{j-1} \in [a,b)\})$

$= lim_{n\rightarrow\infty}\frac{1}{n}Card(\{j \in \mathcal{N} \cap [1,n] : sh_l^{j-1}(\gamma(x))(i) = a_i, 1 \leq i \leq m\})$

$= b - a = \frac{1}{2^m}$

Therefore, by Lemma 2.22(i) of [5], we obtain;

$\frac{1}{2^m} \simeq \frac{1}{\nu}{}^*Card(\{j \in {}^*\mathcal{N} \cap [1,\nu] : sh_l^{j-1}(\bar{t})(i) = a_i, 1 \leq i \leq m\})$

$\simeq \frac{1}{\nu-m}{}^*Card(\{j \in {}^*\mathcal{N} \cap [1,\nu-m] : sh_l^{j-1}(\bar{t})(i) = a_i, 1 \leq i \leq m\})$, ([2])

$= \frac{1}{\nu-m}{}^*Card(\{j \in {}^*\mathcal{N} \cap [1,\nu-m] : sh_{\nu,r}^{j-1}(S(\bar{t}))(\nu-i+1) = a_i, 1 \leq i \leq m\})$

$\simeq \frac{1}{\nu}{}^*Card(\{j \in {}^*\mathcal{N} \cap [1,\nu] : sh_{\nu,r}^{j-1}(S(\bar{t}))(\nu-i+1) = a_i, 1 \leq i \leq m\})$, ([3])

It follows that;

$\frac{1}{\nu}{}^*Card(\{j \in {}^*\mathcal{N} \cap [1,\nu] : sh_{\nu,f}^{j-1}(y) \in [a, b - (\frac{1}{2})^{\nu-2m})\}) \simeq \frac{1}{2^m}$, $(*)$

We claim that, for a given $x \in \overline{\Omega}_\eta$;

$\frac{1}{\nu}{}^*Card(\{j \in {}^*\mathcal{N} \cap [1,\nu] : sh_{\nu,f}^{j-1}(y) \in [\frac{[\eta x]}{\eta}, \frac{[\eta x]+1}{\eta})\}) \simeq 0$, $(**)$

In order to se this, using $(*)$, choose sequences $\{a_m\}_{m\in\mathcal{N}}$, $\{b_m\}_{m\in\mathcal{N}}$, with $[\frac{[\eta x]}{\eta}, \frac{[\eta x]+1}{\eta}) \subset [a_m, b_m - (\frac{1}{2})^{\nu-2m}) \subset \overline{\Omega}_\eta$, $\{a_m 2^m, b_m 2^m\} \subset \mathcal{N}$, and

[2] For infinite κ and finite m, letting $S_\kappa = \frac{1}{\kappa}{}^*Card(\{j \in {}^*\mathcal{N} \cap [1,\kappa] : sh_l^{j-1}(\bar{t})(i) = a_i, 1 \leq i \leq m\})$, $S_{m,\kappa} = \frac{1}{m}{}^*Card(\{j \in {}^*\mathcal{N} \cap [\kappa-m+1,\kappa] : sh_l^{j-1}(\bar{t})(i) = a_i, 1 \leq i \leq m\})$, by the law of weighted averages, we have that $S_{\nu-m} = \frac{1}{\nu-m}(\nu S_\nu - m S_{m,\nu}) \simeq S_\nu$, as $\frac{\nu}{\nu-m} \simeq 1$.

[3] Using a similar argument to footnote 2

$\mu_\nu([a_m, b_m - (\frac{1}{2})^{\nu-2m})) \simeq \frac{1}{2^m}$. As the associated measure ρ_y is monotonic and $(\frac{1}{2})^{\nu-2m} \simeq 0$, for $m \in \mathcal{N}$, we obtain $(**)$. Combining $(*)$ and $(**)$, we have that;

$\frac{1}{\nu}{}^*Card(\{j \in {}^*\mathcal{N} \cap [1,\nu] : sh_{\nu,f}^{j-1}(y) \in [a,b)\}) \simeq \frac{1}{2^m}$, $(***)$

It follows, from $(***)$, that if $\mathcal{B}' \subset L(\mathcal{C}_\eta)$ is the subalgebra generated by finite unions of intervals, having the form $[a_{j,m}, a_{j+1,m})$, with $a_{j,m} = \frac{j}{2^m}$, $a_{j+1,m} = \frac{j+1}{2^m}$, for $0 \leq j \leq 2^m - 1$, then $L(\rho_y)|_{\mathcal{B}'} = L(\mu_\eta)|_{\mathcal{B}'}$.By Caratheodery's Theorem, $L(\rho_y)|_{\sigma(\mathcal{B}')} = L(\mu_\eta)|_{\sigma(\mathcal{B}')}$. In particular, for $\{a,b\} \subset \mathcal{R}$, we have $L(\rho_y)([a,b)) = b-a$, and $\rho_y([a,b)) \simeq b-a$. Hence, we obtain the final result.

□

Lemma 0.21. *For every (all) $m \in \mathcal{N}$, there exists $x_m \in [0,1)$ (x), which is typical for sh_d^m, the sequence $\{sh_d^{m(j-1)}(x_m)\}_{j\in\mathcal{N}}$ is equidistributed. Moreover, for all $\nu \in {}^*\mathcal{N}$ infinite, if $\bar{t} \in \{0,1\}^{{}^*\mathcal{N}}$ transfers $\gamma(x)$, $y = \frac{\theta_\nu^{-1}(S(\bar{t}))}{\eta}$, then $\{{}^*sh_d^{m(j-1)}(x)\}_{1\leq j\leq\nu}$ and $\{sh_{\nu,f}^{m(j-1)}(y)\}_{1\leq j\leq\nu}$ are weakly equidistributed in the sense of* [3].

Proof. Again, the first part follows by combining versions of Lemmas 0.15, 0.16, 0.18 and 0.19. For Lemma 0.15. Let $\{\alpha_n\}_{n\in\mathcal{N}}$ be a sequence of periodic elements with respect to sh_d^m, with period c_n, and corresponding decimal binary expansions $\{\gamma(\alpha_n)\}_{n\in\mathcal{N}}$, such that $lim_{w,n\rightarrow\infty}\rho_{m,\alpha_n} = \mu$. The sequences $\{\tau_n\}_{n\in\mathcal{N}}$, $\tau_n \in [0,1)^{\mathcal{N}}$, defined by $\Phi(\alpha_n)(j) = \tau_n(j) = \sum_{k=mj}^{m(j+1)-1}\gamma(\alpha_n)(k)2^{mj-k-1}$, are periodic with respect to sh_l, and period c_n. As in Lemma 1.11 of [6], the sequence τ, defined from $\{\tau_n : n \in \mathcal{N}\}$, has the property that;

$lim_{n\rightarrow\infty}\frac{1}{n}\sum_{j=0}^{n-1} g(sh_l^j\tau)$

$= lim_{n\rightarrow\infty}\frac{1}{c_n}\sum_{j=0}^{c_n-1} g(sh_l^j\tau_n)$, $(*)$

for $g \in C(Q_m^{\mathcal{N}})$, where $Q_m = \{\frac{j}{2^m} : 0 \leq j \leq 2^m - 1\}$. Hence, if λ is defined by $\lambda(j) = \gamma(\tau([\frac{j}{m}]))(rem(m,j)+1)$, using the fact that, if $f \in C([0,1))$, $(\gamma \circ \Phi)^* f \in C(Q_m^{\mathcal{N}})$, and $(*)$, then, for $f \in C([0,1))$;

$= \int_0^1 f d\mu = lim_{n\rightarrow\infty}\frac{1}{c_n}\sum_{j=0}^{c_n-1} f(sh_d^{mj}\alpha_n)$

$= lim_{n\rightarrow\infty}\frac{1}{c_n}\sum_{j=0}^{c_n-1}(\gamma^{-1} \circ \Phi)^* f(sh_l^j\tau_n)$

$= lim_{n\rightarrow\infty}\frac{1}{n}\sum_{j=0}^{n-1}(\gamma^{-1}\circ\Phi)^{*}f(sh_{l}^{j}\tau)$

$= lim_{n\rightarrow\infty}\frac{1}{n}\sum_{j=0}^{n-1}(\gamma^{-1})^{*}f(sh_{l}^{mj}\lambda)$

$= lim_{n\rightarrow\infty}\frac{1}{n}\sum_{j=0}^{n-1}f(sh_{d}^{mj}\gamma^{-1}(\lambda))$

Hence, $\gamma^{-1}(\lambda)$ is typical for sh_d^m. For Lemmas 0.16 and 0.18, the same proofs hold, replacing the measures ρ_{x_ϵ} and ρ_{x_n} by ρ_{m,x_ϵ} and ρ_{m,x_n}. For Lemma 0.18, observe that the μ is sh_d^m invariant, hence, the condition (ii) in Theorem 1.15 of [6] can be reformulated as;

$\sum_{(\xi_0,\ldots,\xi_{m-1})\in\Sigma^m}\kappa(\xi_0,\ldots,\xi_{m-1},\xi_m,\ldots,\xi_{n-1})$

$= \sum_{(\xi_0,\ldots,\xi_{m-1})\in\Sigma^m}\kappa(\xi_m,\ldots,\xi_{n-1},\xi_0,\ldots,\xi_{m-1})$ $(*)$

for $n>m$. Using the fact that n can be taken arbitrarily large, and $m|n$, in Lemma 0.18 and corresponding Lemma 1.14 of [6], it is sufficient to show in Lemma 0.19, that we can find a periodic element $x_{m,\delta}\in[0,1)$, with respect to sh_d^m, such that;

$|\rho_{m,x_{m,\delta}}(B_{j,n})-\mu(B_{j,n})|<\delta$ $(\dagger)$

for every $\bar{j}\in 2^n$ and $n>m$, $m|n$. Let $\Sigma'=\{\frac{i}{2^m}:0\leq i\leq 2^m-1\}$. Then, with $n=sm$, $(*)$ becomes;

$\sum_{\xi_0\in\Sigma'}\kappa(\xi_0,\ldots,\xi_{s-1})$

$= \sum_{\xi_0\in\Sigma'}\kappa(\xi_1,\ldots,\xi_{s-1},\xi_0)$ $(**)$

where $\kappa(\xi_0,\ldots,\xi_{s-1})=\mu(\pi_n^{-1}(\overline{a}_0{}^\wedge\overline{a}_1{}^\wedge\ldots{}^\wedge\overline{a}_{s-1}))$

$= \mu(B_{\overline{a}_0{}^\wedge\overline{a}_1{}^\wedge\ldots{}^\wedge\overline{a}_{s-1},sm})$

and $\overline{a}_j=\gamma(\xi_j)|_m$, $0\leq j\leq s-1$. Follow proof of Theorem 1.15 in [6], (i). $N\kappa(\xi)\in\mathcal{Z}$ holds for N sufficiently large, $\xi\in\Sigma'$, (ii). κ is a probability measure on Σ'^s, and (iii). $(**)$. Take β to be periodic, $n+r-1$, defined by $(\xi_0^0,\ldots\xi_{sm-1}^0,\xi_{sm-1}^1,\ldots,\xi_{sm-1}^{r-1})$, for longest sequence $(\xi^0,\ldots\xi^{r-1})$. Then corresponding $(\overline{a}_0^0,\ldots\overline{a}_{sm-1}^0,\overline{a}_{sm-1}^1,\ldots,\overline{a}_{sm-1}^{r-1})$ is $n+r-1$ periodic with respect to sh_d^m, and satisfies $(\dagger)$, with $n=rm$.

The rest of the claims follow from the proof of the corresponding results in Lemma 0.21.

□

Remarks 0.22. *An interesting use of the tree image, with a trunk and two sets of branches, representing a vertical line and two semi-diagonals, can be found on a fragment at St. Michael's church, Dowdeswell, in Gloucestershire, see the section on Norman architecture at http://www.magneticstrix.net This proves to be a useful mnemonic for structuring the proof of Theorem 1.15 in* [6]*, and relates to the imagery of the line, cross and circle, which I examined in "Christian Geometry: The Geometry of Light", Chapters 11 and 12, see also Remark 1.8 of "Some Geometry of Nodal Curves".*

Lemma 0.23. *If $n \in \mathcal{N}$ and $x \in [0,1)$ is typical for sh_d^n, then x is typical for sh_d^m, with $m|n$. In particular, if $n \in \mathcal{N}$, then there exists $x \in [0,1)$ which is typical for sh_d^r, with $1 \leq r \leq n$.*

Proof. For the first part, let $\bar{t} = \gamma(x)$, and suppose that $n = mt$, then we have, as x is typical for sh_d^n, using Definition 0.13, Lemma 0.8 and Lemma 0.14, that if $r \in \mathcal{N}$ and $1 \leq k \leq t$, then;

$$lim_{s\rightarrow\infty}\tfrac{1}{s}Card(\{i \in [1,s] \cap \mathcal{N} : \bar{t}(j+(i-1)n) = \delta_{\bar{j}}(j),$$

$$1 \leq j \leq r+(k-1)m\}) = \tfrac{1}{2^{r+(k-1)m}} \ (*)$$

for $\delta_{\bar{j}} \in \Omega_{r+(k-1)m}$. We claim that, for $r \in \mathcal{N}$, and $\theta_{\bar{j}} \in \Omega_r$;

$$lim_{s\rightarrow\infty}\tfrac{1}{s}Card(\{i \in [1,s] \cap \mathcal{N} : \bar{t}(j+(i-1)m) = \theta_{\bar{j}}(j),$$

$$1 \leq j \leq r\}) = \tfrac{1}{2^r} \ (**)$$

Observe that, from $(*)$;

$$lim_{s\rightarrow\infty}\tfrac{1}{s}Card(\{i \in [1,s] \cap \mathcal{N} : \bar{t}(j+(i-1)n) = \theta_{\bar{j}}(j),$$

$$1 \leq j \leq r\}) = \tfrac{1}{2^r} \ (***)$$

If $\theta^1_{\bar{j}} \in \Omega_{r+m}$, with $\theta^1_{\bar{j}}(m+j) = \theta_{\bar{j}}(j)$, for $1 \leq j \leq r$, then, by $(*)$;

$$lim_{s\rightarrow\infty}\tfrac{1}{s}Card(\{i \in [1,s] \cap \mathcal{N} : \bar{t}(j+(i-1)n) = \theta^1_{\bar{j}}(j),$$

$$1 \leq j \leq r+m\}) = \frac{1}{2^{r+m}} \ (****)$$

We have that;

$$Card(\{i \in [1,s] \cap \mathcal{N} : \bar{t}(j+(i-1)n) = \theta_{\bar{j}}(j-r), 1+m \leq j \leq r+m\})$$

$$= \sum_{\theta^1_{\bar{j}} \supset \theta_{\bar{j}}} Card(\{i \in [1,s] \cap \mathcal{N} : \bar{t}(j+(i-1)n) = \theta^1_{\bar{j}}(j),$$

$$1 \leq j \leq r+m\}) \ (*****)$$

Hence, taking limits, and using $(****)$;

$$lim_{s\rightarrow\infty}\frac{1}{s}Card(\{i \in [1,s] \cap \mathcal{N} : \bar{t}(j+(i-1)n) = \theta_{\bar{j}}(j-m),$$

$$1+m \leq j \leq m+r\})$$

$$= \sum_{\theta^1_{\bar{j}} \supset \theta_{\bar{j}}} \frac{1}{2^{r+m}} = \frac{2^m}{2^{r+m}} = \frac{1}{2^r} \ (******)$$

In the same way, taking the last r terms of sequences of length $r+(k-1)m$, for $1 \leq k \leq t$, we can show that;

$$lim_{s\rightarrow\infty}\frac{1}{s}Card(\{i \in [1,s] \cap \mathcal{N} : \bar{t}(j+(i-1)n) = \theta_{\bar{j}}(j-(k-1)m),$$

$$1+(k-1)m \leq j \leq r+(k-1)m\}) = \frac{1}{2^r} \ (\dagger)$$

The result $(**)$ then follows, using $(\dagger)$ by observing that;

$$lim_{s\rightarrow\infty}\frac{1}{s}Card(\{i \in [1,s] \cap \mathcal{N} : \bar{t}(j+(i-1)m) = \theta_{\bar{j}}(j), 1 \leq j \leq r\})$$

$$= lim_{s\rightarrow\infty}\frac{1}{st}Card(\{i \in [1,st] \cap \mathcal{N} : \bar{t}(j+(i-1)m) = \theta_{\bar{j}}(j), 1 \leq j \leq r\})$$

$$= \frac{1}{t}\sum_{1\leq k\leq t} lim_{s\rightarrow\infty}\frac{1}{s}Card(\{i \in [1,s] \cap \mathcal{N} : \bar{t}(j+(i-1)n)$$

$$= \theta_{\bar{j}}(j-(k-1)m), 1+(k-1)m \leq j \leq r+(k-1)m\})$$

$$= \frac{1}{t}\sum_{1\leq k\leq t}\frac{1}{2^r} = \frac{1}{2^r}$$

Again, using Lemmas 0.8 and 0.14, we have that x is typical for sh^m_d, as required.

The last part follows by taking $n_1 = n!$, using Lemma 0.21 to find $x \in [0,1)$ which is typical for $sh^{n_1}_d$, and applying the first part. $\square$

Lemma 0.24. *For any $\nu \in {}^*\mathcal{N}$, infinite, there exists $y \in {}^*[0,1)$ and an infinite $\kappa \in {}^*\mathcal{N}$ such that $\{sh_{\nu,f}^{m(j-1)}(y)\}_{1\leq j\leq \nu}$ is weakly equidistributed, for $1 \leq m \leq \kappa$. In particularly, $\{sh_{\nu,f}^{m(j-1)}(y)\}_{1\leq j\leq \nu}$ is weakly equidistributed, for $m \in \mathcal{N}$.*

Proof. Using Lemmas and 0.21, if ν is infinite we can find a sequence $(y_n)_{n\in\mathcal{N}}$, such that $\{sh_{\nu,f}^{m(j-1)}(y)\}_{1\leq j\leq \nu}$ is weakly equidistributed, for $1 \leq m \leq n$, $(*)$. For $n \in \mathcal{N}$, let;

$$A_n = \{y \in {}^*[0,1) : (\forall z_1 \in {}^*[0,1))(\forall z_2 \in {}^*[0,1))(\forall_{1\leq m\leq n})$$

$$|\tfrac{1}{\nu}{}^*Card(S_{m,\nu,z_1,z_2}) - |z_2 - z_1|| < \tfrac{1}{n}\}$$

where $S_{m,\nu,z_1,z_2} = \{i \in \mathcal{N} \cap [1,\nu] : sh_{\nu,f}^{m(j-1)}(y) \in [z_1, z_2)\}$. Each $A_n \subset {}^*[0,1)$ is internal, $A_{n+1} \subset A_n$, and, by $(*)$, we have that $A_n \neq \emptyset$. By countable comprehension, we can find $(A_n)_{n\in{}^*\mathcal{N}}$, an internal sequence, extending the sequence $(A_n)_{n\in\mathcal{N}}$. By overflow, we can find an infinite $\kappa \in {}^*\mathcal{N}$, with $A_\kappa \neq \emptyset$. Hence, we can find $y \in {}^*[0,1)$, satisfying the requirements of the lemma. Note that the final claim follows just by compactness of the nonstandard model, see Lemma 2.14 of [5]. □

Lemma 0.25. *With notation as in Theorem 0.20, we have that there exists $\kappa \in {}^*\mathcal{N}$, κ infinite, $\kappa \leq \nu$, such that for all $\kappa' \leq \kappa$, and $a_{\kappa'} = \frac{k}{2^{\kappa'}}$, $0 \leq k \leq 2^{\kappa'} - 1$, with associated decimal binary expansion given by $(a_1, \ldots, a_{\kappa'})$, $b_{\kappa'} = \frac{k+1}{2^{\kappa'}}$;*

$$\tfrac{1}{\nu}{}^*Card(\{j \in {}^*\mathcal{N} \cap [1,\nu] : sh_{\nu,f}^{j-1}(y) \in [a_{\kappa'}, b_{\kappa'} - (\tfrac{1}{2})^{\nu-2\kappa'})\}) = \tfrac{1}{2^{\kappa'}} + \delta_{\kappa'}, \quad (*)$$

where $|\delta_{\kappa'}| < \epsilon_{\kappa'} + \frac{2\kappa'}{\nu}$

and $\epsilon_{\kappa'} \simeq 0$.

In particular, for any $n \in \mathcal{N}$, and $a_{n,1} = \frac{k_1}{2^n}$, $a_{n,2} = \frac{k_2}{2^n}$, with $0 \leq k_1 \leq 2^n - 1$, $0 \leq k_2 \leq 2^n - 1$ and associated decimal binary expansion given by $(a_{1,1}, \ldots, a_{n,1})$, $(a_{1,2}, \ldots, a_{n,2})$, we have that;

$$\tfrac{1}{\nu}{}^*Card(\{j \in {}^*\mathcal{N} \cap [1,\nu] : sh_{\nu,f}^{j-1}(S(\bar{t}))(\nu - i + 1) = a_{i,1}, 1 \leq i \leq n,$$
$$. \qquad sh_{\nu,f}^{j-1}(S(\bar{t}))(\nu - \kappa' + n + 2 - i) = a_{i,2}, 1 \leq i \leq n\}) \simeq \tfrac{1}{2^{2n}}$$

for $\kappa' \leq min(\kappa, \frac{log(\nu)}{4log(2)} + n - \frac{1}{2})$, κ' *infinite.*

(then replace $sh_{\nu,f}$ *with* $sh^{n}_{\nu,f}$*)*

Proof. Given a binary sequence, $\overline{a}_\mu$ of length μ, we let;

$$A_{\mu,\overline{a}_\mu,\lambda} = \{j \in {}^*\mathcal{N} \cap [1,\lambda] : sh_l^{j-1}(\overline{t})(i) = a_i, 1 \leq i \leq \mu\}$$

Let;

$$B_{\mu,\nu,1} = max_{L(\overline{a}_\mu)=\mu}(\frac{{}^*Card(A_{\mu,\overline{a}_\mu,\nu})}{\nu})$$

$$B_{\mu,\nu,2} = min_{L(\overline{a}_\mu)=\mu}(\frac{{}^*Card(A_{\mu,\overline{a}_\mu,\nu})}{\nu})$$

By Theorem 0.20, we have that;

$B_{m,\nu,1} \simeq \frac{1}{2^m}$, $B_{m,\nu,2} \simeq \frac{1}{2^m}$, for all $m \in \mathcal{N}$.

In particular, if $\{\epsilon_m\}_{m\in\mathcal{N}}$, with $lim_{m\rightarrow\infty}\epsilon_m = 0$, we have that;

$$max(|B_{m,\nu,1} - \frac{1}{2^m}|, |B_{m,\nu,2} - \frac{1}{2^m}|) < \epsilon_m$$

Hence, by overflow, there exists $\kappa \in {}^*\mathcal{N}$, $\kappa \leq \nu$, κ infinite, with $max(|B_{\kappa',\nu,1} - \frac{1}{2^{\kappa'}}|, |B_{\kappa',\nu,2} - \frac{1}{2^{\kappa'}}|) < \epsilon_{\kappa'}$, for all $\kappa' \leq \kappa$

We have that, for all $\kappa' \leq \kappa$

$$A_{\kappa',\overline{a}_{\kappa'},\nu-\kappa'} = A_{\kappa',\overline{a}_{\kappa'},\nu} - B_{\kappa',\overline{a}_{\kappa'},\nu-\kappa'}$$

$$C_{\kappa',\overline{a}_{\kappa'},\nu} = A_{\kappa',\overline{a}_{\kappa'},\nu-\kappa'} + D_{\kappa',\overline{a}_{\kappa'},\nu-\kappa'}$$

$$= A_{\kappa',\overline{a}_{\kappa'},\nu} + (D_{\kappa',\overline{a}_{\kappa'},\nu-\kappa'} - B_{\kappa',\overline{a}_{\kappa'},\nu-\kappa'})$$

$$max(|C_{\kappa',\nu,1} - \frac{1}{2^{\kappa'}}|, |C_{\kappa',\nu,2} - \frac{1}{2^{\kappa'}}|)$$

$$< \epsilon_{\kappa'} + \frac{{}^*Card(D_{\kappa',\overline{a}_{\kappa'},\nu-\kappa'} \cup B_{\kappa',\overline{a}_{\kappa'},\nu-\kappa'})}{\nu} \leq \epsilon_{\kappa'} + 2\frac{\kappa'}{\nu}$$

giving the result $(*)$, observing that $\epsilon_{\kappa'} \simeq 0$, if κ' is infinite, by Theorem 2.22(i) of [5], taking $\epsilon_{\kappa'}$ arbitrarily small and using compactness otherwise.

We have that;

$E_{n,\kappa',\overline{a}_{2n},\nu}=\sum_{L(\overline{a}_{\kappa'-2n})=\kappa'-2n}C_{\kappa',\overline{a}_{n}\wedge\overline{a}_{\kappa'-2n}\wedge\overline{a}_{n},\nu}$

Hence;

$2^{\kappa'-2n}\nu(2^{-\kappa'}-|\delta_{\kappa'}|)\leq E_{n,\kappa',\overline{a}_{2n},\nu}\leq 2^{\kappa'-2n}\nu(2^{-\kappa'}+|\delta_{\kappa'}|)$

where $|\delta_{\kappa'}|\leq\epsilon_{\kappa'}+\frac{2\kappa'}{\nu}$. Therefore;

$\nu(2^{-2n}-2^{\kappa'-2n}|\delta_{\kappa'}|)\leq E_{n,\kappa',\overline{a}_{2n},\nu}\leq\nu(2^{-2n}+2^{\kappa'-2n}|\delta_{\kappa'}|)$

$|\frac{E_{n,\kappa',\overline{a}_{2n},\nu}}{\nu}-2^{-2n}|\leq 2^{\kappa'-2n}(\epsilon_{\kappa'}+\frac{2\kappa'}{\nu})\simeq 0$

taking $\epsilon_{\kappa'}=3^{-\kappa'}$, $\kappa'\leq\frac{log(\nu)}{4log(2)}+n-\frac{1}{2}$, if κ' is infinite, and, again, using compactness, if κ' is finite.

([4])

□

Definition 0.26. *Given $n\in\mathcal{N}$ and sequences $\{\overline{t}_1,\overline{t}_2\}\subset\Omega_{n+1}$, with $\overline{t}_1(n+1)=\overline{t}_2(n+1)=0$, we define;*

$G_{\overline{t}_1,\overline{t}_2}=\sum_{j=1}^{n+1}[(\overline{t}_1\oplus\overline{t}_2)(j)-\overline{t}_1(j)-\overline{t}_2(j)]$

---

[4] where;

$B_{\kappa',\overline{a}_{\kappa'},\nu-\kappa'}={}^*Card(\{j\in{}^*\mathcal{N}\cap[\nu-\kappa'+1,\nu]:sh_l^{j-1}(\overline{t})(i)=a_i,1\leq i\leq\kappa'\})$

$C_{\kappa',\overline{a}_{\kappa'},\nu}={}^*Card(\{j\in{}^*\mathcal{N}\cap[1,\nu]:sh_{\nu,r}^{j-1}(S(\overline{t}))(\nu-i+1)=a_i,1\leq i\leq\kappa'\})$

$D_{\kappa',\overline{a}_{\kappa'},\nu-\kappa'}={}^*Card(\{j\in{}^*\mathcal{N}\cap[\nu-\kappa'+1,\nu]:sh_{\nu,r}^{j-1}(S(\overline{t}))(\nu-i+1)=a_i,1\leq i\leq\kappa'\})$

$C_{\kappa',\nu,1}=max_{L(\overline{a}_{\kappa'})=\kappa'}(\frac{{}^*Card(C_{\kappa',\overline{a}_{\kappa'},\nu})}{\nu})$

$C_{\kappa',\nu,2}=min_{L(\overline{a}_{\kappa'})=\kappa'}(\frac{{}^*Card(C_{\kappa',\overline{a}_{\kappa'},\nu})}{\nu})$

$$E_{n,\kappa',\overline{a}_{2n},\nu}=\{j\in{}^*\mathcal{N}\cap[1,\nu]:sh_{\nu,f}^{j-1}(S(\overline{t}))(\nu-i+1)=a_{i,1},1\leq i\leq n,\ sh_{\nu,f}^{j-1}(S(\overline{t}))(\nu-\kappa'+n+2-i)=a_{i,2},1\leq i\leq n\}$$
.

$E_n = \frac{1}{2^{2n}}\sum_{\bar{t}_1\in\Omega_n,\bar{t}_2\in\Omega_n} G_{\bar{t}_1\hat{}0,\bar{t}_2\hat{}0}$

Lemma 0.27. *We have that* $lim_{n\rightarrow\infty}\frac{-2E_n}{n} = 1$

Proof. Observe that;

$\Omega_{n+1} = (\bigcup_{\bar{t}\in\Omega_n}\bar{t}\hat{}0)\cup(\bigcup_{\bar{t}\in\Omega_n}\bar{t}\hat{}1)$

Hence;

$2^{2(n+1)}E_{n+1} = \sum_{\bar{t}_3\in\Omega_{n+1},\bar{t}_4\in\Omega_{n+1}} G_{\bar{t}_3\hat{}0,\bar{t}_4\hat{}0}$

$= \sum_{\bar{t}_1\in\Omega_n,\bar{t}_2\in\Omega_n} G_{\bar{t}_1\hat{}00,\bar{t}_2\hat{}00}$

$+2\sum_{\bar{t}_1\in\Omega_n,\bar{t}_2\in\Omega_n} G_{\bar{t}_1\hat{}10,\bar{t}_2\hat{}00}$

$+\sum_{\bar{t}_1\in\Omega_n,\bar{t}_2\in\Omega_n} G_{\bar{t}_1\hat{}10,\bar{t}_2\hat{}10}$

We have that, for $\{\bar{t}_1,\bar{t}_2\}\subset\Omega_n$;

$G_{\bar{t}_1\hat{}00,\bar{t}_2\hat{}00} = G_{\bar{t}_1\hat{}0,\bar{t}_2\hat{}0}$, $(*)$

$G_{\bar{t}_1\hat{}10,\bar{t}_2\hat{}10}$

$= \sum_{j=1}^{n+2}[(\bar{t}_1\hat{}10\oplus\bar{t}_2\hat{}10)(j) - \bar{t}_1\hat{}10(j) - \bar{t}_2\hat{}10(j)]$

$= (\sum_{j=1}^{n+1}(\bar{t}_1\hat{}0\oplus\bar{t}_2\hat{}0)(j)+1) - (2+\sum_{j=1}^{n+1}(\bar{t}_1\hat{}0(j)+\bar{t}_2\hat{}0(j)))$

$= G_{\bar{t}_1\hat{}0,\bar{t}_2\hat{}0} - 1$, $(**)$

using the fact that;

$(\bar{t}_1\hat{}10\oplus\bar{t}_2\hat{}10)$

$= (\bar{t}_1\hat{}00\oplus\bar{t}_2\hat{}00)\oplus(\bar{0}\hat{}10\oplus\bar{0}\hat{}10)$

$= ((\bar{t}_1\hat{}0\oplus\bar{t}_2\hat{}0)\hat{}0)\oplus(\bar{0}\hat{}01)$

We have that;

$G_{\bar{t}_1\hat{}10,\bar{t}_2\hat{}00}=\sum_{j=1}^{n+2}[(\bar{t}_1\hat{}10\oplus\bar{t}_2\hat{}00)(j)-\bar{t}_1\hat{}10(j)-\bar{t}_2\hat{}00(j)]$

$=W_{\bar{t}_1,\bar{t}_2}-(1+\sum_{j=1}^{n+1}(\bar{t}_1\hat{}0(j)+\bar{t}_2\hat{}0(j)))$

where, if $(\bar{t}_1\hat{}0\oplus\bar{t}_2\hat{}0)(n+1)=0$;

$W_{\bar{t}_1,\bar{t}_2}=(\sum_{j=1}^{n+2}(\bar{t}_1\hat{}10\oplus\bar{t}_2\hat{}00)(j))$

$=(\sum_{j=1}^{n+2}((\bar{t}_1\hat{}00\oplus\bar{t}_2\hat{}00)\oplus(\bar{0}\hat{}10))(j))$

$=\sum_{j=1}^{n+1}(\bar{t}_1\hat{}0\oplus\bar{t}_2\hat{}0)+1$

and, if $(\bar{t}_1\hat{}0\oplus\bar{t}_2\hat{}0)(n+1)=1$;

$W_{\bar{t}_1,\bar{t}_2}=(\sum_{j=1}^{n+2}(\bar{t}_1\hat{}10\oplus\bar{t}_2\hat{}00)(j))$

$=(\sum_{j=1}^{n+2}((\bar{t}_1\hat{}00\oplus\bar{t}_2\hat{}00)\oplus(\bar{0}\hat{}10))(j))$

$=\sum_{j=1}^{n+1}(\bar{t}_1\hat{}0\oplus\bar{t}_2\hat{}0)(j)$

using the fact that;

$(\bar{t}_1\hat{}00\oplus\bar{t}_2\hat{}00)\oplus(\bar{0}\hat{}10)$

$=(\bar{t}_1\hat{}0\oplus\bar{t}_2\hat{}0)|_{\Omega_n}\hat{}00\oplus(\bar{0}\hat{}10)\oplus(\bar{0}\hat{}10)$

$=(\bar{t}_1\hat{}0\oplus\bar{t}_2\hat{}0)|_{\Omega_n}\hat{}00\oplus(\bar{0}\hat{}01)$

Hence;

$G_{\bar{t}_1\hat{}10,\bar{t}_2\hat{}00}=G_{\bar{t}_1\hat{}0,\bar{t}_2\hat{}0},\qquad$ if $(\bar{t}_1\hat{}0\oplus\bar{t}_2\hat{}0)(n+1)=0$

$G_{\bar{t}_1\hat{}10,\bar{t}_2\hat{}00}=G_{\bar{t}_1\hat{}0,\bar{t}_2\hat{}0}-1,\quad$ if $(\bar{t}_1\hat{}0\oplus\bar{t}_2\hat{}0)(n+1)=1$, $(***)$

Let $D_n=\{(\bar{t}_1,\bar{t}_2)\in\Omega_n^2:(\bar{t}_1\hat{}0\oplus\bar{t}_2\hat{}0)(n+1)=1\}$

and $C_n=Card(D_n)$. We have that $(\bar{t}_1,\bar{t}_2)\in D_n$ iff $\bar{t}_1(n)=\bar{t}_2(n)=1$, or $\bar{t}_1=\bar{t}_5\hat{}0$, $\bar{t}_2=\bar{t}_6\hat{}1$, with $(\bar{t}_5,\bar{t}_6)\in D_{n-1}$, or $\bar{t}_1=\bar{t}_7\hat{}1$, $\bar{t}_2=\bar{t}_8\hat{}0$, with $(\bar{t}_7,\bar{t}_8)\in D_{n-1}$. Hence;

$C_n = 2C_{n-1} + (2^{n-1})^2 = 2C_{n-1} + 4^{n-1}$

$C_n = 2^{n-1}C_1 + \sum_{j=1}^{n-1} 2^{j-1}4^{n-j}$

Using the fact that $C_1 = 1$, we obtain $C_n = 2^{2n-1}+2^{n-1}-2^n$, $(****)$. By $(*), (**), (***), (****)$, we obtain;

$2^{2(n+1)}E_{n+1} = 2^{2n}E_n + (2^{2n}E_n - 2^{2n}) + 2(2^{2n}E_n - (2^{2n-1} + 2^{n-1} - 2^n))$

$E_{n+1} = E_n - \frac{1}{2} + \frac{1}{2^{n+1}} - \frac{1}{2^{n+2}}$

$E_n = E_1 - \frac{n-1}{2} + (\frac{1}{4} - \frac{1}{2^{n+1}})$

As $E_1 = -\frac{1}{4}$, we have;

$E_n = -\frac{n}{2} + 1 - \frac{1}{2^{n+1}}$

Hence, $lim_{n\rightarrow\infty}\frac{-2E_n}{n} = 1$.

□

Lemma 0.28. *If q is prime, $q \neq 2$, and $0 \leq i \leq q-1$, then $\frac{i}{q}$ has a recurring binary expansion based on $\overline{a}$, with length $q-1$, and $\frac{i}{q}$ has period $r|q-1$, with respect to sh_d. Moreover, $r = q-1$ iff 2 generates the multiplicative group of F_q iff for every $0 < i \leq q-1$, $\frac{i}{q}$ has period $q-1$, with respect to sh_d iff $\overline{a}$ is not the recurrence of a shorter sequence $\overline{b}$. Conversely, if $x \in [0,1)$ has a recurring binary expansion based on $\overline{a}$, with length $q-1$, then $x = \frac{i}{2^{q-1}-1}$, with $0 \leq i < 2^{q-1}-1$.*

Proof. By Fermat's Little Theorem, we have that $q|2^{q-1}-1$. Then;

$\frac{i}{q} = \frac{im}{qm} = \frac{im}{2^{q-1}-1} = \frac{\frac{im}{2^{q-1}}}{\frac{2^{q-1}-1}{2^{q-1}}} = \frac{\frac{im}{2^{q-1}}}{1-\frac{1}{2^{q-1}}}$

where $im \leq (q-1)m \leq 2^{q-1}-1$. Letting $\overline{a}$ denote the finite decinal binary expansion of $\frac{im}{2^{q-1}}$, we obtain that the decimal binary expansion of $\frac{i}{q}$ is the recurrence of $\overline{a}$, and $l(\overline{a}) = q-1$. By Lemma 0.8, we have that $sh_d^{q-1}(\frac{i}{q}) = \frac{i}{q}$, hence $r|q-1$. If $r = q-1$, then, for all $0 \leq s < q-1$, $sh_d^s(\frac{i}{q}) = \frac{2^s i}{q}(mod 1) \neq \frac{i}{q}$, hence $2^s \neq 1(mod\ q)$, and, as $2^{q-1} = 1(mod\ q)$, 2 generates the multiplicative group of F_q. Hence, for every $0 < i \leq q-1$, as $sh_d^s(\frac{i}{q}) = \frac{i}{q}$ iff $2^s = 1(mod\ q)$, we have that $\frac{i}{q}$ has period $q-1$, with respect to sh_d. Then, if $\overline{a}$ is the recurrence of a shorter sequence $\overline{b}$, we obtain, by Lemma 0.8, that $sh_d^r(\frac{i}{q}) = \frac{i}{q}$, for some $r|q-1$, a contradiction. We then have, by Lemma 0.8, that $sh_d^s(\frac{i}{q}) \neq \frac{i}{q}$, for $1 \leq s < q-1$, and $sh_d^{q-1}(\frac{i}{q}) = \frac{i}{q}$, hence $r = q-1$. For the final claim, if $x \in [0,1)$ has a recurring decimal binary expansion based on $\overline{a}$, with length $q-1$, then $sh_d^{q-1}(x) - y = 2^{q-1}x - y = x$, where y is the binary expansion of the sequence $\overline{a}$. As $\overline{a}(j) = 0$, for some $1 \leq j \leq q-1$, we have that $0 \leq i \leq 2^{q-1}-2$, and $x = \frac{i}{2^{q-1}-1}$. □

Remarks 0.29. *For q prime, the multiplicative group of F_q is always cyclic of order $q-1$. Taking an ultraproduct $\prod_{q\ prime, q\neq 2} \overline{F_q}$, and using the fact that 2 is not cyclic in the multiplicative group of a field of characteristic zero, we can see that there cannot exist $r \in \mathcal{N}$ such that $2^{s(q)} = 1$ in F_q, with $s(q) \leq r$. Is it true that there exist infinitely many primes $q \neq 2$, for which 2 is cyclic in F_q^*? In this case, we obtain an easier proof of the main result, by observing that the sequence $\{\frac{2}{q_n}\}_{n\in\mathcal{N}}$, consists of periodic elements of order $q-1$, with respect to sh_d, and, $lim_{w,n\rightarrow\infty}\rho_{\frac{2}{q_n}} = \mu$, by Darboux's Theorem.*

Lemma 0.30. *With notation as in Definition 0.1, we have that, for $\nu \in {}^*\mathcal{N}$ even, $\nu \geq 2$, ${}^*Card(B_\nu) = C^\nu_{\frac{\nu}{2}}$.*

Proof. For $n \in \mathcal{N}$, n even $n \geq 2$, letting F_n consist of sequences of length n, consisting of 1's and -1, and $D_n = \{\overline{t} \in F_n : \sum_{j=1}^n \overline{t}(j) = 0\}$, we show that $Card(D_n) = C^n_{\frac{n}{2}}$. For n even, $n \geq 2$, $-n \leq m \leq n$ even, we let;

$R(n,m) = Card(\{\overline{t} \in D_n : \sum_{j=1}^n \overline{t}(j) = m\})$

$R(0,0) = 1$, $R(n,m) = 0$, if $n \geq 0$ $|m| > n$, m,n even.

We have, for n,m even, $n \geq 0$, that;

$R(n+2,m) = R(n,m+2) + 2R(n,m) + R(n,m-2)$

hence, for $1 \leq k \leq \frac{n}{2}$, $n \geq 2$;

$R(n,m) = \sum_{1\leq r\leq 2k} C^{2k}_r R(n-2k, m-2k+2r)$

$R(n,0) = \sum_{1\leq r\leq 2k} C^{2k}_r R(n-2k, 2r-2k)$

Taking $k = \frac{n}{2}$, we obtain, for $n \geq 2$;

$R(n,0)=\sum_{1\leq r\leq n}C^n_r R(0,2r-n)=C^n_{\frac{n}{2}}$

By transfer, we obtain the result for $\nu\in {}^*\mathcal{N}$ even, $\nu\geq 2$.

$\square$

Definition 0.31. *With notation as in* [4], *for $\nu\in {}^*\mathcal{N}$ even and infinite, $1\leq j\leq\nu$, we let $X_j:\overline{\Omega_\eta}\rightarrow\overline{\Omega_\eta}$ be defined by;*

$X_j={}^*\sum_{1\leq k\leq j}\omega_k$

Observe that $\{X_j:1\leq j\leq\nu\}$ is a nonstandard martingale sequence, in the sense of [4].

Lemma 0.32. *For ν even and infinite, the set $W_\nu=\{\frac{[\eta x]}{\eta}:x\in X_\nu^{-1}(0)\}$ is weakly equidistributed.*

Proof. We have that $W_\nu=\{\theta_\nu^{-1}(B_\nu)\}$, and, using Lemma 0.30;

${}^*Card(W_\nu)={}^*Card(\theta_\nu^{-1}(B_\nu))={}^*Card(B_\nu)=C^\nu_{\frac{\nu}{2}}$

By Definition 0.1, W_ν is $sh_{\eta,f}$ invariant, (†). By Theorem 0.20, there exists $x\in[0,1)$ which is typical for sh_d, with corresponding expansion $\gamma(x)$. Let $\overline{t}\in\{0,1\}^{{}^*\mathcal{N}}$ be the transfer of $\gamma(x)$. We claim that, for any infinite $\kappa\in {}^*\mathcal{N}$, $\frac{1}{\kappa}{}^*\sum_{1\leq j\leq\kappa}\overline{t}(j)\simeq\frac{1}{2}$, $(*)$. As x is typical, we have that the sequence $\{sh_d^{j-1}(x):j\in\mathcal{N}\}$ is equidistributed, in particular;

$lim_{n\rightarrow\infty}\frac{1}{n}Card(\{j\in\mathcal{N}\cap[1,n]:sh_d^{j-1}(x)\in[\frac{1}{2},1)\})$

$=lim_{n\rightarrow\infty}\frac{1}{n}Card(\{j\in\mathcal{N}\cap[1,n]:\gamma(x)(j)=1\})=\frac{1}{2}$

It follows that, for any infinite κ;

$\frac{1}{\kappa}{}^*Card(\{j\in {}^*\mathcal{N}\cap[1,\kappa]:\overline{t}(j)=1\})=\frac{1}{\kappa}{}^*\sum_{1\leq j\leq\kappa}\overline{t}(j)\simeq\frac{1}{2}$

hence, $(*)$ is shown. By $(*)$, it follows that ${}^*\sum_{1\leq j\leq\nu}\overline{t}(j)=\nu(\frac{1}{2}+\epsilon)$, where $\epsilon\simeq 0$ and $\epsilon\nu\in {}^*\mathcal{N}$. If $\epsilon>0$, choose s_1, with ${}^*\sum_{s_1\leq j\leq\nu}\overline{t}(j)=\epsilon\nu$, and let;

$\overline{t}_1(j)=\overline{t}(j)$, if $1\leq j\leq s_1-1$

$\overline{t}_1(j)=0$, if $s_1\leq j\leq\eta$

If $\epsilon<0$, choose s_2, with ${}^*\sum_{s_2\leq j\leq\nu}(1-\bar{t}(j))=\epsilon\nu$, and let;

$\bar{t}_2(j)=\bar{t}(j)$, if $1\leq j\leq s_2-1$

$\bar{t}_2(j)=1$, if $s_2\leq j\leq\eta$

In either case, we obtain that $\bar{t}_i\in W_\nu$, for $1\leq i\leq 2$ (rephrase this). Let $x'=\frac{\theta_\nu^{-1}(rev(\bar{t}_i))}{\eta}$, $\eta=2^\nu$, then, by construction, $x'\in X_\nu^{-1}(0)$. By (†), we have that $\{sh_{\nu,f}^{j-1}(x'):1\leq j\leq\nu\}\subset X_\nu^{-1}(0)$. We claim that $\{sh_{\nu,f}^{j-1}(x'):1\leq j\leq\nu\}$ is weakly equidistributed, $(**)$. By Theorem 0.20, we have that $\{sh_{\nu,f}^{j-1}(y):1\leq j\leq\eta\}$ is weakly equidistributed, where $y=\frac{\theta_\nu^{-1}(S(\bar{t}))}{\eta}$. We modify the proof of this result replacing y by x'. For $m\in\mathcal{N}$, let;

$$S_{\epsilon\nu,m}=\{j\in[1,\nu]:\exists j_0\in[1,\nu],|j-j_0|\leq m+1,S(\bar{t})(j_0)\neq S(\bar{t}_1)(j)\},$$
([5])

then ${}^*Card(S_{\epsilon\nu,m})=\lambda\leq(2m+1)\epsilon\nu$. Using the proof of Theorem 0.20 and footnote 2, with $(a_1,\ldots,a_m)$, as in the proof of Theorem 0.20, we have;

$$2^{-m}\simeq\frac{1}{\nu}{}^*Card(\{j\in{}^*\mathcal{N}\cap[1,\nu]:sh_{\nu,r}^{j-1}(S(\bar{t}))(i)=a_i,1\leq i\leq m\})$$

$$\simeq\frac{1}{\nu-\lambda}{}^*Card(\{j\in{}^*\mathcal{N}\cap S^c_{\epsilon\nu,m}:sh_{\nu,r}^{j-1}(S(\bar{t}))(i)=a_i,1\leq i\leq m\})$$

$$=\frac{1}{\nu-\lambda}{}^*Card(\{j\in{}^*\mathcal{N}\cap S^c_{\epsilon\nu,m}:sh_{\nu,r}^{j-1}(S(\bar{t}_1))(i)=a_i,1\leq i\leq m\})$$

$$\simeq\frac{1}{\nu}{}^*Card(\{j\in{}^*\mathcal{N}\cap[1,\nu]:sh_{\nu,r}^{j-1}(S(\bar{t}_1))(i)=a_i,1\leq i\leq m\})$$

By the same argument, as in Theorem 0.20, we obtain $(**)$. We claim that, for $z\in X_\nu^{-1}(0)$, $z\oplus\{sh_{\nu,f}^{j-1}(x'):1\leq j\leq\nu\}$ is weakly distributed, $(***)$.
.... By Lemma 0.24, for any $\nu\in{}^*\mathcal{N}$, infinite, there exists $z\in{}^*[0,1)$ and an infinite $\kappa\in{}^*\mathcal{N}$ such that $\{sh_{\nu,f}^{m(j-1)}(z)\}_{1\leq j\leq\nu}$ is weakly equidistributed, for $1\leq m\leq\kappa$. As $\{sh_{\nu,f}^{(j-1)}(z)\}_{1\leq j\leq\nu}$ is weakly equidistributed, we have that;

[5]With the convention that $|j-j'|\leq m+1$, if $|j-1|\leq m_1$, $|j'-\nu|\leq m_2$ and $m_1+m_2\leq m+1$

$\frac{1}{2}+\epsilon=\frac{1}{\nu}{}^{*}Card(\{j\in {}^{*}\mathcal{N}\cap[1,\nu]:sh_{\nu,f}^{j-1}(z)\in[\frac{1}{2},1)\})$

$=\frac{1}{\nu}Card(\{j\in {}^{*}\mathcal{N}\cap[1,\nu]:(sh_{\nu,r}^{j-1}\circ\theta_{\nu})([\eta z])(\nu)=1\})$

$=\frac{1}{\nu}Card(\{j\in {}^{*}\mathcal{N}\cap[1,\nu]:\theta_{\nu}([\eta z])(\nu-j+1)=1\})$

$=\frac{1}{\nu}{}^{*}\sum_{1\leq j\leq\nu}\theta_{\nu}([\eta z])(j)$ $(****)$

where $\epsilon\simeq 0$. Now, repeating the modification of the above proof, replacing $\bar{t}|_{[1,\nu]}$ with $\theta_{\nu}([\eta z])$, we can find $z'=\frac{\theta_{\nu}^{-1}((\theta_{\nu,i}))}{\eta}\in X_{\nu}^{-1}(0)$, and, clearly, as W_{ν} is $sh_{\eta,f}$ invariant, $\{sh_{\nu,f}^{m(j-1)}(z')\}_{1\leq j\leq\nu}\subset X_{\nu}^{-1}(0)$, for $1\leq m\leq\kappa$. We claim that $\{sh_{\nu,f}^{m(j-1)}(z'):1\leq j\leq\nu\}$ is weakly equidistributed, for $1\leq m\leq\delta$, $(*****)$, where $(2\delta+1)\delta\epsilon\simeq 0$, $(\dagger\dagger)$, and $\delta\leq\kappa$ is infinite. This follows, repeating the argument above, by observing that, for finite $n\in\mathcal{N}$, $S_{\epsilon\nu,n}\subset S_{\epsilon\nu,\delta}$, and if;

$T_{\epsilon\nu,m,n}=\{j\in[1,\nu]:[\nu-m(j-1)-n,\nu-m(j-1)+n]\cap S_{\epsilon\nu,\delta}\neq\emptyset\}$

$\frac{\lambda}{\nu}\simeq 0$, $\lambda={}^{*}Card(T_{\epsilon\nu,m,n})$, as if $\lambda=\gamma\nu$, with $\gamma\notin\mu(0)$, then, using the fact that $\nu=\frac{\nu}{\delta}\delta$;

${}^{*}Card(S_{\epsilon\nu,\delta})\geq\frac{m\gamma\nu}{2n\delta}$, if $m<n$, ${}^{*}Card(S_{\epsilon\nu,\delta})\geq\frac{\gamma\nu}{2\delta}$, if $m\geq n$, contradicting $(\dagger\dagger)$ and the fact that ${}^{*}Card(S_{\epsilon\nu,\delta})\leq(2\delta+1)\epsilon\nu$.

Hence, for $(a_1,\ldots,a_n)\in C_n$, with $n\in\mathcal{N}$;

$2^{-n}\simeq\frac{1}{\nu}{}^{*}Card(\{j\in {}^{*}\mathcal{N}\cap[1,\nu]:sh_{\nu,r}^{m(j-1)}(\theta_{\nu})([\eta z])(i)=a_{i-\nu+n},\nu-n+1\leq i\leq\nu\})$

$\simeq\frac{1}{\nu-\lambda}{}^{*}Card(\{j\in {}^{*}\mathcal{N}\cap T_{\epsilon\nu,m}^{c}:sh_{\nu,r}^{m(j-1)}(\theta_{\nu})([\eta z])(i)=a_{i-\nu+n},\nu-n+1\leq i\leq\nu\})$

$=\frac{1}{\nu-\lambda}{}^{*}Card(\{j\in {}^{*}\mathcal{N}\cap T_{\epsilon\nu,m}^{c}:sh_{\nu,r}^{m(j-1)}(\theta_{\nu})([\eta z'])(i)=a_{i-\nu+n},\nu-n+1\leq i\leq\nu\})$

$\simeq\frac{1}{\nu}{}^{*}Card(\{j\in {}^{*}\mathcal{N}\cap[1,\nu]:sh_{\nu,r}^{j-1}(\theta_{\nu})([\eta z'])(i)=a_{i-\nu+n},\nu-n+1\leq i\leq\nu\})$

$\square$

Definition 0.33. *We define an addition* $\oplus' : \Omega_\nu \times \Omega_\nu \rightarrow \Omega_\nu$*, as follows. Given* $\{\alpha,\beta\} \subset \Omega_\nu$*, write* $\alpha = \alpha_1 + \gamma 2^{\nu-1}$*,* $\beta = \beta_1 + \delta 2^{\nu-1}$*, with* $\{\gamma,\delta\} \subset \{0,1\}$ *and* $\{\alpha_1,\beta_1\} \subset \Omega_{\nu-1}$*. Then let;*

$$\alpha \oplus' \beta = (\alpha_1+\beta_1)+1, \text{ if } \gamma=\delta=1, \text{ or } \gamma+\delta=1 \text{ and } \alpha_1+\beta_1 < 2^{\nu-1}$$

$$= \alpha_1 + \beta_1, \text{ if } \gamma=\delta=0$$

$$= (\alpha_1+\beta_1-2^{\nu-1})+1, \text{ if } \gamma+\delta=1 \text{ and } \alpha_1+\beta_1 \geq 2^{\nu-1}$$

We define $\oplus : C_\nu \times C_\nu \rightarrow C_\nu$*, by setting* $\bar{t}_1 \oplus \bar{t}_2 = \theta_\nu(\theta_\nu^{-1}(\bar{t}_1) \oplus' \theta_\nu^{-1}(\bar{t}_2))$

If $r \in {}^*\mathcal{N} \cap [1,\nu]$*, we define;*

$C_{\nu,r} = \{\bar{t} \in C_\nu : \bar{t}(\nu-j) = 0, 0 \leq j \leq r-1\}$

with corresponding initial segment $\Omega_{\nu-r} \subset \Omega_\nu$*.*

Given $\bar{t} \in C_\nu$*, we define* $L(\bar{t}) = {}^*Card(\{i \in {}^*\mathcal{N} \cap [1,\nu] : \bar{t}(i) = 1\})$*.*

We define $\oplus : \overline{\Omega}_\eta \times \overline{\Omega}_\eta \rightarrow \overline{\Omega}_\eta$*, by setting* $\frac{j_1}{\eta} \oplus \frac{j_2}{\eta} = \frac{j_1 \oplus j_2}{\eta}$*, where* $\{j_1,j_2\} \subset \Omega_\nu$*, and;*

$z_1 \oplus z_2 = (\frac{[\eta z_1]}{\eta} \oplus \frac{[\eta z_2]}{\eta}) + (z_1 - \frac{[\eta z_1]}{\eta}) + (z_2 - \frac{[\eta z_2]}{\eta})$

Lemma 0.34. *We have that;*

$Im(\oplus'|_{\Omega_{\nu-r}\times\Omega_{\nu-r}}) \subset \Omega_{\nu-r+1}$

$Im(\oplus|_{C_{\nu-r}\times C_{\nu-r}}) \subset C_{\nu-r+1}$

$Im(sh_{\nu,r}|_{C_{\nu-r}}) \subset C_{\nu-r+1}$

and, if $\{\bar{t}_1,\bar{t}_2\} \subset C_{\nu-r}$*, for some* $r \in \mathcal{N}$*;*

$\theta_\nu^{-1}(\bar{t}_1 \oplus \bar{t}_2) = \theta_\nu^{-1}(\bar{t}_1) + \theta_\nu^{-1}(\bar{t}_2)$

Proof. □

Lemma 0.35. *If* $\{\bar{t}_1,\bar{t}_2,\bar{t}_3\} \subset C_\nu$*, then;*

$(\bar{t}_1 \oplus \bar{t}_2) \oplus \bar{t}_3 = \bar{t}_1 \oplus (\bar{t}_2 \oplus \bar{t}_3)$

Proof. □

Lemma 0.36. *If $\{\bar{t}_1, \bar{t}_2\} \subset C_\nu$, then;*

$L(\bar{t}_1) = L(sh_{\nu,r}(\bar{t}_1))$

$L(\bar{t}_1 \oplus \bar{t}_2) \leq L(\bar{t}_1) + L(\bar{t}_2)$

Proof. It is simple to see that $L(\bar{t}) = L(sh_{\nu,r}(\bar{t}))$. We claim that $L(\overline{x} \oplus \overline{y}) \leq L(\overline{x}) + L(\overline{y})$, $(**)$. By induction on $L(\overline{y})$. The case $L(\overline{y}) = 0$ is clear. Suppose that $L(\overline{y}) = \kappa \leq \nu$, and let $\overline{y} = \overline{y}_1 \oplus \overline{y}_{final}$, with $L(\overline{y}_1) = \kappa - 1$ and $L(\overline{y}_{final}) = 1$, where $\overline{y}_{final}(i_{max}) = 1$, $\overline{y}(i_{max}) = 1$, $\overline{y}(i) = 0$, for $i > i_{max}$. Then;

$L(\overline{x} \oplus \overline{y})$

$= L(\overline{x} \oplus (\overline{y}_1 \oplus \overline{y}_{final}))$

$= L((\overline{x} \oplus \overline{y}_1) \oplus \overline{y}_{final})$

We claim that if $L(\overline{z}) = 1$, $L(\overline{w} \oplus \overline{z}) \leq L(\overline{w}) + 1$, $(***)$. By induction on $L(\overline{w})$. The base case is clear. If $\overline{z}(i) = 1$ and $\overline{w}(i) = 0$, we clearly have that $L(\overline{w} \oplus \overline{z}) = L(\overline{w}) + L(\overline{z}) = L(\overline{w}) + 1$. Otherwise, if $\overline{w}(i) = 1$, let $\overline{w} = \overline{w}_1 \oplus \overline{w}_i$, then;

$L(\overline{w} \oplus \overline{z})$

$= L((\overline{w}_1 \oplus \overline{w}_i) \oplus \overline{z})$

$= L(\overline{w}_1 \oplus (\overline{w}_i \oplus \overline{z}))$

$= L(\overline{w}_1 \oplus \overline{s})$

$\leq L(\overline{w}_1) + 1 = L(\overline{w})$

where $L(\overline{s}) = 1$, $\overline{s}(i+1) = 1$, and, using the induction hypothesis on $\overline{w}_1$. Hence, using $(* * *)$;

$L((\overline{x} \oplus \overline{y}_1) \oplus \overline{y}_{final})$

$\leq L(\overline{x} \oplus \overline{y}_1) + 1$

$\leq L(\overline{x}) + L(\overline{y}_1) + 1$

$= L(\overline{x}) + L(\overline{y})$

so that $(**)$ is shown.

□

Lemma 0.37. *If* $\{\bar{t}_1, \bar{t}_2\} \subset C_\nu$*, then;*

$sh_{\nu,r}(\bar{t}_1 \oplus \bar{t}_2) = sh_{\nu,r}(\bar{t}_1) \oplus sh_{\nu,r}(\bar{t}_2)$

Proof. We claim that, if $L(\overline{z}) = 1$, then $sh(\overline{x} + \overline{z}) = sh(\overline{x}) + sh(\overline{z})$, $(****)$, by induction on $L(\overline{x})$. The base case is clear. Suppose that $\overline{x}(i) = 0$, where $\overline{z}(i) = 1$, then, clearly $sh(x \oplus z) = sh(x) \oplus sh(z)$, as the sequences are concatenated. Otherwise, let $x = x_1 + z$, with $length(x_1) = length(x) - 1$. Then;

$sh(x + z) = sh(x_1 + (z + z)) = sh(x_1 + w) = sh(x_1) + sh(w)$

$sh(x) + sh(z) = sh(x_1 + z) + sh(z) = (sh(x_1) + sh(z)) + sh(z) = sh(x_1) + sh(z + z) = sh(x_1) + sh(w)$

using the induction hypothesis, and the fact that $sh(z+z) = sh(z) + sh(z)$, when $L(z) = 1$. Hence, $(****)$ is shown. We claim that $sh(\overline{x}+\overline{y}) = sh(\overline{x}) + sh(\overline{y})$, $(*****)$. By induction on $L(\overline{y})$. Base case is clear. Otherwise $\overline{y} = \overline{y}_1 + \overline{y}_{final}$ and;

$sh(\overline{x} + \overline{y})$

$= sh((\overline{x} + \overline{y}_1) + \overline{y}_{final})$

$= sh(\overline{x} + \overline{y}_1) + sh(\overline{y}_{final})$, using $(****)$

$= sh(\overline{x}) + (sh(\overline{y}_1) + sh(\overline{y}_{final}))$, by induction hypothesis

$= sh(\overline{x}) + sh(\overline{y})$, by (****)

□

Lemma 0.38. *The forward shift map* $sh_{\nu,f}$ *is* S*-continuous.*

Proof. Observe, first, that $sh_{\nu,r}(\bar{t}_1 \oplus \bar{t}_2) = sh_{\nu,r}(\bar{t}_1) \oplus sh_{\nu,r}(\bar{t}_2)$, $(*)$. We claim that, if $j \in \Omega_\nu$, then $\frac{j}{\eta} \simeq 0$ iff $j \in \Omega_{\nu-r}$ for all $r \in \mathcal{N}$, $(**)$. If $\frac{j}{\eta} \simeq 0$, then, for all $\epsilon \in \mathcal{R}$, $j < \epsilon\eta = \epsilon 2^\nu$. In particularly, taking $\epsilon = 2^{-r}$ for $r \in \mathcal{N}$, we have that $j < 2^{\nu-r}$, hence, $j \in \Omega_{\nu-r}$ for all $r \in \mathcal{N}$. Conversely, if $j \in \Omega_{\nu-r}$ for all $r \in \mathcal{N}$, then $j < 2^{\nu-r}$, for all $r \in \mathcal{N}$, and, by overflow, we can find an infinite $\lambda \in {}^*\mathcal{N}$ such that $j < 2^{\nu-\lambda}$. Then $\frac{j}{\eta} < 2^{-\lambda} \simeq 0$. By Lemma 0.34, we have that $sh_{\nu,r}(\theta_\nu(j)) \in C_{\nu-r}$, for all $r \in \mathcal{N}$, hence $sh'_{\nu,r}(j) \in \Omega_{\nu-r}$, for all $r \in \mathcal{N}$, and $\frac{sh'_{\nu,r}(j)}{\eta} \simeq 0$. It follows that, if $j \in \Omega_\nu$ and $\frac{j}{\eta} \simeq 0$, then $sh_{\nu,f}(\frac{j}{\eta}) \simeq 0$, $(***)$. Then, using $(*), (***)$, if $\{j_1, j_2\} \subset \Omega_\nu$, with $\frac{j_1}{\eta} \simeq \frac{j_2}{\eta}$, $j_1 \leq j_2$, we have that $\frac{j_2 - j_1}{\eta} \simeq 0$, and;

$sh_{\nu,f}(\frac{j_2-j_1}{\eta}) = \frac{sh'_{\nu,r}(j_2-j_1)}{\eta}$

$= \frac{sh'_{\nu,r}(j_2) - sh'_{\nu,r}(j_1)}{\eta}$

$= sh_{\nu,f}(\frac{j_2}{\eta}) - sh_{\nu,f}(\frac{j_1}{\eta}) \simeq 0 \ (****)$

Finally, if $\{x, y\} \subset \overline{\Omega_\eta}$, with $x \simeq y$, $x \leq y$. Then;

$|sh_{\nu,f}(x) - sh_{\nu,f}(y)|$

$= |(sh_{\nu,f}(\frac{[\eta x]}{\eta}) - sh_{\nu,f}(\frac{[\eta x]}{\eta})) + (x - \frac{[\eta x]}{\eta}) - (y - \frac{[\eta y]}{\eta})| \simeq 0$

as $\{\frac{[\eta x]}{\eta}, \frac{[\eta y]}{\eta}\} \subset \mathcal{V}_0$. □

Lemma 0.39. *Left Cancellation and Surjection If $\{\alpha, \beta, \epsilon, \theta, \kappa\} \subset \Omega_\nu$, with $\alpha \oplus' \beta = \alpha \oplus' \epsilon$, then $\beta = \epsilon$, and there exists $\theta \in \Omega_\nu$, with $\alpha \oplus' \theta = \kappa$.*

Proof. Let $\alpha = \alpha_1 + \gamma 2^{\nu-1}$, $\beta = \beta_1 + \delta 2^{\nu-1}$, $\epsilon = \epsilon_1 + \iota 2^{\nu-1}$

with $\{\gamma, \delta, \iota\} \subset \{0, 1\}$ and $\{\alpha_1, \beta_1, \epsilon_1\} \subset \Omega_{\nu-1}$. Then let;

.... $\alpha \oplus' \beta = (\alpha_1 + \beta_1) + 1$, if $\gamma = \delta = 1$, or $\gamma + \delta = 1$ and $\alpha_1 + \beta_1 < 2^{\nu-1}$

$= \alpha_1 + \beta_1$, if $\gamma = \delta = 0$

$= (\alpha_1 + \beta_1 - 2^{\nu-1}) + 1$, if $\gamma + \delta = 1$ and $\alpha_1 + \beta_1 \geq 2^{\nu-1}$

...

It follows, as the operation $\oplus'$ is internal on the *-finite set $\overline{\Omega}_\nu \times \overline{\Omega}_\nu$, and for any $\alpha \in \Omega_\nu$, $\theta_\alpha : \Omega_\nu \to \Omega_\nu$, $\theta_\alpha(x) = \alpha \oplus' x$ is injective, that, θ_α is surjective. Hence, the last result holds. □

Definition 0.40. *Given $\{\alpha, \beta\} \subset \Omega_\nu$, we define $\alpha \ominus \beta$ to be the unique γ such that $\beta \oplus \gamma = \alpha$.*

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