NEWTON AND THE SUBLIME GEOMETRY OF LINES

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This chapter will be concerned with the way in which geometric thinking is able to draw on the aesthetic ideas outlined above. In the second chapter, I referred to the intelligence which mediates between the images of the Crucifixion and the Throne of God as the Ascension. In some sense, the geometric ideas that we will consider in this chapter conform to this general notion. However, in order to illustrate the specific geometric ideas, I will prefer to rely on the more detailed aesthetic terminology which I introduced above. The geometric ideas which I will consider here are all concerned with the theory of algebraic curves. One does not need to be an expert mathematician, in order to understand such objects at a rudimentary level. Anyone who has studied mathematics to secondary school level in England, will have come across the idea of using functions to draw the graphs of curves or lines in a plane. A simple example is given by a line, with the equation;

$y = ax + b$

which we represent using cartesian coordinates as shown in the diagram (insert image). The simplest example of a curve (a parabola) is given by the equation;

$y = ax^2 + bx + c = 0$

We are all taught the formula for solving such an equation at school;

$x = -b \pm \frac{\sqrt{b^2-4ac}}{2a}$

Geometrically, this is interpreted as finding the two points where the curve meets the x-axis or the line $y = 0$, ([6] is a good reference for secondary level mathematics.). One may generalise the theory of functions, by considering equations in which the y -variable may appear as a higher power. For example, the equation;
$y^2 = x^3 - x$

may be represented as follows (insert image). In modern terminology, this is referred to as an elliptic curve. Generalising further, one may consider arbitrary polynomials $p(x, y) = 0$ in the variables x and y;

$p(x, y) = \sum_{i+j \leq d} a_{ij} x^i y^j = 0$ $(*)$

The solutions of such polynomial equations are referred to as algebraic curves. The important point to remember in the definition of an algebraic curve is that the polynomial equation defining it has a finite number of terms. The index d appearing in the polynomial equation above is referred to as the degree of the curve and measures its complexity.

In order to make the theory more precise, one has to clarify the domain in which the coefficients of the equation (*), and the variables $\{x, y\}$ are allowed to vary. At secondary school, we confine our attention to the real numbers, intuitively any number that can be represented by a possibly infinite decimal expansion, and denoted in modern mathematics by the symbol $\mathcal{R}$. Such numbers are convenient, as they can be programmed easily on a computer or a calculator. However, they also suffer from the following drawback, that there are many polynomial equations which have coefficients in the real numbers, that admit no real solutions. The simplest example is given by the equation;

$y^2 + 1 = 0$

This has no real solutions, and, in order to solve it, we are forced to introduce the imaginary number, denoted by i. The domain obtained, by extending real numbers, to numbers of the form;

$a + bi$ with $\{a, b\} \subset \mathcal{R}$

is referred to as the complex numbers, denoted in modern mathematics by the symbol $\mathcal{C}$.

For the purposes of algebraic curves, it is more convenient to work with the domain $\mathcal{C}$, rather than $\mathcal{R}$. For example, let us consider the algebraic curve, defined by the polynomial equation $y^2 + 1 = 0$. Over the real numbers, the solution set is empty, but over the complex numbers, it consists of 2 lines defined by $y = i$ and $y = -i$, (see diagram). Intuitively, we would like any polynomial equation to define a "curve" rather than the empty set, hence, the complex numbers seem to be a better choice of domain. It might be objected, however, that there could still be polynomial equations, with coefficients in $\mathcal{C}$, that admit no complex solutions. However, remarkably, this objection turns out to be false, as a consequence of the following property of $\mathcal{C}$, often called "The Fundamental Theorem of Algebra";

Any polynomial equation of the form $y^n + a_1 y^{n-1} + \ldots + a_n = 0$, with coefficients in $\mathcal{C}$, admits n complex solutions (possibly with multiplicity). In particular, any such equation admits at least one complex solution. (**)

Here, one should clarify what is meant by a solution with multiplicity, the equation $y^2 - 2y + 1 = 0$ factors as $(y - 1)^2 = 0$, hence, 1 is counted as a solution *twice*. As a consequence of this theorem, any polynomial $p(x, y)$ with coefficients in $\mathcal{C}$ admits an infinite number of solutions in the plane $\mathcal{C}^2$. Moreover, we have the following further property of the solution set of $p(x, y)$, which I will denote by C;

Either the solution set C consists of finitely many lines of the form $x = a$, for $a \in \mathcal{C}$, *or*, for all but finitely many elements $\{a_1, \ldots, a_n\} \subset \mathcal{C}$, if $x \in \mathcal{C} \setminus \{a_1, \ldots, a_n\}$, there exist finitely many solutions of the form (x, y) in C.

If we represent the complex numbers $\mathcal{C}$ as a line, corresponding to the horizontal x-axis, then this theorem tells us that any algebraic curve C consists of either a finite number of vertical lines, or is essentially a finite cover of the x-axis, see diagram. Intuitively, this is exactly the property that we would expect a "curve" drawn in the plane to have. In modern mathematics, this property is referred to, by saying that the dimension of the curve C is 1, abbreviated as $dim(C) = 1$. In contrast, we say that a finite number of points in the plane has dimension 0, while, the complement of an algebraic curve or the complement of finitely many points, has dimension 2. In particular, the whole plane $\mathcal{C}^2$ has dimension 2.

There is still, however, another consideration to be taken into account, for the representation of algebraic curves over the complex numbers. It is more natural to represent the real numbers $\mathcal{R}$ as a line, in which case, the domain of the complex numbers corresponds to a real plane, given by $\mathcal{R}^2$. In this interpretation, any algebraic curve may be viewed as a surface, in the sense that it looks 2-dimensional, when considered in relation to the real line $\mathcal{R}$. In the aesthetic representations of algebraic curves that we will consider, both interpretations of an algebraic curve as a "curve" and a "surface" will prove to be useful.

For the reader interested in a rigorous account of the mathematical construction of the real numbers, see, for example, [5]. The construction of the complex numbers and more advanced properties of functions on them, is dealt with in [28]. The history behind the construction of the real numbers is also interesting, and usually accredited to Dedekind in the nineteenth century. The mathematical development of the complex numbers is also an important field of study, and is associated with the names of a number of great French mathematicans from the nineteenth century, such as Cauchy and Liouville. If the reader is interested in the basic construction of algebraic curves, excellent modern accounts can be found in [35] or [7]. For a series of pictures of algebraic curves, the reader is strongly recommended to consult the St. Andrew's directory of algebraic curves, a link can be found on my website at http://www.curveline.net

Although the rigorous construction of algebraic curves using algebra is a fairly modern development, ([1]), the basic theory of polynomials has been known for a long time, probably with the invention of algebra by the Arabs. The intuitive representation of algebraic curves, in terms of a planar coordinate system, is, accredited to the French philosopher Rene Descartes. However, it seems feasible that this representation was known, in some form, even before. In order to illustrate the aesthetic ideas of the previous chapter, however, I wish to confine my attention to, without doubt, the greatest of all English geometers, Isaac Newton.

Isaac Newton was born on 4 January 1643, in the tiny village of Woolsthorpe, Lincolnshire, ([2]). An amusing anecdote records that he was so small at birth, that he could be fitted into a pint pot, a clear demonstration, if one was needed, that a towering intellect has nothing to do with physical size. In 1661, he attended Trinity College, Cambridge, graduating in 1665 without honours or distinction. In the middle of 1665, due to plague which had broken out in London, Newton returned to his family home, Woolsthorpe Manor. In the course of the next two years, Newton made a series of extraordinary mathematical breakthroughs.

The first of these, in 1665, was his discovery of the generalised Binomial Theorem. The

[1]Part of the modern subjects of algebraic geometry and commutative algebra, good references are [3], [20], [24] and [33].

[2]There are a number of good biographies of Newton, see, for example, [2] and [37]

binomial theorem gives a procedure for finding the expansion of a *positive* power of a polynomial expression. It is normally stated in the following form;

$(x+y)^n = \sum_{k=0}^{n} C_k^n x^{n-k} y^k$ where $C_k^n = \frac{n!}{(n-k)!k!}$, $n \geq 0$

This result was known in some form, possibly as early as the 3rd century BC. What is remarkable about Newton's generalised version of this theorem, is that it includes negative and even rational powers and introduces the idea of an infinite power series, an innovation also due to Newton. It is now formally stated in the following way;

$(x+y)^n = \sum_{k=0}^{\infty} C_k^n x^{n-k} y^k$ where $C_k^n = \frac{n(n-1)\dots(n-(k-1))}{k!}$, n rational

In order to give simple demonstrations of this theorem, in the style that Newton originally gave, consider the problem of finding an inverse to the polynomial $(1+x)$, that is an expression $q(x)$ involving x, such that $q(x)(1+x) = 1$. A simple calculation shows that any finite expression of the form $a_0 + a_1 x + \ldots + a_n x^n$, for some $n \geq 0$ is insufficient. The correct answer, provided by the generalised binomial theorem, is;

$1 - x + x^2 - \ldots + (-1)^n x^n + \ldots$

where the summation is taken over an infinite number of terms. Consider the problem of finding a square root to the polynomial $(1+y^2)$, that is an expression $q(y)$ such that $q(y)^2 = 1+y^2$. Again, a simple calculation shows that a finite expression is inadequate. This time, the correct answer is provided by;

$1 + \frac{1}{2}y^2 - \frac{1}{8}y^4 + \ldots + \frac{1}{2}\frac{-1}{2}\frac{-3}{2}\ldots\frac{1-2(n-1)}{2}y^{2n} + \ldots$

where, again, an infinite summation is used. Both these problems are considered and solved successfully by Newton in his first major work "Analysis of Equations of an Infinite Number of Terms", ([3]).

The second achievement of the years between 1665 and 1667, when Newton returned to Cambridge, was his development of the infinitesimal calculus, probably the single greatest mathematical breakthrough in human history. Newton's mathematical writings on calculus of this time, which were never published, still survive and can be found in the prodigious "Mathematical Papers of Isaac Newton", by D.T Whiteside ([4]). Together, these papers are an invaluable source for understanding the evolution of his work in this field. The most important papers are collated under the headings;

(i). "Normals, Curvature and the Resolution of the General Problem of Tangents", (1664-1665)

[3] He, in fact, considers the more general cases of inverting the expression $b+x$ and finding the square root of a^2+y^2

[4] Sadly, D.T Whiteside died on 22 April, 2008.

(ii). "The Calculus Becomes an Algorithm", (1665), (Whiteside's terminology)

(iii). "The General Problems of Tangents, Curvature and Limit Motion Analysed by the Method of Fluxions", (1665-1666)

(iv). "The October 1666 Tract on Fluxions", (1666)

We will take the opportunity to consider them, in greater detail, later in the chapter. In 1669, Newton wrote;

(v). "Analysis of Equations of an Infinite Number of Terms".

which we mentioned above. Not only does this deal with the innovation of infinite series, but also develops a general method for finding infinite series solutions to polynomial equations, now known as Newton's Theorem, and work on the theory of integration. However, for some reason, Newton decided against publication of his manuscript and it was only circulated amongst a close circle of friends. The complete manuscript did not appear in print for decades later, ([5]). Shortly afterwards, in 1671, Newton wrote

(vi). "On the Method of Fluxions and Infinite Series".

This work not only incorporated his results on infinite series from his work of 1669, but also developed his ideas on the infinitesimal calculus, in particular his work on the theory of differentiation. Again, Newton decided to withold the manuscript from publication, a full version being printed much later.([6]) In 1676, Newton completed his last major work on calculus;

(vii). "The Quadrature of Curves".

([7]), which gave his most rigorous exposition of the theory of fluxions and tangency.

In the period between 1670 and 1700, Newton made a number of pioneering breakthroughs in the field of physics. The first of these was in the subject of Optics, the theory of light. Newton's first writings on this subject date back to his years as an undergraduate and his return to Woolsthorpe Manor, collated in Whiteside's "Mathematical Papers of Isaac Newton";

(viii). "Early Notes on Reflection and Refraction", (1664)

(ix). "The Essay 'Of Refractions'", (1665-1666)

[5] A translated version was printed in 1745 along with another publication, "The Quadrature of Curves", edited by Stewart, together referred to as "Geometria Analitica"

[6] The latin manuscript "Analysis per quantitem series, fluxiones ac differentias; cum enumeratione lineares tertius ordinis" appeared in 1711, the translated version, edited by Colson, "The method of fluxions and infinite series, with its application to the geometry of curve lines" appearing in 1736.

[7] A published version appeared in 1704, appended to his "Opticks"

(x). "Refraction of Light at a Spherical Surface", (1666?)

Between 1670 and 1671, Newton collated the results of this research in "The Lucasian Lectures on Optics", (1670-1671), delivered in 1672, after his appointment in 1669, as successor to Isaac Barrow, to The Lucasian Professorship of Mathematics. In 1672, he published his first paper on Optics, in the journal "Philosophical Transactions", entitled "A New Theory About Light and Colours". Two other important unpublished optical papers survive from this period;

(xi). "Miscellaneous Researches into Refraction at a Curved Interface", (1671)

(xii). "Miscellaneous Optical Calculations", (1670)

A final version of these ideas appeared in 1704, in;

(xiii). "Opticks: A Treatise of the Reflections, Refractions, Inflexions and Colours of Light", ([8]).

Here, Newton formulated his corpuscular theory of light, as consisting of particles propogated along straight line paths, and developed his theories on refraction and reflection. Newton conducted experiments with prisms to show that light could be refracted into a continuous spectrum of coloured light, and knives, observing the hyperbolic nature of the fringes of their shadows, an optical phenomenon now known as diffraction, ([9].) The work also contained the design of the first reflecting telescope and numerous results on the theory and construction of lenses for focusing light. Appended to his treatise on Optics was a new geometric work;

(xiv). "Enumeration of Lines of the Third Order"

dealing with the classification of cubic curves, that is algebraic curves of degree 3. This involved a number of new geometrical ideas, such as the use of asymptotes to understand such curves in terms of hyperbolas, a connection that I will argue was intimately connected to his research into optics.

The second important development occurred in the theory of motion and gravitation, which Newton published in;

(xv). "Philosophiae Naturalis Principia Mathematica", (1687), (See [26]).

Newton formulated three laws of motion, summarised as follows;

[8] See [25]

[9] Newton was unable to give a clear explanation of diffraction. This led to his corpuscular theory being largely rejected outside England, in favour of the circular wave model, proposed by his contemporary Huygens, in [22]

(i). A body's velocity will remain constant if the forces acting on it are balanced.

"Every body perseveres in its state of rest, or of uniform motion in a right line, unless it is compelled to change that state by forces impressed thereon", (Book I, Axioms, or Laws of Motion).

(ii). A body's acceleration is proportional to the net force acting on it, and described by the formula $F = ma$, where F denotes force, m denotes mass and a denotes acceleration.

"The alteration of motion is ever proportional to the motive force impressed; and is made in the direction of the right line in which that force is impressed", (Book I, Axioms, or Laws of Motion).

(iii). Every action has an equal and opposite reaction. That is any applied force meets with an equal force in the opposite direction.

"To every action there is always opposed an equal reaction: or, the mutual actions of two bodies upon each other are always equal, and directed to contrary parts", (Book I, Axioms, or Laws of Motion).

Newton formulated a Universal Principle of Gravitation in (Book III, Proposition VII, Theorem VII);

"That there is a power of gravity pertaining to all bodies, proportional to the several quantities of matter which they contain"

and the inverse square law of gravitation in (Book III, Proposition VIII, Theorem VIII) (see also (Book I, Section XII);

"In two spheres mutually gravitating each towards the other, if the matter in places on all sides round about and equi-distant from the centres is similar, the weight of either sphere towards the other will be reciprocally as the square of the distance between their centres"

These laws are still accepted as fundamental to modern physics. Newton used the inverse square law of gravitation to explain the elliptical orbits of planets around the sun, confirming in a striking way earlier observations made by Johannes Kepler in 1660. He was one of the first people to consider the mathematics behind three body gravitational problems in (Book I, Proposition LXVI, Theorem XXVI). The discovery of the law of gravitation is usually associated with the famous story of an apple falling on Newton's head. It is not known whether this event is apocryphal or not.

As well as scientific studies, Newton was deeply interested in theological issues. After 1690, he wrote a number of religious tracts, concerned with the literal interpretation of the Bible. Perhaps the most important of these is his "Observations on the Prophecies of Daniel, and the Apocalypse of St. John", published in 1733. Here, he makes the interesting observation, which, is not contained in Revelations, that the sign of the Antichrist is "+++", (three crosses), rather than the more traditionally held "666". Newton conducted his research into theology with the same precision that characterised his scientific writings, his Observations are full of historical details and numerology. These clearly show that Newton believed the Bible to be the unchangeable word of God, although subject to rational interpretation. Newton was able to reconcile his rational, scientific thinking with his theological views by taking God to be a supremely rational creator, designing the universe according to prescribed, mechanical laws. Many later writers and artists, such as Blake, took this to mean that Newton enjoyed a unique status of being priviliged to the working of God's mind. His portrait of Newton (1795) is comparable to his depiction of Urizen in "Ancient of Days", measuring the world with a compass.

The rationalist philosophy to which Newton subscribed, in many ways, shaped the thinking of much of the eighteenth century. However, part of the subject of this chapter is to argue that Newton's work in geometry is underpinned by a number of aesthetic ideas, which we explored previously. In this sense, there is a latent spirituality inherent in Newton's work, which, perhaps, he was not consciously aware of, but, nevertheless, informed his highly creative scientific work. The picture of Newton drawing on aesthetic and spiritual ideas to inform his rational thinking, as opposed to his pure rationality accessing the mind of God, is, in my opinion, a truer and more attractive image of him. Given his unquestionably devout nature, Newton himself admitted that he spent more time reading the Bible than any other book, perhaps, it is the way in which he, himself, would have wanted posterity to remember him.

The later years of Newton's life were mainly involved with what has become known as the Newton-Liebniz controversy. Liebniz, a German mathematician and philosopher, began working on a variant of the infinitesimal calculus, usually known as the differential calculus in 1674, and published his first paper on the subject in 1684. As we observed earlier, Newton didn't give a full printed account of his version of the calculus until 1704, but Newton's circle of friends claimed that he had obtained the results earlier, around 1667, and before Liebniz. In 1711, members of the Royal Society accused Liebniz of plagiarism, based on previous allegations of Facio and Kiel, and the fact that he had obtained a copy of Newton's manuscript (v) from a colleague of Newton, Tschirnhaus, in 1675. It is not known whether Liebniz made use of this manuscript in his work on calculus, or, whether he had, in fact, already invented the calculus previously. However, those who question Liebniz's good faith allege that, to a man of his ability, the manuscript sufficed to give him a clue to the methods of calculus. We will consider the question of the priority dispute in greater detail shortly.

Newton moved away from Cambridge in 1696, after obtaining a position as Warden and Master of the Mint in London. It is during this time that he is alleged to have carried out much of his alchemical research, some of which can be found in the final sections of his "Optics". He was elected as President of the Royal Society in 1703, an office that he retained for the rest of his life, and was knighted in 1705. He died in London on March 31st, 1727, at the age of 84, and is buried in Westminster Abbey. His tomb, the work of the artists William Kent and Michael Rysbrack, was completed in 1731. A good description of the tomb can be found in The Gentlemen's Magazine of the same year;

"On a Pedestal is placed a Sarcophagus (or Stone Coffin) upon the front of which are Boys in Basso-relievo with instruments in their hands, denoting his several discoveries, viz.

one with a Prism on which principally his admirable Book of Light and Colours is founded; another with a reflecting Telescope, whose great Advantages are so well known; another Boy is weighting the Sun and Planets with a Stillard, the Sun being near the Centre on one side, and the Planets on the other, alluding to a celebrated Proposition in his Principia; another is busy about a Furnace, and two others (near him) are loaded with money as newly coined, intimating his Office in the Mint.

Behind the Sarcophagus is a Pyramid; from the middle of it a Globe arises in Mezzo Relievo, on which several of the Constellations are drawn, in order to shew the path of the Comet in 1681, whose period he has with the greatest Sagacity determin'd. And also the Position of the solstitical Colure mention'd by Hypparchus, by which (in his Chronology) he has fixed the time of the Argonautic Expedition - On the Globe sits the Figure of Astronomy weeping, with a Sceptre in her Hand, (as Queen of the Sciences) and a Star over the Head of the Pyramid."

The frontispiece of the tomb displays a trefoil design, characteristic of the English innovations in Gothic design. As I hope to demonstrate later in this chapter, Newton's work draws heavily on the aesthetic ideas inherent to this period, hence, his tomb seems to be an entirely fitting memorial to his life and work.

Newton's work in calculus, and the priority dispute over its invention with Liebniz, provide an excellent introduction to the deep geometric and aesthetic ideas behind his thinking. A good survey of this subject can be found in my paper [10], but I will repeat a number of the essential ideas here.

The science of calculus is well known to any college student of mathematics. Its modern rigorous formulation is primarily due to the work of the 19th century mathematicians Cauchy, Riemann and Weierstrass. However, the geometrical ideas underlying the theory are, essentially, due to Newton. I hope to make this point of view clearer in the course of this chapter. The foundations of the calculus are the ideas of differentiation and integration. In order to express the modern formulations of these ideas, we need to first briefly introduce the mathematical idea of a *limit*, the modern formulation of which is essentially due to Weierstrass. Roughly speaking, we define the limit of a function $f(x)$, at a given point x_0, to be the value obtained by the function as the variable x approaches infinitely close to x_0. In the diagram,(insert diagram "limit"), we are given two elementary functions, $y = x^2$ and $y = sign(x)$, defined by;

$sign(x) = \ \ 1$ if $x > 0$
$sign(x) = -1$ if $x < 0$

The former function has a well defined limit at $x = 0$, namely $y = 0$. We express this by saying that $lim_{x \to 0} x^2 = 0$, ([10]). The latter function, however, has no well defined limit at $x = 0$, as when x approaches 0 from values strictly greater than 0, the function $sign(x)$

[10]It is possible for a function $f(x)$ to have a well defined limit at x_0, without this actually being the value of the function at x_0. In the given example of the function $y = x^2$, we could declare the function to take the value 1 at 0, without effecting the limit calculation. If $f(x_0) = lim_{x \to x_0} f(x)$, we say that the function is continuous at x_0

approaches 1, whereas, when x approaches 0 from values strictly less than 0, the function $sign(x)$ approaches -1. The concept of differentiation originated in the problem of finding the tangent line to a curve. In the diagram, insert diagram "tangents", we are given an ellipse and want to find the equation of the tangent line to the ellipse at the marked point O. The critical geometrical observation, we will discuss its origination in greater detail later, is that the tangent line is obtained as an approximation of lines of the form OP, where P approaches infinitely close to O. The modern formulation of this idea, due to the German mathematician Karl Wierstrass, is the following;

Definition 0.1. *Differentiation*

Let $f(x)$ be a real-valued continuous function on the open interval (a,b), then we say that $f(x)$ is differentiable on (a,b) if;

$lim_{h\rightarrow 0}\frac{f(x+h)-f(x)}{h}$ $(*)$

exists for every $x \in (a,b)$. In this case, we define the derivative of f to be $\frac{df}{dx}$.

It is possible that this limit does not exist, a simple example is given by the function $y=|x|$, insert diagram "modulus". It is not possible to draw a tangent line to this function at the marked point 0, equivalently, the limit defined in $(*)$ is undefined. The idea of integration originated in computing the area under the arc of a curve, as a sequence of approximations of areas under a series of rectangles. The modern formulation is mainly due to the mathematicians Bernhard Riemann, and Henri Lebesgue, (see [4] and [36]);

Definition 0.2. *Integration*

Let $f(x)$ be a real-valued continuous function on the closed interval $[a,b]$, then, if $\epsilon_n = \frac{b-a}{n}$, and;

$s_n = \epsilon_n \sum_{j=0}^{n-1} f(a+j\epsilon_n)$

we define the integral;

$\int_a^b f(x)dx = lim_{n\rightarrow\infty} s_n$

The fundamental theorem of calculus relates the notions of differentiation and integration, showing that they are inverse procedures. Roughly speaking, if I begin with a given function $f(x)$, and then proceed to integrate and differentiate it, I return to the original one. The modern formulation and proof is credited to the French mathematician Augustin Louis Cauchy;

Theorem 0.3. *Fundamental Theorem of Calculus*

Let $f(x)$ be a real-valued continuous function on the closed interval $[a,b]$, then, if;

$F(x) = \int_a^x f(y)dy$

$F(x)$ is differentiable on the open interval (a,b) and continuous on the closed interval $[a,b]$. Moreover;

$\frac{dF}{dx}(x) = f(x)$ *for* $x \in (a,b)$

Proof:
The proof is a simple consequence of the definitions, we refer the reader to the paper [10] for more details.

As a result of the theorem, one obtains a simple method of computing integrals;

Theorem 0.4. *Suppose that $f(x)$ is a continuous function on a closed interval $[a,b]$, and $G(x)$ is an antiderivative of $f(x)$, that is a continuous function on $[a,b]$, with the property that $\frac{dG}{dx}(x) = f(x)$, on (a,b). Then;*

$\int_a^b f(x)dx = G(b) - G(a)$

Proof:
If F is the function given by the previous theorem, then the function $F - G$ is continuous and;

$\frac{d(F-G)}{dx} = \frac{dF}{dx} - \frac{dG}{dx} = f(x) - f(x) = 0$ on (a,b)

It follows easily that $F - G = c$, where c is a constant, and;

$\int_a^b f(x)dx = F(b) - F(a) = (G(b) - c) - (G(a) - c) = G(b) - G(a)$

Newton's approach to calculus is, in many ways, radically different from the modern approach. His method can be unravelled from the unpublished papers (ii), (iii) and (iv) that he wrote between 1665 and 1666, referred to above, collected in [38], and his published papers, (v),(vi),(vii),(xiv) and (xv). As we explained above, Newton's major work in the field of calculus occurred in close proximity to that of his contemporary, the German mathematician, Gottfried Liebniz, who published his results in 1684.

The first point of departure, in the methods of both Newton and Liebniz, with the modern approach is the replacement of the notion of a limit with that of an infinitesimal quantity. Roughly speaking, an infinitesimal quantity, which Newton usually denoted by o or δ, is a quantity which is non-vanishing, and yet smaller than any other finite quantity. Newton essentially justified such quantities on a geometric level, by using them to find the tangent line to a curve at a given point. In the paper (vii), sections 5 and 6 of the Introduction, we find his explanation of this method, see figure 1;

"5. Let the Ordinate BC advance from its Place into any new Place bc. Complete the Parallelogram BCEb, and draw the right Line VTH touching the curve in C, and meeting the two lines bc and BA produced in T and V: and Bb, Ec and Cc will be the Augments now generated of the Abciss AB, the Ordinate BC and the Curve Line ACc; and the Sides of the Triangle CET are in the *first Ratio* of these Augments considered as nascent, therefore the fluxions of AB, BC and AC are as the Sides CE, ET and CT of the triangle CET, and may be expounded by these same Sides, or, which is the same thing, by the sides of the Triangle VBC, which is similar to the triangle CET.

6. It comes to the same purpose to take the Fluxions in the *ultimate Ratio* of the evanescent Parts. Draw the right line Cc, and produce it to K. Let the Ordinate bc return into it's former place BC, and when the points C and c coalesce, the right line CK will coincide with the tangent CH, and the evanescent triangle CEc in its ultimate Form will become similar to the Triangle CET, and its evanescent Sides CE, Ec and Cc will be *ultimately* among themselves as the Sides CE, ET and CT of the other triangle CET, are, and therefore the Fluxions of the lines AB, BC and AC are in the same Ratio. If the points C and c are distant from one another by any small Distance, the right line CK will likewise be distant from the Tangent CH by a small Distance. That the right Line CK may coincide with the Tangent CH, and the ultimate Ratios of the lines CE, Ec and Cc may be found, the Points C and c ought to coalesce and exactly coincide. The very smallest Errors in mathematical Matters are not to be neglected".

Newton argues that, in order to find the slope of the tangent line to the curve at C, given by the ratio $\frac{ET}{EC}$, it is necessary to find the "ultimate" ratio $\frac{Ec}{EC}$, as c "coalesces and exactly coincides" with C. Clearly, if c were identified with C, there would be no "evanescent triangle" CEc for which to compute such a ratio. Whereas, if c and C are "distant from one another by any small Distance", one still only obtains a line CK, "distant from the tangent CH by a small Distance". Newton is suggesting at an infinitesimal quantity to solve the problem, and, indeed, in Section 11 of the same Introduction, he shows how to compute the gradient (or the derivative of Definition 0.1) of the function $f(x) = x^n$, which he refers to as its Fluxion;

"11. *Let the Quantity x flow uniformly, and let it be proposed to find the Fluxion of* x^n.

In the same Time that the Quantity x, by flowing, becomes $x + o$, the Quantity x^n will become $(x + o)^n$, that is, by the Method of infinite Series's, $x^n + nox^{n-1} + \frac{n^2-n}{2}oox^{n-2}+$ etc. And the Augments o and $nox^{n-1} + \frac{n^2-n}{2}oox^{n-2}+$ etc are to one another as 1 and $nx^{n-1} + \frac{n^2-2}{2}ox^{n-2} + etc.$ Now let these Augments vanish, and their ultimate Ratio will be 1 to nx^{n-1}."

In other words, if the curve in the previous figure is given by the graph of the function $y = x^n$, Newton computes the gradient of the line Cc as the ratio;

$\frac{y(x+o)-y(x)}{o} = \frac{(x+o)^n-x^n}{o}$

He then expands the expression in the numerator, and cancels the quantities involving o. In order for this calculation to make sense, it is clearly necessary that o should represent a non-zero quantity. After arriving at the final expression;

$nx^{n-1} + \frac{n^2-n}{2}ox^{n-2} + \dots$

Newton then supposes that o may be taken to be so small, that it can be set to 0 in the above expression, leaving the final fluxion to be(correctly) nx^{n-1}.

On a purely logical level, there is a problem with Newton's argument in 11. Newton concerns himself with a single infinitesimal quantity o, which he needs to be both zero and non-zero at different stages of the calculation in 11. This logical paradox was heavily criticized by the philosopher George Berkeley, in his tract, "The Analyst; Or, A Discourse Addressed to an Infidel Mathematician" (1734);

"XIV...Hitherto I have supposed that x flows, that x hath a real increment, that o is something. And I have proceeded all along on that Supposition, without which I should not have been able to have made so much as one single Step. From that Supposition it is that I get at the increment of x^n, that I am able to compare it with the Increment of x, and that I find the Proportion between the two Increments. I now beg leave to make a new Supposition contrary to the first, i.e I will suppose that there is no Increment of x, or that o is nothing; which second Supposition destroys my first, and is inconsistent with it, and therefore with every thing that supposeth it. I do nevertheless beg leave to retain nx^{n-1}, which is an Expression obtained in virtue of my first Supposition, which necessarily presupposeth such Supposition, and which could not be obtained without it: All which seems a most inconsistent way of arguing, and such as would not be allowed of in Divinity"

There is also a similar latent logical paradox in Newton's argument $(5, 6)$ on tangent lines. On the one hand, Newton needs to find a non-vanishing triangle CEc, in order to compute the ratio $\frac{Ec}{EC}$, while, on the other hand, he needs this triangle to collapse to the point C, in order for this ratio to coincide with the slope of the tangent line given by $\frac{ET}{EC}$. In the same tract, Berkeley again observes this paradox;

"XXXIV...It is supposed that the Ordinate bc moves into the place BC, so that the Point c is coincident with the Point C; and the right Line CK, and consequently the Curve Cc, is coincident with the Tangent CH. In which case the mixtilinear evanescent Triangle CEc will, in its last form, be similar to the triangle CET: And its evanescent Sides CE,Ec and Cc will be proportional to CE, ET and CT the Sides of the Triangle CET. And therefore it is concluded, that the Fluxions of the lines AB,BC, and AC, being in the last Ratio of their evanescent Increments, are proportional to the Sides of the triangle CET, or, which is all one, of the triangle VBC similar thereunto. [NOTE: Introd. ad Quad. Curv.] It is particularly remarked and insisted on by the great Author, that the points C and c must not be distant from one another, by any the least interval whatsoever: But that, in order to find the ultimate Proportions of the Lines CE, Ec, and Cc (i.e. the Proportions of the Fluxions or Velocities) expressed by the finite sides of the triangle VBC, the points C and c must be accurately coincident, i.e one and the same. A Point therefore is considered as a

Triangle, or a Triangle is supposed to be formed in a Point. Which to conceive seems quite impossible. Yet some there are, who, though they shrink at all other Mysteries, make no difficulty of their own, who strain at a Gnat and swallow a Camel."

Leaving aside the additional vitriol that Berkeley pours into his argument, ([11]), the logical inconsistencies that he observes are valid. However, there is clearly a sense that Newton's geometrical intuition is correct. The rigorous formulation of the calculus in the 19th century was able to resolve the logical problems by replacing the notion of an infinitesimal quantity with that of a limit, arriving at Definition 0.1 to replace Newton's arguments on tangents that we have considered ([12]). However, the notion of a limit draws on certain topological properties of real numbers, that they are in a sense infinitely divisible, which, perhaps, Newton was keen to avoid. Indeed, he gives the following description of Time in (vi);

" 3. Fluxions are very nearly as the Augments of the Fluents generated in equal but very small Particles of Time,..."

In the last fifty years, the theory of infinitesimals has enjoyed a modern renaissance, in the field of what is now referred to as non-standard analysis. The first major pioneer in this area was Abraham Robinson, whose book "Non-Standard Analysis" is still a definitive account of the subject. More recently, the mathematicians Zilber and Hrushovski have further developed the theory of infinitesimals in the context of Zariski structures, finding new approaches to current problems in algebraic geometry, [39] is an excellent reference.

Robinson was able to resolve the paradox, observed by Berkeley, by finding a logically consistent structure $\mathcal{R}^*$, extending the real numbers $\mathcal{R}$, which contains infinitesimal elements. ([13]). In such a structure, every bounded element r^* has a uniquely defined standard part,

[11] Berkeley also attacks the logical contradictions involving succession of infinities, which are introduced by the use of infinitesimals. This problem is resolved for the case of complex algebraic curves in [15]. With some effort, one could provide similar arguments in the case of real algebraic curves.

[12] There is no possibility that Newton could have derived his arguments on tangents from Liebniz's work, even though his paper (vii) was published later than Liebniz's work on calculus. The reader can find one of Newton's first use of the infinitesimal o notation in his unpublished paper (iii), where he uses the method outlined above, to compute the tangent line of the curve defined by $rx + x^2 - y^2 = 0$;

"Now if the Equation expressing the relation of the lines x and y be $rx + xx - yy = 0$. I may substitute $x + o$ and $y + \frac{qo}{p}$ into the place of x and y because (by the lemma) they as well as x and y doe signify the lines described by the bodys A and B. By doeing so there results $rx + ro + xx + 2ox + oo - yy - \frac{2qoy}{p} - \frac{qqoo}{pp} = 0$. But $rx + xx - yy = 0$ by supposition: there remaines therefore $ro + 2ox + oo - \frac{2qoy}{p} - \frac{qqoo}{pp} = 0$. Or divideing it by o tis $r + 2x + o - \frac{2qy}{p} - \frac{oqq}{pp} = 0$. Also those terms in which o is are infinitely less than those in which o is not therefore blotting them out there rests $r + 2x - 2\frac{qy}{p} = 0$. Or $pr + 2px = 2qy$."

Here, the quantities p and q denote the fluxions $\frac{dx}{dt}$ and $\frac{dy}{dt}$. Hence, Newton derives the correct formula for the derivative $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{q}{p} = \frac{r+2x}{2y}$.

[13] The reader should look at the paper [11], for the construction of $\mathcal{R}^*$. For technical reasons, it is convenient to work with what I refer to as analytic non-standard extensions, see [11], this assumption was in force throughout the paper [10]. The construction is carried out more rigorously in [16], where an important logical step is the use of compactness, see [21], and saturation, see [27], to guarantee the existence of infinitesimals and other technical logical properties such as "overflow" and "underflow".

which he denoted $st(r^*)$, with the property that the difference $r - st(r^*)$ is an infinitesimal. Real analytic functions $f(x)$, defined on $\mathcal{R}$, extend to well-defined functions on the non-standard structure $\mathcal{R}^*$. Each element $r \in \mathcal{R}$ has a set of elements which are "infinitely close" to it, in $\mathcal{R}^*$, which is now called an infinitesimal neighborhood $\mathcal{V}_r$;

Definition 0.5. *Infinitesimal Neighborhood*

If $r \in \mathcal{R}$, we define its infinitesimal neighborhood $\mathcal{V}_r$, to be;

$\{r^* \in \mathcal{R}^* : st(r^*) = r\}$

One may then give a consistent definition of differentiation, using infinitesimals, in the following way;

Definition 0.6. *Non-Standard Differentiation*

Let $f(x)$ be a real-valued analytic function on the open interval (a,b), then we say that $f(x)$ is non-standard differentiable on (a,b), if, for every $x \in (a,b)$, there exists $c_x \in \mathcal{R}$, such that;

$\frac{f(x+o)-f(x)}{o} \in \mathcal{V}_{c_x}$, *for every infinitesimal $o \in \mathcal{V}_0$.*

We then define the non-standard derivative of f at x to be c_x.

It is not hard to show that the non-standard definition of differentiation is equivalent to the modern Definition 0.1. The interested reader should look at Robinson's book [29] for a detailed account of his construction or my notes [11]. With this definition, it is easy to see that Newton's calculation, in (5,6) of (vii), is no longer paradoxical. If the ordinate b is taken an infinitesimal distance away from B, the gradient of the corresponding line Cc lies infinitely close to the gradient of the tangent line CT. By Robinson's construction, the gradient of the tangent CT is then determined uniquely from this information.

Although the logical problems with Newton's original method are now resolved, there is still something geometrically unsatisfactory about the resulting use of infinitesimals. This originates in Newton's comment from (6) of (vii), that "the points C and c ought to coalesce and exactly coincide". In the Definition 0.1 of differentiation using limits, given any prescribed neighborhood of B, it is possible to take the ordinate b to any point within this neighborhood. In this sense the line Cc genuinely approaches and converges to the tangent line CT. This is not the case with infinitesimals. If I was to choose two distinct infinitesimal quantities, say $\{o, o'\}$, consider the ordinates $B+o$ and $B+o'$, and the corresponding values $\{c, c'\}$, then, although the gradients of the lines Cc and Cc' both lie infinitely close to the gradient of the line CT, it is not important that one gradient lies closer to the gradient of CT than the other. In this picture, there is no *motion* of the line Cc towards the tangent line CT.

That Newton intended the geometrical picture of a sequence of lines converging towards the true tangent line, is supported by his description of mathematical quantities in 1. of (vii);

"I consider mathematical Quantities in this Place not as consisting of very small Parts; but as decrib'd by a continued Motion. Lines are describ'd, and thereby generated not by the Apposition of Parts, but by the continued Motion of Points..."

his description of fluxions as velocities in 2. of (vii);

"Therefore considering that Quantities, which increase in equal Times, and by increasing are generated, become greater or less according to the greater or less Velocity with which they increase and are generated; I sought a Method of determining Quantities from the Velocities of the Motions or Increments, with which they are generated; and calling the Velocities of the Motions or Increments, with which they are generated; and calling these Velocities of the Motions or Increments Fluxions, and the generated Quantities Fluents, I fell by degrees upon the Method of Fluxions..."

and his description of the Method of Fluxions in (vi);

"In Finite Quantities so to frame a calculus, and thus to investigate the Prime and Ultimate Ratios of Nascent or Evanescent Finite Quantities, is agreeable to the Ancients; and I was willing to shew, that in the Method of Fluxions there's no need of introducing Figures infinitely small into Geometry. For this Analysis may be performed in any Figures whatsoever, whether finite or infinitely small, so that they are imagined to be similar to the Evanescent Figures..."

Professor Goldblatt makes the important observation in [19], that this last passage supports the view that Newton was prepared to dispense with the use of infinitesimals, if he had a coherent notion of a limit available. In his later paper (xv), Newton developed what he referred to as "The method of first and last ratios of quantities", in which he comes close to formulating a reasonable definition of a limit, and, therefore, avoiding the logical paradoxes of infinitesimals;

From (viii), Lemma 1 of Section 1, Book 1; "Quantities, and the ratios of quantities, which in any finite time converge continually to equality, and before the end of time approach nearer to each other than by any given difference, become ultimately equal.

If you deny it, suppose them to be ultimately unequal, and let D be their ultimate difference. Therefore they cannot approach nearer to equality than by that given difference D; which is against the supposition."

In the case of ratios of quantities, if one takes the definition of "ultimately equal" to be the one provided by his definition of fluxions in Section 11 of (vi), that we considered above, and the definition of a limit as that provided by the remaining statement of the lemma, then Newton's proof attempts to show that these definitions coincide. Again, both the definition and proof are unrigorous by modern standards, but further support the view that Newton favoured the use of the geometrical model using limits, introducing infinitesimals as a practical technical solution.

In my opinion, the geometrical limit model that Newton uses draws on the aesthetic idea of the sublime that we considered in the previous chapter. Namely, Newton relies on the notion of an infinity, "an ultimate ratio", which is, in itself, unattainable, but may be approached through a series of finite calculations. As we will see later in the chapter, Newton refines this aesthetic idea, in his use of asymptotes as a method of analysing algebraic curves.

There is an alternative, more geometrically satisfying picture of tangency, which preserves the use of infinitesimals, see figure 2. The aesthetic idea behind this geometric model belongs to the following chapter, and we will consider its geometric implications in greater detail, when we consider Severi's work. However, as will become clearer below, Newton also considered this model, ([14]) and, hence, it is natural to consider it now, to the extent that it features in Newton's work. In this sense, the remark that he made on Time from (vi), which we considered above, reflects a more profound understanding of the geometric significance of the use of infinitesimals. In this example, I consider two complex ([15]) plane algebraic curves C and D, that is curves defined by the polynomials;

$p(x,y)=\sum_{(i+j)\leq n} a_{ij}x^iy^j$, with $a_{ij}\in\mathcal{C}$

$q(x,y)=\sum_{(i+j)\leq m} b_{ij}x^iy^j$, with $b_{ij}\in\mathcal{C}$

in the complex plane $\mathcal{C}^2$, intersecting in a point O, and sharing a common tangent line l. ([16]). I now consider what happens if I vary the coefficients of the polynomials by infinitesimal amounts([17]), that is I choose infinitesimal quantities $\{\epsilon_{ij}: i+j\leq n\}$, $\{\delta_{ij}: i+j\leq m\}$ and consider the curves C_ϵ and D_δ, defined by the polynomials;

$p_\epsilon(x,y)=\sum_{(i+j)\leq n}(a_{ij}+\epsilon_{ij})x^iy^j$

$q_\delta(x,y)=\sum_{(i+j)\leq m}(b_{ij}+\delta_{ij})x^iy^j$

In general, ([18]), one would expect to obtain 2 points of intersection, marked by the points $\{u,v\}$ in the diagram, which are at an infinitesimal distance from the point O. This is, in fact, an intuition that Newton observes in his discussion of curvature from (vi);

[14]Newton, therefore, had two geometric approaches to the theory of tangency

[15]This is a different situation to real plane curves, which Newton studied extensively in the cited papers (ii)-(vii). We will consider this case later in the chapter

[16]If a plane algebraic curve C, (real or complex), is defined by a polynomial $p(x,y)$, with $p(x_0,y_0)=0$, then we say that C is singular at (x_0,y_0), if $\frac{\partial p}{\partial x}(x_0,y_0)=\frac{\partial p}{\partial y}(x_0,y_0)=0$, otherwise we say that C is non-singular. If C is non-singular at (x_0,y_0), we define its tangent line l, to be the line defined by the equation $\frac{\partial p}{\partial x}(x_0,y_0)x+\frac{\partial p}{\partial y}(x_0,y_0)y=0$. In the particular case of a real function $y=f(x)$, which we have considered, this rule gives the tangent line l to be $\frac{df}{dx}(x_0)x-y=0$, as we saw in the explanation of differentiation above.

[17]The construction of non-standard extensions of the complex numbers $\mathcal{C}$ may be done algebraically. The reader should look at my paper [12] for more details.

[18]By which I mean, the infinitesimal quantities are chosen generically, that is there are no algebraic relations between the elements of the tuple $\{\bar{a}+\bar{\epsilon},\bar{b}+\bar{\delta}\}$, and the tuple $\{\bar{a},\bar{b}\}$ defining the original curves C and D is generic in the space of curves, tangent to the line l at O

"54. And now I have finish'd the Problem; but having made use of a Method which is pretty different from the common ways of operation, and as the Problem itself is of the number of those which are not very frequent among Geometricians: For the illustration and confirmation of the Solution here given, I shall not think much to give a hint of another, which is more obvious, and has a nearer relation to the usual Methods of drawing Tangents. Thus if from any Center, and with any Radius, a Circle be conceived to be describd, which may cut any Curve in several points; if that Circle be suppos'd to be contracted, or enlarged, till two of the Points of intersection coincide, it will there touch the Curve. And besides, if its Center be suppos'd to approach towards, or recede from, the Point of Contact, till the third Point of intersection shall meet with the former in the Point of Contact; then will that Circle be aequicurved with the Curve in that Point of Contact..." ([19])

A more precise formulation of this geometric intuition is the following;

Theorem 0.7. *Let C and D be plane complex algebraic curves, intersecting at a point O, which is non-singular for both curves. Then C and D share a common tangent line at O iff;*

For a generic choice of infinitesimal quantities $\{\bar{\epsilon},\bar{\delta}\}$;

$Card(C_{\bar{\epsilon}}\cap D_{\bar{\delta}}\cap\mathcal{V}_O)\geq 2$

The proof of this result may be found in my paper ([13]), ([20]) One can reformulate this result in the particular case of a curve intersecting a line;

Theorem 0.8. *New Geometric Formulation of Tangency*

If C is an irreducible complex algebraic curve, in particular, if C is the graph of a polynomial function $f(x)$, that is an algebraic curve of the form $y-f(x)=0$, passing through the point $O=(0,0)$, then a line of the form $y=c_O x$ is tangent([21]) to the curve at O, iff, for any non-zero choice of infinitesimal $\epsilon\in\mathcal{V}_\prime$, there exists a non-empty collection of points $\{O_1(\epsilon),\ldots,O_n(\epsilon)\}$, distinct from O, such that;

$C\cap(y-(c_0+\epsilon)x=0)\cap\mathcal{V}_O=\{O_1(\epsilon),\ldots,O_n(\epsilon)\}$

The proof of this result may be found in the paper [14]. The reader should consider figure 3, in which, by moving the tangent line l to the curve C at O, to the new position l_ϵ, one

[19]Newton defines the centre of curvature in (i) and (iii), as the limit meet of normals to a curve C. The proof that this is well defined is intrinsically connected to his proof of "The Fundamental Theorem of Calculus", that we will consider below.

[20]In this paper, I show an even more general result; that the notion of algebraic intersection multiplicity between curves C and D, coincides with a non-standard notion of intersection multiplicity, defined, as in Theorem 0.7.

[21]One may still formulate a coherent definition of tangency for singular points on an irreducible curve C, the reader should look at the paper [14] for such a definition.

obtains two new intersection points $\{A(\epsilon), B(\epsilon)\}$ infinitely close to the original point O. In this geometric picture, it is not important that these points converge to the fixed point O, but only that they are *released* from the original position O. The distinction is a subtle one, but important for more advanced geometrical constructions. Moreover, it is the most natural geometric picture to use, in conjunction with infinitesimal quantities, and one which Newton, at least superficially, seems to have considered. The method is explored, in greater detail, in the paper [14].

In order to extend these considerations to the case of real plane curves, which Newton considered, that is curves defined by a polynomial $p(x, y)$ in the plane $\mathcal{R}^2$, one needs to overcome certain technical difficulties resulting from the existence of real solutions to such polynomials. A simple example is given by the polynomial $x^2+1 = 0$, which has *no* solutions in $\mathcal{R}^2$. One would, therefore, hesitate to call this a curve. Another problematic example is given by a polynomial such as $x^2+y^8+x^4+y^2 = 0$, which only has one real solution at $(0, 0)$, due to the fact that this is a singular point for the polynomial. In such cases, it is impossible to formulate a coherent notion of tangency. The technical solution to this problem may be found in my paper [10], which the reader is encouraged to read, as it uses Newton's method of constructing power series solutions to polynomial equations, first developed in (v), which we will consider later in this chapter. The final result obtained in [10] is the following;

Theorem 0.9. *Let C be a real plane curve, such that $O = (0, 0)$ is a non-singular point for C, then a line of the form $y = c_O x$ is tangent to the curve at O, iff, in a non-standard extension of $\mathcal{R}$;*

Either*, for any positive infinitesimal ϵ, there exists at least one solution $O(\epsilon)$, distinct from O in;*

$$C \cap (y - (c_O + \epsilon)x = 0) \cap \mathcal{V}_O$$

Or*, for any positive infinitesimal ϵ, there exists at least one solution $O(-\epsilon)$, distinct from O in;*

$$C \cap (y - (c_O - \epsilon)x = 0) \cap \mathcal{V}_O$$

In this theorem, one should think of a line passing through the curve at O as rotating, either "clockwise" or "anti-clockwise" about the fixed point O. Tangency is then characterised by an intersection $O(\epsilon)$, moving away continuously from the fixed point O, along the curve, in one of these cases. This form of definition seems to improve on the slightly clumsy use of infinitesimals appearing in Newton's original definition of tangency, and, from previous remarks, is one which we he considered.

The methods of Newton and Liebniz also differ considerably from the modern approach to theory of integration. Both formulated a non-standard definition of integration, based on the idea of finding the area under a curve by summation over a series of rectangles of infinitely small width, see figure 4. The formal non-standard definition is the following;

Definition 0.10. *Non-Standard Integration*

Let $f(x)$ be a continuous function on the closed interval $[a,b]$, and let R_f be the Riemann sum, defined for a real number with $0 < c < (b-a)$, by;

$R_f(c) = \sum_{j=0}^{N(c)} f(a+jc)c$

where $N(c)$ is the greatest positive integer n such that $(a+nc) < b$. Then, we define;

$\int_a^b f(x)dx = st(R_f(\epsilon))$

where ϵ is a positive infinitesimal.

The proof that this is a good definition, that is doesn't depend on the choice of infinitesimal ϵ, and gives the same value as Definition 0.2, can be found in my notes [11]. Arguably, Liebniz was the first to formulate this definition in what would be called a rigorous way. However, that the geometric idea was known to Newton, before Liebniz's publications on calculus, is clear from his proof of the Fundamental Theorem of Calculus, given in his unpublished documents (i), see p304 of [38], which we will consider shortly.

Newton's first published explanation of integration can be found in (vii). Newton uses the term quadrature to mean integration, the term quadrature referring obliquely to approximating the area under a curve by a series of quadrangles. However, the paper is essentially a tabulation of integrals for various curves, and there is no recognisable proof of the Fundamental Theorem of Calculus (Theorem 0.3) and its corollary (Theorem 0.4), which allows one to compute explicit integrals. Newton gives a much clearer explanation of quadrature in his earlier paper (v), see figure 5;

" The Demonstration of the Quadrature of Simple Curves belonging to Rule the first.

Preparation for demonstrating the first Rule.

54. Let then $AD\delta$ be any curve whose Base $AB = x$, the perpendicular Ordinate $BD = y$, and the area $ABD = z$, as at the Beginning. Likewise put $B\beta = o$, $BK = v$; and the Rectangle $B\beta HK(ov)$ equal to the Space $B\beta\delta D$.

Therefore it is $A\beta = x+o$, and $A\delta\beta = z+ov$: Which Things being premised, assume any Relation betwixt x and z that you please, and seek for y in the following Manner.

Take at Pleasure $\frac{2}{3}x^{\frac{3}{2}} = z$; or $\frac{4}{9}x^3 = z^2$. Then $x+o$ $(A\beta)$ being substituted for x, and $z+ov$ $(A\delta\beta)$ for z, there arises $\frac{4}{9}$ into $x^3+3xo^2+3xo^2+o^3 =$ (from the Nature of the Curve) $z^2+2zov+o^2v^2$. And taking away Equals ($\frac{4}{9}x^3$ *and* z^2) and dividing the Remainders by o, there arises $\frac{4}{9}$ into $3x^2+3xo+oo = 2zv+ovv$. Now if we suppose $B\beta$ to be diminished infinitely and to vanish, or o to be nothing, v and y, in that Case will be equal, and the Terms which are multiplied by o will vanish: So that there will remain $\frac{4}{9}\times 3x^2 = 2zv$, or $\frac{2}{3}x^2(=zy) = \frac{2}{3}x^{\frac{3}{2}}y$; or $x^{\frac{1}{2}} = \frac{x^2}{x^{\frac{3}{2}}} = y$. Wherefore conversely if it be $x^{\frac{1}{2}} = y$, it shall be $\frac{2}{3}x^{\frac{3}{2}} = z$.

55. Or universally, if $\frac{n}{m+n}\times ax^{\frac{m+n}{n}} = z$; or, putting $\frac{na}{m+n} = c$, and $m+n = p$, if $cx^{\frac{p}{n}} = z$; or $c^nx^p = z^n$: Then, by substituting $x+o$ for x, and $z+ov$ (or which is the same $z+oy$) for

z, there arises c^n into $x^p + pox^{p-1}$, etc $= z^n + noyz^{n-1}etc$, the other Terms, which would at length vanish being neglected. Now taking away $c^n x^p$ and z^n which are equal, and dividing the Remainders by o, there remains $c^n px^{p-1} = nyz^{n-1}(= \frac{nyz^n}{z}) = \frac{nyc^n x^p}{cx^{\frac{p}{n}}}$, or, by dividing by $c^n x^p$, it shall be $px^{-1} = \frac{ny}{cx^{\frac{p}{n}}}$; or $pcx^{\frac{p-n}{n}} = ny$; or by restoring $\frac{na}{m+n}$ for c, and $m+n$ for p, that is m for $p-n$, and na for pc, it becomes $ax^{\frac{m}{n}} = y$. Wherefore conversely, if $ax^{\frac{m}{n}} = y$, it shall be $\frac{n}{m+n}ax^{\frac{m+n}{n}} = z$. Q.E.D"

Here, Newton demonstrates how to find the quadrature of simple curves. If a curve is given by the equation $y = x^{\frac{m}{n}}$, in Article 55, he deduces correctly the formula for the integral of the curve as $\frac{n}{m+n}x^{\frac{m+n}{n}}$. The argument he gives for this formula, however, would not be called rigorous by modern standards. In the diagram pertaining to the problem which Newton gives, (calculus5), Newton sets the area $B\beta\delta D$ to be $o.v$, where $o = length(B\beta)$ and $v = length(BK) = y$ (†). Assuming the quadrature(area) of the curve (area(ABD)) can be expressed as a polynomial expression $z(x) = x^q$, q rational, of the base $x = length(AB)$, Newton then forms the equation;

$$z + o.v = z(x+o) = (x+o)^q$$

By expanding this expression and setting the term o to be nothing, (††), he then derives the expression $y(x) = qx^{q-1}$. Finally, he argues that these steps may be reversed to give the formula for quadrature, given only the equation of the curve $y(x)$, (†††). The first argument (†) relies on the intuition that, as $length(B\beta)$ becomes sufficiently small, the quadrangle $B\beta HK$ is a good approximation to the area under the curve $BD\delta\beta$. In order to make the idea rigorous, one needs to formulate a definition of integration, using infinitesimals, similar to that given in Definition 0.10. However, the intuition is still one of two main geometric components behind such a definition, and, as we will find the other geometric component in an earlier unpublished paper, it is reasonable to say that Newton had, at this stage, formulated a clear geometric idea of integration, although, he fails, here, to formulate the idea in its entirety. The second argument (††) contains the germ of the idea of fluxions, but the exposition of it, here, is unclear. Finally, the third argument (†††), suffers from the same deficiencies as the first (†), that of a clear definition of integration, which is required, here, to make the argument rigorous.

The contents of this paper, in particular the argument given here, was the centre of the later dispute, as to whether Liebniz had obtained the idea of the differential calculus from Newton, previous to his publication of 1684. However, it is interesting to note that in Liebniz's "Excerpta from Newton's De Analisi",([22]), October 1676, he makes no comment whatsoever on this fundamental passage. Given the lack of justification in certain steps of Newton's argument, it seems reasonable that Liebniz's formulation of a rigorous definition of integration and differentiation, using infinitesimals, was a genuine independent achievement. This last conclusion is similar to that which Charles Bossut gives in "A General History of Mathematics from the Earliest Times to the Middle of the Eighteenth Century" (1802);

[22]De Analisi was the latin title of Newton's paper (iv)

"All these considerations appear to me to evince that, if the piece De Analysi per Aequationes and the letter of 1672 contain the method of fluxions, it was at least enveloped in great darkness." ([23])

I wish to now consider an argument that Newton gives in his paper (ii), see figure 6, which I will, first, quote in full;

Prop:

Haveing an equation of 2 dimensions to find w^t crooke line it is whose area it dothe expresse. suppose y^e equation is $\frac{x^3}{a}$. naming y^e quantitys, $a = dh = kl$. $bg = y$. $db = mk = x = gp$. y^e superficies $dbg = \frac{x^3}{a}$. supose y^e square $dhkl$ is equall to y^e superficies gbd; y^n $dk = z = bm = lh = \frac{x^3}{aa}$, and $aaz = x^3$. w^{ch} is an equation expressing y^e nature of y^e line fmd.
Next makeing $nm = s$ a line w^{ch} cutteth dmf at right angles. $nd = v$.

$ss - vv + 2vx - xx = \frac{x^6}{a^4} = mb$ squared.

0. 0. 1. 2. 6.

(w^{ch} is an equation haveing 2 equall rootes and therefore multiplyed according to Huddenius his method, produceth another.)

$2vx = 2xx + \frac{6x^6}{a^4}$.

$v = x + \frac{3x^5}{a^4}$. and $nb = v - x = \frac{3x^5}{a^4}$.

Now supposeing $mb : bn :: dh : bg$. that is, $\frac{3x^5}{a^4} : \frac{x^3}{aa} :: y : a$. $3xx = ay$ and $3xxa = a^2y$. Which is y^e nature of y^e line dgw. and y^e area $dbg = dklh = \frac{x^3}{aa}$, makeing $db = x$. $dh = a$. Or. $diw = deoh = \frac{x^3}{a}$, determining (di) to be (x), etc.

The Demonstration whereof is as followeth.

Suppose $w\Pi\Omega$, Ωmz, zfv etc are tangents of y^e line dmf. and from theire intersections z, Ω, v, w draw $va, zq, \Omega s, wx$ and from theire touch points draw $fw, mg, \Pi\xi$ all parallell to kp. also from y^e same points[s] of intersection draw $v\sigma, z\lambda, \Omega v, \omega\zeta$. And $mb : nb :: bt : bm :: \Omega\beta : \beta m :: kl : bg$. wherefore $\Omega\beta \times bg = \beta m \times kl$. that is y^e rectangle $kl\nu\mu = b\rho sg$. And $\pi\rho s\breve{o} = \theta\lambda\nu\mu$. in like manner it may be demonstrated y^t $aq\pi n = \theta\lambda\sigma\rho$, and $\rho\omega xy = \mu d\nu h$ etc so y^t y^e rectangle $\rho\sigma hd$ is equall to any number of such like squares inscribed twixt y^e line $n\psi$ and y^e point d, w^{ch} squares if they bee infinite in number, they will bee equall to y^e

[23]The letter referred to is one that Newton wrote to Collins, claiming to have found a general method of finding the tangent to a curve, but without giving any demonstration. As we have also observed, the method of quadrature(integration), is also obscure. Given the argument which we will establish below, that Newton had, by this stage, a full understanding of the methods of calculus, it seems likely that this obscurity was deliberate.

superficies $dn\psi\omega g\xi$, $(*)$. ([24])

The argument is essentially a proof of The Fundamental Theorem of Calculus. However, before considering it in detail, we will make a preliminary observation. This is the argument $(*)$, at the end, that the area $dn\psi\omega g\xi$ (that is the area above the lower curve between d and n) is equal to the sum of an infinite number of rectangles inscribed between the lower curve and the axis rdh. This observation, together with the argument we considered above, from Newton's paper (v), suggest that Newton had, at least, an intuitive idea of a formal definition of integration, using infinitesimals, by the time he wrote (v), ([25]). Given his use of infinitesimal arguments in the context of differentiation, see footnote 9, and his proofs, here and in (v), of The Fundamental Theorem of Calculus, it seems clear that Newton had formulated his own version of the calculus, by 1669, independently from Liebniz, and, even at the same level of precision. (The logical paradoxes observed by Berkeley are problematic for both the work of Liebniz and Newton).

Now, considering the argument in more detail, I will show that not only does Newton give a reasonable proof of The Fundamental Theorem of Calculus, but also one which is geometrically superior to the modern proofs of the result. Newton begins his argument, by setting the fixed length dh to be a, the area dbg to be $\frac{x^3}{a}$, the length of the ordinate db to be x, the length of the coordinate bg to be y and the length of the coordinate bm to be $\frac{x^3}{a^2}$. In modern terminology, Newton takes the bottom curve to be described by the equation $y(x)$ and the function;

$z(x) = \frac{1}{a}\int_0^x y(x)dx$ $(\dagger)$

which describes the top curve. He assumes that $z(x) = \frac{x^3}{a^2}$ $(**)$. Newton now makes the unusual step of drawing the *normal* to the top curve at m. Setting $length(nm) = s$ and $length(nd) = v$, he calculates;

$(v-x)^2 + length(mb)^2 = s^2$ (Pythagoras' Theorem)

deriving the formula;

$length(mb)^2 = s^2 - (v-x)^2 = s^2 - v^2 + 2vx - x^2 = (\frac{x^3}{a^2})^2 = \frac{x^6}{a^4}$ $(\dagger\dagger)$

He then differentiates the expression $(\dagger\dagger)$, to obtain;

[24]Newton uses the shorthand notation w^t for which, y^e for the, y^t for that, and y^n for then. I have made the occasional modifications to Newton's original text, as done in footnotes 48-50 of Whiteside's commentary on the paper. Finally, one should observe that the letter ζ in Newton's argument coincides with the letter h of the attached diagram. In an original, cancelled version of the figure, Newton drew the line ωd distinct from rdh, and $\omega\zeta$ a little above the horizontal axis rdh, see footnote 47 of Whiteside. Also the letter n denotes both the intersection of the vertical line $va\psi$ with the horizontal axis rdh, *and* the intersction of the normal to the curve at m with the base rdh, see Whiteside's footnote 45 in [38]

[25]In his paper (xv), Lemmas 2,3 of Section 1, Book 1, Newton gives a much clearer account of a formal definition of integration. Although he previously introduces the notion of a limit in this paper, his definition is closer to the previous Definition 0.10 using infinitesimals, than Definition 0.2, using limits

$(v-x)=\frac{3x^5}{a^4}$ ([26])

In modern terminology, $v-x$ gives the length of the subnormal to the top curve defined by $z(x)$. The general formula for the length of the subnormal is $z\frac{dz}{dx}$, (***),([27]) as, indeed, Newton deduces correctly in this particular case.

Newton continues by assuming that $\frac{length(bn)}{length(bm)}=\frac{length(bg)}{length(dh)}$. By $(***)$, this is equivalent to;

$$\frac{dz}{dx}=\frac{z\frac{dz}{dx}}{z}=\frac{y}{a}\ (****)$$

By Newton's previous argument, which we reformulated in (†), the assumption is exactly a statement of The Fundamental Theorem of Calculus, Theorem 0.3. From this assumption, and the explicit equation of the function $z(x)$, given in $(**)$, Newton is then able to derive the formula for the original "crooked line" $y(x)$, namely $y(x)=\frac{3x^2}{a}$, which was the purpose of the original proposition.

The crux of Newton's argument is, then, contained in his proof of $(****)$, "The Demonstration whereof is as followeth". Newton begins by drawing a series of tangent lines to the top curve, centred at the points $\{f,m,\Pi\}$. From the intersections of these tangent lines and the points of tangency themselves, he draws a series of lines, parallel to the lines defined by kp and dh. He then claims that the ratio $\Omega\beta:\beta m$ is equal to the ratio $kl:bg$, from which he deduces that $\Omega\beta\times bg=\beta m\times kl$, (†††). ([28]) The rectangles formed by the series of lines form a partition, which approximates the area under the bottom curve. Using the equality (†††), he shows that the area defined by this partition is equal to the area defined by the rectangle $\rho\sigma hd$. By refining the partition, that is taking an infinite number of tangent lines and intersections, he deduces the formula (†).

The circularity in Newton's argument, see footnote 25, is easily remedied by, instead, defining the function $z(x)$ by the formula $(****)$ and deducing the formula (†). A more precise version of Newton's geometric idea is formulated and proved in [10], which the reader is encouraged to read. For convenience, we state the result, although the interest is in the mechanism of the proof.

Theorem 0.11. *Newton's Version of The Fundamental Theorem of Calculus*

[26] The reference to Huddenius' method is unclear. The sequence of numbers 0.0.1.2.6 refer to the weights of x in the expression (††)

[27] The proof is an elementary exercise in differentiation and trigonometry, which we give for the convenience of the reader. Suppose that $z(x)$ defines a differentiable function and let (x_0,z_0) be a fixed coordinate on the curve C defined by $z-z(x)=0$. The vector defining the tangent to C at (x_0,z_0) is given by $(1,\frac{dz}{dx}|_{(x_0,z_0)})$. Hence, if (α,β) is the vector defining the normal to C at (x_0,z_0), we obtain, by the use of the dot product, $\alpha+\beta\frac{dz}{dx}|_{(x_0,z_0)}=0$. Hence, $\frac{\beta}{\alpha}=\frac{-1}{\frac{dz}{dx}|_{(x_0,z_0)}}$. The equation of the normal through (x_0,z_0) is given by $z-z_0=-\frac{(x-x_0)}{\frac{dz}{dx}|_{(x_0,z_0)}}$. The point of intersection x_1 of this line, with the axis $z=0$, is, therefore, $x_1=x_0+z_0\frac{dz}{dx}|_{(x_0,z_0)}$, hence, the length of the subnormal is $x_1-x_0=z_0\frac{dz}{dx}|_{(x_0,z_0)}$, as required.

[28] Unfortunately, this claim is equivalent to the assumption $(****)$ that he is trying to demonstrate. However, we will show presently how to remedy Newton's argument.

Let $y(x)$ be an analytic, ([29]), function, defined on the closed interval $[a,b]$, and let $z(x)$ be an analytic function, defined on $[a,b]$, with the additional property that $\frac{dz}{dx}=y$. Then;

$\int_a^b y(x)dx = z(b) - z(a)$

Newton's Version of the Fundamental Theorem of Calculus, Theorem 0.11, although slightly different in structure to Theorem 0.3, is equivalent for analytic functions. We give the proof, here, for the convenience of the reader;

Theorem 0.12. *Newton's Version of The Fundamental Theorem of Calculus and Theorem 0.3 are equivalent.*

Proof:
Suppose that $f(x)$ is an analytic function on the closed interval $[a,b]$, and we have shown Theorem 0.11. Let $F(x)$ be the function defined in Theorem 0.3, and let $G(x)$ be an antiderivative of $f(x)$, that is $G'(x) = f(x)$ on $[a,b]$, ([30]). By Theorem 0.11, we have that $F(x) = G(x) - G(a)$. In particular, we obtain $F'(x) = G'(x) = f(x)$, for $x \in (a,b)$. Hence, Theorem 0.3 holds for such an analytic function f. Conversely, suppose that Theorem 0.3 is shown for f, let $G(x)$ and $F(x)$ be as defined above, then we obtain that both $G'(x) = F'(x) = f(x)$ on (a,b). Applying elementary results, not depending on the theory of integration, we have that $G(x) = F(x) + c$, where c is a constant. Then $G(b) - G(a) = F(b) - F(a) = \int_a^b f(x)dx$, by definition of F. Therefore, Theorem 0.11 holds for the analytic function f as well.

The attentive reader may, at this stage, be wondering why Newton makes the step of introducing the normal to the curve defined by $z(x)$ in his Proposition which we considered above. Although unnecessary for his calculation to go through, Newton makes a connection between his definition of curvature, see footnote 16, and his proof of The Fundamental Theorem of Calculus. The relationship is the following;

(i). His calculus proof depends on the fact that;

For a given analytic function $f(x)$, if l_{x_0} denotes the tangent line to the curve C at $O = (x_0, f(x_0))$, defined by f, then, for an infinitesimal ϵ, if $l_{x_0+\epsilon}$ denotes the tangent line to the curve at $(x_0+\epsilon, f(x_0+\epsilon))$, the intersection $O_\epsilon = l_{x_0} \cap l_{x_0+\epsilon}$ lies in the infinitesimal neighborhood $\mathcal{V}_O$.

(ii). In his definition of curvature;

With the same conditions on f, if n_{x_0} denotes the normal to the curve at $O = (x_0, f(x_0))$, defined by f, then, for an infinitesimal ϵ, if $n_{x_0+\epsilon}$ denotes the normal to the curve at

[29] The formal definition of analytic is given in [10], but, roughly speaking, it is a function defined by power series, which Newton introduced in (v)

[30] Such a function is easily constructed for an analytic function f, by integrating each term of its power series expansion. This technique was also developed by Newton in (vi).

$(x_0 + \epsilon, f(x_0 + \epsilon))$, the intersection $N_\epsilon = n_{x_0} \cap n_{x_0+\epsilon}$ lies in the infinitesimal neighborhood $\mathcal{V}_K$, where K is the centre of curvature of the curve C at O.

This connection supports the idea that Newton's work in geometry is deeply related to his research into Optics, a connection which will develop further later in the chapter. The interested reader can find out more about Newton's work on curvature in, for example, the mathematical papers (i),([31]), and (iii) , and his more scientific papers $(xiii)$ and (xv).

Although the mechanism of Newton's proof of The Fundamental Theorem of Calculus is longer than the modern approach, Theorem 0.3, the geometric idea behind it is more elegant, for the following reasons. First, his argument relies on a "global" geometric relationship between the function $y(x)$ and its antiderivative $z(x)$, thus the global definition of integration is incorporated directly into the proof. In Theorem 0.3, a local property of the integrated function $y(x)$ is used, which is, on a geometric level, slightly unsatisfactory. Secondly, as we have just noted, Newton explores, in his argument, a connection between curvature and integration. The notion of locus of curvature may be defined for any algebraic curve C, using Newton's method, ([32]). Understanding the geometry of the locus of curvature was, for Newton, see footnote 29, intrinsically related to understanding the geometry of the curve C itself, and Newton could be said to establish some geometric link with his theory of integration, here. This is not only interesting in itself, but, further confirms the primary role that aesthetic and geometric connections play in Newton's work.

Although Newton's work in calculus is fundamentally important in the history of scientific ideas, in my opinion, his deepest geometric work can be found elsewhere. This work draws heavily on the aesthetic terminology that we considered in the previous chapter, in particular the images of ascent, crucifixion, fragmentation and the sublime.

The most important motivating idea towards this connection is Newton's introduction of power series, modern terminology for infinite series, in his papers (v) and (vi). We briefly encountered the idea of power series in the earlier discussion of the Generalised Binomial Theorem. Newton's discovery of power series methods for algebraic curves, which he refers to as "species", partially resulted from his consideration of "affected equations", that is polynomial equations $p(y)$ in a single variable. In Sections 19 and 20 of (vi), we find an explanation of his method of solving such equations;

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[31]In this paper, one can find Newton's first definition and computations involving the centre of curvature. He first computes the subnormal to a given curve, as in footnote 23, which is done on p278 of [38], "The Perpendiculars to crooked lines and also y^e Theorems for finding them may otherwis more conveniently be found thus." Then, using an argument, involving similar triangles, and an infinitesimal o-argument, see footnote 8, he correctly deduces the coordinates (x_1, y_1) of the centre of curvature K, at $(x, f(x))$ for the graph of $y = f(x)$ as;

$y_1 = f(x) + \frac{1+[f'(x)]^2}{f''(x)}$, $x_1 = x + \frac{f'(x)(1+[f'(x)]^2)}{f''(x)}$ $(*)$

in particular cases, see, for example, p245-248 of [38]. In Newton's diagram for the problem, see figure 7, $y_1 = length(gh)$ and $x_1 - x = length(hf)$ after setting $length(hi) = o$ to be zero in the limit calculation. The formula $(*)$ is given in full generality, using Newton's rather cumbersome differential notation, on p290 of [38]. Newton also considers the method that we discussed above, section 54 of (vi), on p262 of [38]

[32]Newton's interest in curvature is related to his idea that, if the curve C could be considered as a perfect refractive surface, light shining on the curve would be focused along the locus of curvature

" Of the Reduction of affected Equations.

19. As to affected Equations, we must be something more particular in explaining how their Roots are to be reduced to such Series as these; because their Doctrine in Numbers, as hitherto deliver'd by Mathematicians, is very perplexed, and incumber'd with superfluous Operations, so as not to afford proper Specimens for performing the Work in Species. I shall therefore first shew how the Resolution of affected equations may be compendiously perform'd in Numbers, and then I shall apply the same to Species.

20. Let this Equation $y^3 - 2y - 5 = 0$ be proposed to be resolved, and let 2 be a Number (any how found) which differs from the true Root less than by a tenth part of itself. Then I make $2 + p = y$, and substitute $2 + p$ for y in the given Equation, by which is produced a new Equation $p^3 + 6p^2 + 10p - 1 = 0$, whose Root is to be sought for, that it may be added to the Quote. Thus rejecting $p^3 + 6p^2$ because of its smallness, the remaining Equation $10p - 1 = 0$; or $p = 0.1$, will approach very near to the truth. Therefore I write this in the Quote, and suppose $0.1 + q = p$, and substitute this fictitious Value of p as before, which produces $q^3 + 6.3q^2 + 11.23q + 0.061 = 0$. And since $11.23q + 0.061 = 0$ is near the truth, or $q = -0.0054$ nearly, (that is, dividing 0.061 by 11.23, till so many figures arise as there are places between the first Figure of this, and of the principal Quote exlusively, as here there are two places between 2 and 0.005) I write -0.0054 in the lower part of the Quote, as being negative, and supposing $-0.0054 + r = q$, I substitute this as before. And thus I continue the Operation as far as I please, in the manner of the following Diagram:" ([33])

In other words, Newton makes a series of approximations to solutions of the original equation $y^3 - 2y - 5 = 0$. At the second and third stages of the approximation, the remainder terms, 0.1 and -0.0054 become smaller, hence, one is guaranteed that the infinite sum obtained by adding the remainders to the original quote 2, will converge to a genuine solution of the original equation. The reader can think of such a solution as an infinite decimal expansion on a calculator.

Newton extends this method to find power series solutions to polynomial equations $G(x, y)$ in two variables, in Section 36 of (vi), "The Praxis of Resolution";

"36. These things being premised, it remains now to exhibit the Praxis of Resolution. Therefore, let the Equation $y^3 + a^2y + axy - 2a^3 - x^3 = 0$ be proposed to be resolved. And from its Terms $y^3 + a^2y - 2a^3 = 0$, being a fictitious Equation, by the third of the foregoing Premises, ([34]), I obtain $y - a = 0$, and therefore I write $+a$ in the Quote. Then because $+a$ is not the complete Value of y, I put $a + p = y$, and instead of y, in the Terms of the Equation written in the Margin, I substitute $a + p$, and the Terms resulting ($p^3 + 3ap^2 + axp$, etc) I again write in the Margin; from which again, according to the third of the Premises, I select the Terms $+4a^2p + a^2x = 0$ for a fictitious Equation, which giving $p = \frac{-1}{4}x$, I write $\frac{-1}{4}x$ in the Quote. Then because $\frac{-1}{4}x$ is not the accurate Value of p, I put $\frac{-1}{4}x + q = p$, and in the marginal Terms for p I substitute $\frac{-1}{4}x + q$, and the resulting Terms ($q^3 - \frac{3}{4}xq^2 + 3aq^2$, etc) I again write in the Margin, out of which, according to the foregoing Rule, I again select the

[33]The diagram is omitted, as the written text is a clear enough explanation of Newton's method.

[34]We explain this reference below.

Terms $4a^2q - \frac{1}{16}ax^2 = 0$ for a fictitious Equation, which giving $q = \frac{x^2}{64a}$, I write $\frac{x^2}{64a}$ in the Quote. Again, since $\frac{x^2}{64a}$ is not the accurate Value of q, I make $\frac{x^2}{64a} + r = q$, and instead of q I substitute $\frac{x^2}{64a} + r$ in the marginal Terms. And thus I continue the Process at pleasure, as the following Diagram exhibits to view." ([35])

Here, Newton finds a power series solution to the equation defined by $y^3 + a^2y + axy - 2a^3 - x^3 = 0$. After finding a as an initial approximation, the remainder terms, $\frac{-1}{4}x$ and $\frac{x^2}{64a}$, at the second and third stages, involve successively higher powers of x. Although Newton doesn't give a completely rigorous proof of the fact, he argues that the process will continue to yield successively higher powers of x as remainder terms. Hence, one eventually obtains a power series solution of the form $\sum_{i=0}^{\infty} y_i(x)$, where $y_i(x)$, the i'th remainder term, is of the form $a_i x^{j(i)}$ with $a_i \in \mathcal{R}$, $j(i) \geq i$ is an integer, and $j(i') > j(i)$, for $i' > i$. There is an obvious parallel with the previous example, the higher powers of x correspond to the successively higher decimal places of the numerical approximation, ([36]). A completely rigorous proof of Newton's construction is given in the following lemma;

Lemma 0.13. *Let C be a real (or complex) plane curve, defined by a polynomial $G(x, y)$, passing through the origin O, such that O is a non-singular point for C. Then, possibly after a linear change of variables, one can find a power series of the form;*

$s(x) = \sum_{j=1}^{\infty} a_j x^j$, with $a_j \in \mathcal{R}$ (or $a_j \in \mathcal{C}$ respectively)

such that $G(x, s(x)) \equiv 0$.

Proof:

After making a linear change of variables, one can assume that the tangent line to C at $(0, 0)$, defined algebraically as in Footnote 13, does not coincide with the axis $x = 0$. Equivalently, one may assume that $G(0, 0) = 0$ and $\frac{\partial G}{\partial y}(0, 0) \neq 0$, $(\dagger)$. We follow Newton's method in Section 36 above. The assumption $(\dagger)$ allows us to write;

$G(x, y) = p_m(x)y^m + \ldots + p_1(x)y + p_0(x)$ $(*)$

with $p_0(0) = 0$ and $p_1(0) \neq 0$. For (x, y) "small", Newton observes that it, therefore, makes sense to take as a first approximate solution to this equation;

$y_0 = \frac{-\lambda_0 x^{i_0}}{p_1(0)}$

where $\lambda_0 x^{i_0}$, $(i_0 \geq 1)$, is the first term in the expression for $p_0(x)$, this is the content of his "third of the foregoing Premises", ([37]).

[35] Again, I have omitted the accompanying diagram.

[36] The interested reader can find similar calculations on "affected equations" and "species" in Sections 21-32 of (v)

[37] The content of the third premise in (vi) is the following;

"27. Thirdly, when the Equation is thus prepared, the work begins by finding the first Term of the Quote; concerning which, as also for finding the following Terms, we have this general Rule, when the indefinite

Now, Newton makes the substitution $y = (y' + y_0)$ in $(*)$, this results in a further polynomial equation of the same form;

$$q_m(x)y'^m + \ldots + q_1(x)y' + q_0(x) = 0 \ (**)$$

By a straightforward algebraic calculation, using the fact that $i_0 \geq 1$, one checks that $q_1(0) \neq 0$ and $ord(q_0(x)) > ord(p_0(x))$. Hence, one can take as the second quote;

$$y_1 = \frac{-\lambda_1 x^{i_1}}{q_1(0)}$$

where $\lambda_1 x^{i_1}$, $(i_1 > i_0)$, is the first term in the expression for $q_0(x)$. Continuing in this way, one obtains a sequence of approximate solutions;

$$s_n(x) = y_0(x) + y_1(x) + \ldots + y_n(x), \text{ for } n \geq 0$$

Either this process terminates after a finite number of approximations, giving a polynomial solution to $(*)$, or one obtains an infinite power series;

$$s(x) = \sum_{i \geq 0} y_i(x)$$

A straightforward algebraic calculation shows that;

$$ord(G(x, s_{n+1}(x))) \geq ord(G(x, s_n(x))) + 1, \ ([38])$$

Species (x or z) is supposed to be small; to which Case the other two Cases are reducible.

28. Of all the Terms, in which the Radical Species (y,p,q, or r, etc.) is not found, chuse the lowest in respect of the Dimensions of the indefinite Species (x or z, etc) then chuse another Term in which that Radical Species is found, such as that the Progression of the Dimensions of each of the fore-mentioned Species, being continued from the Term first assumed to this Term, may descend as much as may be, or ascend as little as may be. And if there are any other Terms, whose Dimensions may fall in with this Progression continued at pleasure, they must be taken in likewise. Lastly, from these Terms thus selected, and made equal to nothing, find the Value of the said Radical Species, and write it in the Quote."

(In the proof of the lemma, x is the indefinite species and y is the radical species.)

[38] *ord* is the modern abbreviation for *order*, that is the value of the highest power of x appearing in a polynomial $p(x)$

Hence, by elementary arguments, we are guaranteed that $G(x, s(x)) \equiv 0$, ([39]).

The assumption of "non-singularity" in Lemma 0.13 is necessary, in order to ensure that the obtained power series does not involve fractional powers of x, that is powers of the form $x^{\frac{m}{n}}$, for $n \geq 2$. If we consider the example of a curve C, defined by $y^2 - x^3 = 0$, then the only possible power series solution of this equation is given by $y = x^{\frac{3}{2}}$, the curve in this case has a "cusp" singularity at the origin O. However, Newton also gives a general method for finding power series solutions to polynomial equations of the form $G(x, y) = 0$, without the simplifying assumption that $\frac{\partial G}{\partial y}(0, 0) \neq 0$. This is done by the introduction of "Newton's parallelogram", in Paragraph 29 of his paper (vi), and fractional exponents. However, although Newton's method is ingenious, as I will demonstrate below, the introduction of what are now referred to as Puiseux series, is not particularly useful, in the sense of understanding the global geometry of algebraic curves. It is to this subject that I now wish to turn.

In my opinion, Newton's finest contribution to the study of algebraic curves and also his most important geometric work is the paper (*xiv*), which will be my major source for the appraisal of his work in this field. Before considering this paper in greater detail, I wish to give a geometric interpretation of Newton's construction of power series in relation to algebraic curves. This interpretation begins with what is today known as "Newton's Theorem".

The modern formal statement of Newton's theorem, given in [1], is the following;

Theorem 0.14. *Let $\mathcal{C}$ be an algebraically closed field, and let;*

$G(x, y) = y^n + a_1(x)y^{n-1} + \ldots + a_n(x)$ *in* $\mathcal{C}(x)[y]$

[39]Newton's method of constructing a solution to the polynomial equation $G(x, y) = 0$, given in the Lemma, is closely related to what is now known as the Newton-Raphson method. Namely, one considers the function;

$G : \mathcal{R}[x] \to \mathcal{R}[x]$, $G(y) = p_m(x)y^m + \ldots + p_1(x)y + p_0(x)$ (†)

Having obtained a first approximation $y_0 = s_0$ to the equation $G(y) = 0$, the Newton-Raphson method gives the further approximation;

$s_1 = y_0 - \frac{G(y_0)}{G'(y_0)} = y_0(x) - \frac{q_0(x)}{q_1(x)}$

where $q_0(x)$ and $q_1(x)$ are obtained from the transformed polynomial (∗∗) in the Lemma. By a similar argument to the above, replacing the successive approximations $-\frac{\lambda_1 x^{i_1}}{q_1(0)}$ by $-\frac{q_0(x)}{q_1(x)}$ and noting that;

$\frac{q_0(x)}{q_1(x)} = \frac{\lambda_1 x^{i_1}}{q_1(0)} u(x)$ for a unit $u(x) \in \mathcal{R}[[x]]$

one is similarly guaranteed that this method also yields the same power series solution $s(x)$ to the equation $G(y) = 0$ in (†). The Newton-Raphson method is usually applied to functions of a real variable, rather than the rings $\mathcal{R}[x]$ or $\mathcal{R}[[x]]$. However, that Newton intended his method of finding power series solutions to polynomial equations of the form $G(x, y) = 0$, ("species"), to be a partial generalisation of this method is borne out by his consideration of the case of "affected equations", that we considered above. Newton's work in (v), and similar calculations in (iv), should, therefore, be considered as a principal origination of this idea.

be a monic polynomial of degree $n > 0$ in y, with coefficients $\{a_1(x), \ldots, a_n(x)\}$ in $\mathcal{C}(x)$. Then, there exists a positive integer m, such that;

$$G(t^m, y) = \prod_{i=1}^{n}[y - \eta_i(t)], \text{ with } \eta_i(t) \text{ in } \mathcal{C}((t))$$

On an algebraic level, one may view this theorem as a generalisation of "The Fundamental Theorem of Algebra" to polynomials in 2 variables, see also Footnote 36 and the above consideration of "affected equations". Just as any polynomial in 1 variable, of degree n, admits n complex number solutions, any polynomial in 2 variables of degree n, as defined above, admits n infinite series solutions, if we allow for fractional exponents and a finite number of inverse powers. Let us give some simple examples to illustrate the theorem. Consider the polynomial $y^2 - x^3$. This factorises as $(y - x^{\frac{3}{2}})(y + x^{\frac{3}{2}})$, demonstrating the necessity for fractional powers. The polynomial $y^2 - (1 + x)$, on the other hand, factorises as $(y - \eta(x))(y + \eta(x))$, where $\eta(x)$ is given by;

$$\eta(x) = 1 + \frac{x}{2} - \frac{x^2}{8} + \ldots + \frac{(1.(-1).\ldots.(1-2(n-1)))}{2^n n!}x^n$$

demonstrating, here, the necessity for infinite series. Finally, consider the polynomial $x^2y^2 - 1$, which factors as $(y - \frac{1}{x})(y + \frac{1}{x})$, showing that inverse exponents are needed.

The proof of the theorem is a straightforward generalisation of Newton's method, given in Section 36 of (vi), for finding power series solutions to curves, see also Lemma 0.13, his invention of the parallelogram method for fractional exponents, and his analogy with "affected equations", in Sections 19 and 20 of (vi). However, I wish to argue that the real justification for the theorem is reflected not in this numerical analogy, ([40]), but, in the way it is intuitively used by Newton, at a deeper, more profound geometric level, in the study of plane cubic curves.

In order to do this, it is necessary to find geometric interpretations of some of the terminology that Newton uses in (vi) and (xiv). In doing so, we will find deep connections with the aesthetic ideas that we explored in the previous chapter.

The interpretation of Newton's power series solutions, that we discussed in Lemma 0.13, can be found in [13]. Here, the notion of an etale cover of the plane is introduced. Such a cover can be thought of as a surface S, which locally imitates the geometry of the plane P near a given point O, but, on a global level, is more complicated, ([41]). An infinite series of the form $y - s(x)$, as in Lemma 0.13, corresponds to a curve $C' \subset S$, passing through the point O^{lift}, which projects onto the curve $C \subset P$, defined by the polynomial $G(x, y)$, see figure 8. By the nature of the ambient surface S, the curve C' reflects the geometry of C in a local neighborhood of O^{lift}, though, again, the global geometry of C' is, in general, more complex, ([42]).

The intuition of lifting a curve C, lying in the plane, to a surface in 3-dimensional space, recalls the aesthetic idea of ascent that we considered in the previous chapter. The curve C' is a 3-dimensional manifestation of the planar curve C. In Christian terminology, see

[40] Indeed, over non-algebraically closed fields, such as $\mathcal{R}$, the theorem fails. As we explain below, Newton uses a geometric version of the result, in his study of real plane cubic curves.

[41] A formal definition is given in [13]

[42] An excellent discussion of the etale topology can be found in [23]

my paper [9], it can be seen as a spiritual 3-dimensional interpretation of a 2-dimensional physical reality. It is, perhaps, no coincidence that the modern invention of etale covers occurred in Paris, one of the major medieval centres in the development of the aesthetic of ascent.

In the context of Newton's Theorem, the power series solution guaranteed by Lemma 0.13, corresponds to a factorisation of the polynomial $G(x,y)$ into the form;

$G(x,y) = (y - s(x))u(x,y)$ (†)

where $u(x,y)$ is an infinite series in the variables $\{x,y\}$ such that $u(0,0) \neq 0$. As before, we can interpret the infinite series $u(x,y)$ as a curve $C'' \subset S$, which projects onto C but does not pass through the point O^{lift}, see figure 9.

In this example, we see the aesthetic idea of fragmentation, the curve C breaks into 2 components on the surface S. More specifically, the configuration of the curves C' and C'' is reminiscent of the image of the crucifixion. This idea becomes even more geometrically intuitive when we consider S as part of a continuous family of surfaces S_t, for $t \in [0,1]$, moving between the plane $P = S_0$ and the surface $S = S_1$. For $t \in [0,1)$, the curve C lifts to a single curve C'''. At the critical point $t = 1$, we observe the breaking effect into the component curves C' and C''.

In this interpretation, we can see the geometric subtlety of Newton's Theorem in the effective combination of the aesthetics of ascent and fragmentation. It is this form of geometric intuition, paralleled by aesthetic considerations, which I suggested, in the previous chapter, motivated such creative and innovative constructions as the rib vault of Durham Cathedral.

One can refine this intuition further, by introducing an algebraic result that generalises the decomposition, given in (†), known as Weierstrass preparation, ([43]). This is formally stated in [1] as follows;

Theorem 0.15. *Weierstrass Preparation*

Let $\mathcal{C}$ be an algebraically closed field, and let $G(x,y)$ be a polynomial with $G(0,y) \neq 0$ and $d = ord_y G(0,y)$. Then there exist unique series $U(x,y)$ and $N(x,y)$, with $U(0,0) \neq 0$, such that;

$G(x,y) = N(x,y)U(x,y)$

and

$N(x,y) = y^d + c_1(x)y^{d-1} + \ldots + c_d(x)$

with $c_i(x) \in \mathcal{C}[[x]]$ and $c_i(0) = 0$, for $1 \leq i \leq d$.

In the case considered by Lemma 0.13, the infinite series $N(x,y)$ is given by $y - s(x)$. However, in general, the curve C may posess a more complicated singularity at the origin

[43] Weierstrass was a German mathematician of the 19th century, best known for his work in the field of analysis, which we briefly alluded to at the beginning of the chapter

O, for which Newton's construction, summarised in Lemma 0.13, no longer applies. The relevant case to our discussion occurs when the singularity is a node(ii), a particular example of a double point(i), both definitions are given rigorously in the paper [17]. Intuitively, a double point occurs when any infinitesimal variation of a line l, passing through O, intersects the curve C in two points, (see footnotes 14 and 15), while a node occurs when the curve C intersects itself in a cruciform shape, see figure 10. Newton could be said to consider both definitions in (xiv);

(i). Section V; Of Double Points in Curves.

"We have remarked that curves of the second genus can be cut by a straight line in three points. Sometimes two of these three points coincide, as in the case when the straight line passes through an infinitely small oval, or through the intersection of two parts of the curve cutting each other, or meeting in a cusp. And whenever all the straight lines, extending in the direction of any infinite branch, cut the curve in only one point (as occurs in...(text omitted)...) we must conceive that those straight lines pass through two points in the curve at an infinite distance. Two intersections of this sort when they coincide, whether at a finite or an infinite distance, we shall call a double point."

(ii). Section III; The Names of the Curves.

"that which intersects and returns in a loop upon itself, the nodate hyperbola."

Section IV; The Enumeration of Curves.

Newton's description of the 2nd species as a "nodate figure", see figure 11.

Newton's description of the 7th and 8th species, "the figure will be cruciform", see figure 12.

Newton's description of the 34th species, using the term "node", see figure 13.

When a node of $C \subset P$ occurs at the origin O, being a double point, the Weierstrass Theorem guarantees that $d = 2$ for the infinite series $N(x, y)$. Using the elementary method for solving quadratic equations and geometric properties of a node, one can obtain the factorisation;

$$N(x, y) = (y - s_1(x))(y - s_2(x))$$

where $s_1(x)$ and $s_2(x)$ are distinct infinite series, with the properties of $s(x)$, given in Lemma 0.13. Using the method above, we then obtain the following geometric interpretation of this result. The infinite series $s_1(x)$ and $s_2(x)$ correspond to curves C_1 and C_2 on an etale cover S, passing through O^{lift}. As the geometry of S imitates the geometry of the plane P near O, the curves C_1 and C_2 also intersect in a node, see figure 14. In this example, we see that the process of fragmentation occurring on the surface S, imitates the geometry of C itself, the curve fragments over the nodal point at O. This phenomenon can be used to understand the geometry of a planar curve C in a neighborhood of such points, by reducing a potentially difficult problem concerning singularities to that of a simpler problem concerning the intersection of distinct curves.

This method is used in the paper [17] to deduce local properties of a family of nodal curves, namely that one obtains 2 intersections between "infinitely close" curves, in a neighborhood of a node, see figure 15. We will see later how generalisations of this result are important in the study of certain types of degenerations of plane curves. The proof of this result introduces a new idea, that of conic projections, to relate the geometry of the 3-dimensional "cruciform figure", determined by C_1 and C_2, with the plane "nodate figure", determined by C, see figure 14. The method of conic projections is a subject that we will consider further, in a later chapter. Newton was also aware of this method, which he refers to in (xiv) as "The Generation of Curves by Shadows";

Section V. The Generation of Curves by Shadows.

" If the shadows of curves caused by a luminous point, be projected on an infinite plane, the shadows of conic sections will always be conic sections; those of curves of the second genus will always be curves of the second genus; those of the third genus will always be curves of the third genus; and so on ad infinitum.

And in the same manner as the circle, projecting its shadow, generates all the conic sections, so the five divergent parabolas, by their shadows, generate all other curves of the second genus. And thus some of the more simple curves of other genera might be found, which would form all curves of the same genus by the projection of their shadows on a plane."

The use of conic projections introduces a new aesthetic, that of light, which we will discuss at greater depth in the following chapter. The reader is referred to the paper [17], where some aesthetic remarks on light and fragmentation are made, in relation to the use of circular windows with cruciform arches. This combination of designs occurs frequently in medieval abbeys and cathedrals, particularly in France and Italy.

The considerations we have made on the geometrical interpretation of Newton's Theorem, have, so far, taken into account only local properties of plane curves. However, there is a sense that the theorem provides information on a curve's global geometry. In order to make this idea precise, it is necessary to introduce the geometric concept of asymptotes.

A formal definition of asymptotes can be found in the paper [18](Section 2), along with a method of computing the asymptotes of any given plane algebraic curve. Newton was the first writer to make extensive geometric use of asymptotes in his paper (xiv). Intuitively, an asymptote is the limiting direction of the tangent to a point on a plane curve C, as the point moves towards infinity. In the following example, when the curve C is defined by the equation $y = \frac{1}{x}$, the asymptotes are given by the lines $x = 0$ and $y = 0$. It is clear from figure 16 how these correspond to the limiting directions of tangent lines to the curve. Newton gives such a definition in (xiv), when he discusses hyperbolic and parabolic branches;

Section II. 5. Of Hyperbolic and Parabolic Branches, and their Directions.

"All infinite branches of curves of the second and higher genus, like those of the first, are either of the hyperbolic or parabolic sort. I define a hyperbolic branch, as one which constantly approaches some asymptote, a parabolic branch to be that which, although infinite, has no asymptote. These branches are easily distinguished by their tangents, for supposing the point of contact to be infinitely distant, the tangent of the hyperbolic branch will coincide with the asymptote, but, the tangent of the parabolic branch, being at an infinite distance, vanishes, and, is not to be found"

In this passage, Newton makes a distinction between two types of behaviour of a curve C at infinity. In the hyperbolic case, the curve C approaches a limiting direction, defined by a line in the (x, y) plane, as in the previous example, the curve C posesses 2 hyperbolic branches at infinity. In the parabolic case, there is no such limiting direction, a simple example is given by the curve $y = x^2$. This distinction is developed in the paper [18](Section 3), when we consider covers of the line, having certain geometrical properties, which we consider later. Such covers have the feature that the curve intersects itself numerous times at a given point O_∞ at infinity. Using Newton's interpretation of asymptotes and his distinction between hyperbolic and parabolic branches at infinity, one can adopt a more convenient representation of such curves, see figure 17. A famous example of such a representation is "Newton's Trident", fig 74. of (xiv), see figure 18;

Section IV. 12. Of the Trident.

"In the second cited case of the equations, we had the equation $xy = ax^3 + bx^2 + cx + d$. In this case, the curve will have four infinite infinite branches, of which two are hyperbolic about the asymptote AG, extending on contrary sides, and two are parabolic, converging and making with the other two a sort of trident-shaped figure."

In the paper [18], the representation is used to study the behaviour of such curves C, when projected to the x-axis. Formally speaking, such projections are not defined at the point O_∞. Using vertical asymptotic lines, one can still, however, analyse the behaviour of the projection at O_∞, as a limit of points on the curve.

The notion of an asymptote recalls the aesthetic of the sublime that we considered in the previous chapter, the eye is drawn towards a fixed point at infinity along the curve. A particularly geometric use of this device can be found in the numerous examples of lancet windows, belonging to English medieval buildings. There is also the connection with the spiritual image of the Throne of God, represented by an inaccessible point at infinity.

Perhaps, Newton's greatest achievement in (xiv), is his use of the asymptotic method, to understand the global geometry of cubic curves. This begins with the following striking observation;

Section III. The Reduction of all Curves of the Second Genus to Four Cases of Equations.

"All line of the first, third, fifth, seventh, or odd orders, have at least two infinite branches extending in opposite directions; and all lines of the third order have two branches of the same kind, proceeding in opposite directions, towards which no other of their infinite branches proceed, (except only the Cartesian parabola).

Case I (fig 1.)

If the branches be hyperbolic, let GAS be their asymptote, and let CBc be any line drawn parallel to it, meeting the curve on each side(if possible); let this line be bisected in X, which will be the locus of a hyperbola, say $X\Phi$, one of whose asymptotes is AG. Let its other asymptote be AB and the equation defining the relation between abciss AB and ordinate BC, if $AB = x, BC = y$, will be of the form always;

$xy^2 + ey = ax^3 + bx^2 + cx + d$", (i), ($\sharp$), see figure 19, labels (a) and (b) are not part of Newton's original sketch, see Footnote 41 below) ([44])

In this passage, Newton argues that any cubic curve C must have at least 2 infinite(opposite) branches. This is due to the observation that any real cubic polynomial in 1 variable has a real root. Hence, the curve C must have a limit as $x \rightarrow \infty$ or $x \rightarrow -\infty$. In the case that these branches are hyperbolic ,by taking the asymptote to these branches as a coordinate axis, we obtain the following picture, see figure 20. At $x = 0$, the curve C has a solution with multiplicity 2 at ∞, due to tangency of the axis, hence, at most 1 real(complex) solution in finite position. This reduces the equation of C to the following form;

$xy^2 - (ex^2 + fx + g)y = ax^3 + bx^2 + cx + d$ (†)

as if there were any isolated terms of the form y^2 or y^3 in the equation, then, setting $x = 0$, one would obtain a polynomial of degree at least 2 in y, implying the existence of at least 2 finite solutions along the axis $x = 0$. The remainder of Newton's argument implicitly uses the method of completing the square, namely, we obtain;

$y = \frac{ex^2+fx+g}{2x} \pm \sqrt{(\frac{ex^2+fx+g}{2x})^2 + \frac{ax^3+bx^2+cx+d}{x}}$ (††)

The equation of the hyperbola $X\Phi$ that Newton refers to, is then given by $y = \frac{ex^2+fx+g}{2x}$. Newton then simplifies the equation of the hyperbola $X\Phi$ to the form $y' = \frac{e}{x'}$, by taking the asymptotes of the hyperbola as coordinate axes. This reduces the equation (†) to the form (i), under generic assumptions on distinct real roots appearing in the equation (††). Equation (ii) follows from consideration of the degenerate case, when the expression under the square root is set identically to zero. Equations (iii) and (iv) occur from consideration of the case when there are two infinite(opposite) parabolic branches.

The major part of Newton's work in (xiv) is concerned with analysing the different cases

[44]Newton makes a further analysis, reducing to the further equations;

$xy = ax^3 + bx^2 + cx + d$ (ii)
$y^2 = ax^3 + bx^2 + cx + d$ (iii)
$y = ax^3 + bx^2 + cx + d$ (iv)

The four cases of equations account for Species 1-65, 66, 67-71 and 72 in Newton's classification.

that occur for the equation (i). Curves satisfying the Case I equation all have the property that they possess at least 1 asymptote. Newton is able to make some straightforward distinctions between such curves, by considering, first, the number of asymptotes, either one or three, and, in the case of three asymptotes, their arrangement in the plane. Examples of Case I curves with a single asymptote are given in Species 33-56 (fig. 43-64) of Newton's classification, described in Section IV, 5-8. A description of some of these, parabolic hyperbolas, can be found in the paper [18], the following diagram figure 21 (fig. 64), is an example of such a curve. For Case I curves with 3 distinct asymptotes, Newton's analysis proceeds by considering their possible configurations. One would usually expect that 3 such asymptotes intersect in a triangular configuration, which I will refer to as lines in general position. Indeed, these account for Species 1-23 (fig. 5-34) of Newton's classification, described in Section IV, 1-3. A description of some of these, triple hyperbolas, can be found in [18], the previous diagram, figure 19, appearing as (fig. 1 and 5) in Newton's sketches, is an example "Of the nine redundant Hyperbolas, having no Diameter, and three Asymptotes, making a Triangle". Occasionally, one obtains the exceptional configuration of three asymptotes intersecting in a point, these account for Species 24-32 (fig. 35-42) of Newton's classification, described in Section IV, 4, "Of the Nine Redundant Hyperbolas, with three Asymptotes, converging to a common point", the following diagram, figure 22, appears as (fig 42) in Newton's sketches. Another exceptional configuration is that of 3 parallel lines in the plane, accounting for Species 57-60 (fig. 65-68), described in Section IV, 9, "Of the four Hyperbolisms of the Hyperbola", the following diagram, figure 23, appears as (fig 68) in Newton's sketches. Curves satisfying the Cases II-IV equations are more straightforward to analyse, accounting for Species 66-71 (fig. 74-81), described in Section IV, 12-14. Newton's Trident, (fig. 74), which we considered above, is the only example of a Case II curve, the following diagram, figure 24, (fig. 81), is an example "Of the Cubic Parabola", the only Case IV curve. The use of asymptotes, however, is not the only method by which Newton distinguishes between cubic curves. Newton is able to use a more geometrically subtle method to distinguish, for example, between the Species 1-23, having 3 asymptotes in general position. The basis for this method is the use of hyperbolas to decompose the global geometry of the curve. In the previous passage, ($\sharp$), where Newton derives the Case I equation, we find a description of the hyperbola $X\Phi$, given by the equation $y = \frac{ex^2+fx+g}{2x}$ in ($\dagger$). Further hyperbolas, corresponding to the actual locus of the curve C, are obtained, by using Newton's method to expand the root term as an infinite series. ([45])

Newton makes use of this interpretation intuitively. It seems likely, from previous considerations and the confident geometric manner in which the text is written, that Newton, in fact, carried out a number of algebraic calculations with such series, which were never

[45] Rigorous calculations can be found in Talbot's analysis of Newton's text, [34]. From the Case I equation $xy^2 - ey = ax^3 + bx^2 + cx + d$, he derives;

$$y = \frac{e}{2x} \pm \sqrt{ax^2 + bx + c + \frac{d}{x} + \frac{e^2}{4x^2}}$$

$$y = \frac{e}{x} + \frac{d}{e} + Ax + Bx^3 + \ldots \text{ (a)}$$

$$y = \frac{-d}{e} - Ax - Bx^2 - \ldots \text{ (b)}$$

describing the hyperbolas, labelled (a) and (b) respectively, in figure 19.

included. In the following passage, he gives a geometric description of the hyperbolas which are used;

"Section III The Names of the Curves.

In the enumeration of these cases of curves, we shall call that which is included within the angle of the asymptotes in like manner as the hyperbola of the cone, the *inscribed* hyperbola; that which cuts the asymptotes, and includes within its branches the parts of the asymptotes so cut off, the *circumscribed* hyperbola; that which, as to one branch, is inscribed, and, as to the other, circumscribed, we shall call the *ambigenous* hyperbola; that which has branches concave to each other, and proceeding towards the same direction, the *converging* hyperbola; that which has branches convex to each other, and proceeding towards contrary directions, the *diverging* hyperbola; that which has branches convex to contrary parts and infinite towards contrary sides, the *contrary branched* hyperbola; that which, with reference to its asymptote, is concave at the vertex, and has diverging branches, the *conchoidal* hyperbola; that which cuts the asymptote in contrary flexures, having on both sides contrary branches, the *serpentine* hyperbola; that which intersects its conjugate, the *cruciform* hyperbola; that which intersects and returns in a loop upon itself, the *nodate* hyperbola; that which has two branches meeting at an angle of contact, and then stopping, the *cusped* hyperbola; that which has an infinitely small conjugate oval, i.e a conjugate point, the *punctate* hyperbola; that which from the impossibility of two roots, has neither oval, node, cusp or conjugate point, the *pure* hyperbola."

Newton distinguishes between the behaviour of several types of hyperbola. This relies on his analysis of the behaviour of the hyperbola towards its asymptotes, in the description of the terms *inscribed, circumscribed, ambigenous, diverging, contrary branched* and *conchoidal*, on local properties of the hyperbola, at finite position, away from infinity, in the description of the terms *serpentine, cruciform, nodate, cusped, punctate* and *pure*. In the following of Newton's sketches, we see examples of different types;

(i). 3 pure hyperbolas. (fig 14), see figure 28.

(ii).1 pure hyperbola and 2 punctate hyperbolas, see figure 25.

(iii). 2 pure hyperbolas and 1 nodate hyperbola, (fig 8), see figure 11.

(iv). 2 pure hyperbolas and 1 cusped hyperbola, (fig 10), see figure 26.

(v).1 pure hyperbolas and 2 cruciform hyperbolas, (fig 17), see figure 12.

(vi). 2 pure hyperbolas and 1 serpentine hyperbola, (fig 20), see figure 27.

As we have seen in Newton's original text, and Talbot's commentary, the analysis of the behaviour of the hyperbola, at infinity, depends on finding series that describe the behaviour of the curve, towards the relevant asymptote. The local properties of the hyperbola, away from infinity, depend on an analysis of the relationship between two distinct series, obtained from an asymptotic calculation, (†). This idea is suggested by the following passages from Section IV, The Enumeration of Curves;

"If two roots are equal, and the other two, either impossible, (figs 16,18) or real, and with different signs from the equal roots, (figs 17,19), the figure will be cruciform, two of the hyperbolas *intersecting* one another, either at the vertex of the asymptotic triangle, or at its base, (figs 16, 17)." (figure 12 corresponds to Fig 17)

"If the two greatest roots $A\pi$, Ap (Fig 8), or the two least (AP,$A\omega$, fig 9), are equal to each other, and all of the same sign, the oval and circumscribed hyperbola will coalesce, their points of contact η and t, or T and t, and the *branches* of the hyperbola, *intersecting* one another, run on into the oval, making the figure nodate." (see figure 11 corresponding to fig 8)

"If the three greatest roots Ap, $A\pi$, $A\omega$ (fig 10), or the three least roots $A\pi$, $A\omega$, AP (fig 11), are equal to each other, the node becomes a sharp cusp, because the two *branches* of the circumscribed hyperbola *meet* at an angle of contact, and extend no further." (see figure 26 for fig 10) (††)([46])

Newton is continuously aware of the relationship between the global geometry of the curve and its three asymptotes. Although never formally stated in the text, he intuitively uses the idea that a continuous family of degenerations of the curve, will reduce it to these asymptotes. In this degeneration, the local singularities that we have considered, and the vertices of the hyperbolas, (points on the hyperbolas, furthest from infinity), reduce to the intersections of the lines themselves. This is evidenced by his sketches, perhaps, in his description of cubic curves as "Lines of the Third Order", and his reference to the relation of hyperbolas with the angles formed by the asymptotes, both in (††), and the following passage from Section IV;

"If two roots are impossible and the other two unequal, and of the same sign, three pure hyperbolas result, without oval, node or conjugate point, and these hyperbolas will either lie at the sides of the triangle made by the asymptotes, or at its angles, thus forming either the 5th (figs 12,13) or the 6th species (figs 14,15)." (see figure 28 for fig 14)

There is also a sense of linear symmetry in the way that Newton relates a curve to certain configurations of lines.

In order to understand the, perhaps, four distinct geometric ideas that Newton employs in "the method of hyperbolas", I refer the reader to [18], where it is used in the study of plane nodal curves,([47]), without restriction on the degree of the curve.

Suppose, then, that we are given such a curve C of degree m. We begin, by observing that there exists a line l, which cuts C transversely in m distinct non-singular points of the curve. We generalise Newton's construction in (†) and (††) of "The Reduction of all Curves of the Second Genus to Four Cases of Equations", as follows. We take the line l as a coordinate

[46] The italics are not in the original text, but illustrate the argument (†). Newton's use of the term branch in the final two passages, together with his previous definition in Section II, 5, further support this argument, and the choice of labelling $(a),(b)$ of the hyperbolas in figure 19. In the paper [18], we interpret Newton's branches as "flashes", this interpretation will become clearer, later in the chapter.

[47] The assumption that the curve has at most nodes as singularities is probably unnecessary

axis $x = 0$, ensuring that all of the intersections between C and the line are in finite position, not at infinity, (†), see figure 29. The condition of transversality to the axis $x = 0$, allows us to apply Newton's Theorem in the following form;

Theorem 0.16. *Let $\mathcal{C}$ be an algebraically closed field, and let $G(x, y)$ define the plane curve C in the coordinate system defined by (†). Then we can find infinite series $\{\eta_1(x), \ldots, \eta_m(x)\}$ in $\mathcal{C}[[x]]$, with $\{\eta_1(0), \ldots, \eta_m(0)\}$ distinct, such that;*

$$G(x, y) = (y - \eta_1(x)) \cdot \ldots \cdot (y - \eta_m(x))$$

The proof may be found in the paper [18]. By ensuring the line $x = 0$ avoids any singular points or infinite points of the curve C, and is not tangent to it, we avoid the technical problems of fractional series (Puiseux series), and inverse terms (Cauchy series), ([48]). We now draw the tangent lines to the curve C, at each intersection with the axis $x = 0$, see figure 29. These have the form;

$y = n_j x + c_j$, for $1 \leq j \leq m$

corresponding to the first two terms of the infinite series defined by $\eta_j(x)$ in the previous theorem, (⋆). In line with Newton's construction, we can visualise the line $x = 0$ as the line at infinity in the projective plane. The tangent lines that we have drawn now correspond to the asymptotes of C. In this representation, the observation (⋆) supports the idea that the infinite series $\eta_j(x)$ imitate the geometry of C towards the corresponding asymptote. Indeed, the fact that the infinite series $\eta_j(x)$ define functions of x, suggest that they describe the locus of C between infinity and the marked points (∗) in the following diagram, figure 30. This supports the first of Newton's geometric methods on the asymptotic behavior of hyperbolas.

The problem of understanding the behaviour of the curve C, at the vertices of the hyperbolas, which we previously suggested was due to the intersection of loci, described by infinite series, requires a more sophisticated geometric interpretation, which is achieved in the paper [18]. Again, using the method of etale covers, we construct a projective surface $S \subset P^3$, for which the infinite series $\eta_j(x)$ correspond to curves $C_j \subset S$, projecting onto C, see figure 31. We then consider the intersections of the family of curves $\{C_1, \ldots, C_m\}$ on the surface S. In the case that C is a nodal curve, at these intersections, the corresponding projected point is indeed either a vertex or a node of the original plane curve C. Conversely, for any vertex or node of C, there exists a corresponding intersection in the family of curves on S. By the nature of the curve C, it follows that no three of the curves $\{C_1, \ldots, C_m\}$ can intersect in a point, giving a geometrically appealing representation of the family as a net, ([49]),(♯). This, perhaps, clarifies the second of Newton's geometric intuitions in his discussion of hyperbolas. In the case of a general plane curve C, the intersecting geometry of the corresponding

[48] The modern terminology for an infinite series without fractional powers or inverse terms is a Taylor series, denoted in the previous theorem by $\mathcal{C}[[x]]$

[49] The intersecting geometry of the curves $\{C_1, \ldots, C_m\}$ partly motivated the terminology "flash" in the paper [18].

curves $\{C_1, \dots, C_m\}$ imitates the curve's singularities. More precisely, we have the following;

Theorem 0.17. *If C is a plane curve of degree m and $p \in C$ is a singular point with coordinates (a, b), then;*

$I_p(C, x = a) = r < m$, ([50])

iff there exists a corresponding intersection between exactly r of the curves $\{C_1, \dots, C_m\}$.

The proof can be found in the paper [18]. The result is useful in the subsequent discussion, see also figure 33.

In order to understand Newton's third intuition, we introduce the notion of an *asymptotic degeneration* of a plane curve C. In the case that C is a nodal curve, we mean by this a continuous family of curves $\{C_t\}_{t \in P^1}$, such that;

(i). $C_0 = C$.

(ii). $deg(C_t) = m \; \forall t \in P^1$.

(iii). The asymptotes of each curve C_t are fixed.

(iv). For $t \neq \infty$, C_t has at most nodes as singularities. ([51]) (♯♯)

The condition (iii) ensures that the family of curves $\{C_t\}_{t \in P^1}$ all bear the same relation to the fixed set of asymptotes, see figure 32. Moreover, we obtain the following uniformity in Newton's Theorem;

Theorem 0.18. *Let $G(x, y, t)$ define the plane curves C_t in the definition of an asymptotic degeneration, and let $\{\eta_1(x, t), \dots, \eta_m(x, t)\}$ be the power series given by Theorem 0.16, then there exists a corresponding continuous family of curves $\{C_{1,t}, \dots, C_{m,t}\}_{t \in P^1}$, with the properties outlined in (♯) above and in* [18].

The proof can be found in [18]. With the results of the previous two theorems, the problem of understanding the limit C_∞ of the degeneration, is equivalent to understanding the intersection geometry of the limit series of flashes. In the limit of such a series, the intersections of flashes may become more complex, for example, we may obtain that 3 of the flashes in the set of curves $\{C_{1,\infty}, \dots, C_{m,\infty}\}$ intersect in a point. In this case, the corresponding projected point of the limit curve C_∞ defines a more complex singularity, see figure 33. However, we

[50] The reader should look at [13] for a definition of I_p using infinitesimals, generalising the description above of tangency between curves, see Theorem 0.7. In this case, it is clear how such a definition measures the complexity of the singularity p.

[51] More precisely, we require that the d nodes of C are preserved throughout the degeneration. Formally, this requires the parametrisation to take place inside the space $V_{d,m}$, consisting of the closure of irreducible curves of degree m, having d nodes as singularities. The mathematically more advanced reader should look at [32] for the original construction of this space, or [31] for a modern approach.

maintain a geometric understanding of the problem by keeping track of the intersections of flashes. This is made possible by imposing condition (iv), any point of intersection between two flashes is preserved throughout an asymptotic degeneration.

By repeating the process of asymptotic degenerations, ([52]), we can force the limit curve C_∞ to have a higher contact with any given asymptote l_j, ([53]). By a geometrical result, known as Bezout's Theorem, we are eventually guaranteed that the limit curve, C_∞, splits into $(l \cup C'_\infty)$, where l is the corresponding asymptote and C'_∞ has lower degree $m-1$, see figure 34, with corresponding flashes $\{C_l, C'_{1,\infty}, \ldots, C'_{m-1,\infty}\}$.

We then repeat the process for the plane curve C'_∞. In the flash interpretation, the $m-1$ flashes $\{C'_{1,\infty,t}, \ldots, C'_{m-1,\infty,t}\}$, corresponding to degenerations of C'_∞, may now separate from the flash C_l, corresponding to l. However, again we solve the problem by tracking the original intersection throughout the degeneration, see figure 35. Eventually, we obtain a degeneration to the original set of asymptotes.

The method of asymptotic degenerations is connected to a conjecture of Severi that any plane nodal curve $C \subset P$ of degree m can be degenerated to a series of m lines in general position, that is there exists a degeneration of C satisfying conditions $(i), (ii)$ and (iv) of $(\sharp\sharp)$ above. Severi's argument proceeds in the following stages;

(a). Construct a cone $Cone(C)$ over C.

(b). Slice the cone through a continuous family of plane sections;

$(Plane_{(t\in P^l)} \cap Cone(C))$

with $Plane_0 = P$ and $Plane_\infty$ passing generically through the vertex of the cone.

(c). Observe that the intersection $(Plane_\infty \cap Cone(C))$ consists of a series of m distinct lines passing through a point, the vertex of the cone.

(d). Move the m distinct lines passing through a point to m lines in general position.

Unfortunately, Severi's argument fails as the degeneration in the arguments $(a)-(c)$ is incompatible with the degeneration in (d), the reason being that the nodes of the original curve C may not converge to nodes formed by the lines in (c). This problem was observed by Zariski. The difficulty is resolved in the case of asymptotic degenerations, by Theorem 0.18 , and the observation that the curves in the family $\{C_{1,t}, \ldots, C_{m,t}\}_{t\in P^1}$ remain irreducible and non-coincident, $(\sharp\sharp\sharp)$, hence the nodes converge to intersections of lines, a precise statement of this result can be found in [**?**]. For degenerations which are not asymptotic, such as the example given by Zariski, it may still be possible to construct a corresponding continuous family, but the property $(\sharp\sharp\sharp)$ fails, that is the limit curves $C_{i,\infty}$ may become reducible and

[52]For a general curve C, one needs to modify condition (iv), to allow for more complex singularities. The existence of such degenerations follows from dimension calculations on the space of certain adjoint curves to C. The reader should look at the paper [17], to generalise the remarks above on "infinitely close" intersections between curves, in relation to (newton8.jpg)

[53]This implies that the corresponding infinite series, $\eta_{j,\infty}(x)$, has the form $a_j + b_j x + O(x^n)$, for large n.

components of the limit curves $C_{i,\infty}$ and $C_{j,\infty}$, for $i \neq j$, may coincide. In such cases, the intersection geometry of the limit series may change, making it impossible to track the position of the nodes throughout the degeneration, and use Theorem 0.17. However, Severi's observation (d), his argument in [32], and the property of asymptotic degenerations, that the positions of the original nodes of C vary continuously, ([54]), allows us to potentially solve his conjecture, by either altering the final configuration of the asymptotes, or choosing a set of asymptotes for C that are in general position. Clearly, a solution to the conjecture supports Newton's original geometrical thinking in [34].

Newton's fourth intuitive use of symmetry in his work on plane cubics, is also an important feature of the geometrical method used in the construction of flashes and asymptotic degenerations. More specifically, we have the following result from [18];

Theorem 0.19. *Let C be a plane(nodal) curve with $\{C_1, \ldots, C_m\}$ constructed as in the discussion ($\sharp$) above. Then there exists a finite group G and a definable, transitive action of G on this set of curves.*

The finite group G is obtained as the Galois ([55]) group of the polynomial defined by the original curve C. Such an action is useful in understanding the intersection geometry of the set of curves $\{C_1, \ldots, C_m\}$, as, for any point p belonging to $C_i \cap C_j$, and $g \in G$, we have that $g.p \in (C_{g.i} \cap C_{g.j})$. In certain cases, this allows us to find bounds on the number of intersections between two distinct flashes C_i and C_j. The reader should consider the following diagram, see figure 36, of typical intersections between three flashes, in which the Galois group fixes the curve C_2 and permutes the curves C_1 and C_3, the corresponding intersections p_{12} and p_{23} are permuted, while p_{13} is fixed. Such considerations and Theorem 0.17 are useful in understanding the symmetry of the original plane curve C, in terms of the arrangement of its singularities or vertices, in the asymptotic interpretation, (see newton30.jpg). The use of Galois symmetry also simplifies the study of the structure of individual curves appearing in the set $\{C_1, \ldots, C_m\}$, namely that one can assume such curves have no singularities outside their set of intersections, the reader should look at [18] for more details.

The aesthetic of fragmentation in a plane is also reflected in medieval art. Particularly good examples are given by the work of the Cosmati artists in Sicily, which we considered briefly in the previous chapter. The pavements of the Palazzo Normanni and Monreale Cathedral in Palermo are excellent examples of configurations of lines in general position, that is no three of which intersect in a point. There is a highly developed sense of linear symmetry in such patterns, reflecting the advanced geometric intuition of Norman design. There is also an inherent 3-dimensionality in the arrangements, as if the lines are projected from an ambient space. This partly motivated the author's use of the flash construction in the problem of curve degenerations. Interesting examples of the geometry behind asymptotic degenerations can be found in the medieval art of Sweden, particularly around the island of Gotland. One can find extensive use of symmetric linear designs, such as the hexagonal

[54]This last property allows one to vary the families of curves within an irreducible component of $V_{d,m}$, see footnote 47 and [32]

[55]The theory of such groups is based on two papers by the French mathematician Evariste Galois, "Memoire sur les conditions de resolubilite des equations par radicaux" and "Des equations primitives qui sont solubles par radicaux", published in 1846, 14 years after his death. The interested reader can find out more about Galois theory in [8]

vault, and cusped windows, both highly suggestive of the aesthetic of sublime. The reader is referred to the previous chapter.

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