A SIMPLE PROOF OF THE UNIFORM CONVERGENCE OF FOURIER SERIES IN SOLUTIONS TO THE WAVE EQUATION

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Abstract. Using methods of [2], we show that the time dependent Fourier series of any $F \in C^{\infty}(0,L)$, solving the wave equation, with $F(0,t) = F(L,t) = 0$, converges uniformly to F, on $[0,L]$, and find an explicit formula for such series.

Definition 0.1. *We let;*

$$C^n([0,L]) = \{f \in C([0,L]) : f|_{(0,L)} \in C^n(0,L),$$

$$(\forall i \leq n)\exists r_i \in C[0,L], r_i|_{(0,L)} = f^{(i)}\}, \text{ (}^1\text{)}$$

$$C_0^n([0,L]) = \{f \in C^n([0,L]) : \ f(0) = f(L) = 0\}$$

$$C^{\infty}([0,L]) = \{f \in C([0,L]) : f|_{(0,L)} \in C^{\infty}(0,L)$$

$$\forall(i \leq n)\exists r_i \in C([0,L]), r_i|_{(0,L)} = f^{(i)}\}$$

$$C_0^{\infty}([0,L]) = \{f \in C^{\infty}([0,L]) : \ f(0) = f(L) = 0\}$$

$$C^n([0,L] \times \mathcal{R}) = \{F \in C([0,L] \times \mathcal{R}) : F|_{(0,L)\times\mathcal{R}} \in C^n((0,L)\times$$

$$\mathcal{R}), (\forall i \leq n)\exists r_i \in C([0,L] \times \mathcal{R}) \ r_i|_{(0,L)\times\mathcal{R}} = \tfrac{\partial^i F}{\partial x^i}\}$$

$$C_0^n([0,L] \times \mathcal{R}) = \{F \in C^n([0,L] \times \mathcal{R}) : \ (\forall t \in \mathcal{R}), F(0,t)$$

[1]This definition is equivalent to, $(\forall i \leq n)\{f_+^{(i)}(0), f_-^{(i)}(L)\}$ exist, where, for $i \leq n$, $f_+^{(i)}(0)$ is defined inductively, by ,$f_+^{(i)}(0) = lim_{s\rightarrow 0}\frac{f^{(i-1)}(s)-f_+^{(i-1)}(0)}{s}$, and, similarly, for $f_-^{(i)}(L)$. In order to see this, just observe that, for $i \leq n$, $lim_{s\rightarrow 0}\frac{f^{(i-1)}(s)-f_+^{(i-1)}(0)}{s} = lim_{s\rightarrow 0}f^{(i)}(s)$, by L'Hopital's Rule and the Intermediate Value Theorem.

$= F(L,t) = 0\}$

$C^\infty([0,L]\times\mathcal{R}) = \{F \in C([0,L]\times\mathcal{R}) : F|_{(0,L)\times\mathcal{R}} \in C^\infty((0,L)\times$

$\mathcal{R}), \forall(i \leq n)\exists r_i \in C([0,L]\times\mathcal{R})\ r_i|_{(0,L)\times\mathcal{R}} = \frac{\partial^i F}{\partial x^n}\}$

$C_0^\infty([0,L]\times\mathcal{R}) = \{F \in C^\infty([0,L]\times\mathcal{R}) :\ (\forall t \in \mathcal{R}), F(0,t) =$

$F(L,t) = 0\}$

We let $\{T, M, L\}$ denote the tension,mass and length of a string, with $\mu = M/L$, the mass per unit length. The wave equation;

$\frac{\partial^2 F}{\partial t^2} = \frac{T}{\mu}\frac{\partial^2 F}{\partial x^2}$ $(*)$

with boundary condition $F(0,t) = F(L,t) = 0$, for $t \in \mathcal{R}$, describes the motion of a vibrating string under tension, fixed at the endpoints, ([2]).

We say that $h \in C([-L,L])$ is symmetric, if $h(-x) = h(x)$, for $x \in [-L,L]$, (with endpoints identified). We say that $h \in C([-L,L])$ is asymmetric if $h(-x) = -h(x)$,for $x \in [-L,L]$, (with endpoints identified). We use the same notation as above for functions on $[-L,L]$, (with endpoints identified). We define;

$C^n((-L,0)\cup(0,L)) = \{f \in C((-L,0)\cup(0,L)) : \exists(r_1 \in C^n([-L,0]), r_2) \in C^n([0,L]), r_1|_{(-L,0)} = f|_{(-L,0)}, r_2|_{(0,L)} = f|_{(0,L)}\}$

We require the following results;

Lemma 0.2. *Let $h \in C([-L,L])$ be asymmetric, with $h|_{(-L,0)\cup(0,L)} \in C^1((-L,0)\cup(0,L))$, $(*)$, then $h(0) = h(L) = h(-L) = 0$, $h'_+(-L) = h'_-(L)$, $h'_+(0) = h'_-(0)$, $h' \in C([-L,L])$, and h' is symmetric. Let $h \in C([-L,L])$ be symmetric, with $h|_{(-L,0)\cup(0,L)} \in C^1((-L,0)\cup(0,L))$, $(**)$, and $h'_+(-L) = h'_-(L) = 0$, $h'_+(0) = h'_-(0) = 0$, then $h' \in C([-L,L])$ is asymmetric.*

Proof. For the first part, we have, if h is asymmetric, satisfying $(*)$, then $h(L) = -h(-L) = -h(L)$ and $h(0) = -h(-0) = -h(0)$, so

[2] By a solution to the wave equation, we mean $F \in C_0^\infty([0,L]\times\mathcal{R})$, satisfying the equation $(*)$ on $(0,L)\times\mathcal{R}$

$h(0) = h(L) = h(-L) = 0$. We have that;

$h'_+(-L) = lim_{s\to 0}\frac{h(-L+s)-h(-L)}{s}$

$= lim_{s\to 0}\frac{h(-L+s)}{s} = lim_{s\to 0}\frac{-h(L-s)}{s}$ (by asymmetry and $h(-L) = 0$)

$= lim_{s\to 0}\frac{h(L)-h(L-s)}{s} = h'_-(L)$ (as $h(L)$=0)

Similarly, $h'_+(0) = h'_-(0)$. By L'Hopital's rule, and the fact that $h|_{(-L,0)\cup(0,L)} \in C^1((-L,0)\cup(0,L))$, we have that $lim_{s\to 0}h'(s) = lim_{s\to 0}\frac{h(s)-h(0)}{s}$ $= h'_+(0)$, and, similarly, $lim_{s\to 0}h'(-s) = h'_-(0)$, $lim_{s\to 0}h'(L-s) = h'_-(L)$, $lim_{s\to 0}h'(-L+s) = h'_-(L)$. Hence, $h' \in C([-L,L])$, and h' is symmetric by the fact that $h(x) = -h(-x)$, and, therefore, $h'(x) = h'(-x)$, for $x \in (-L,0)\cup(0,L)$, and, automatically, $h'(L) = h'(-L)$, $h'(0) = h'(-0)$, as these points are fixed.

Let $h \in C([-L,L])$ be symmetric, satisfying $(**)$. By L'Hopital's rule, and the fact that $h|_{(-L,0)\cup(0,L)} \in C^1((-L,0)\cup(0,L))$, we have that $lim_{s\to 0}h'(s) = lim_{s\to 0}\frac{h(s)-h(0)}{s} = h'_+(0) = 0 = h'_-(0) = lim_{s\to 0}\frac{h(0)-h(-s)}{s} =$ $lim_{s\to 0}h'(-s)$ and, similarly, $lim_{s\to 0}h'(L-s) = h'_-(L)$, $lim_{s\to 0}h'(-L+s) = h'_-(L)$. Hence, $h' \in C([-L,L])$, and h' is symmetric by the fact that $h(x) = h(-x)$, and, therefore, $h'(x) = -h'(-x)$, for $x \in (-L,0)\cup(0,L)$, and, automatically, $h'(L) = h'(-L) = 0$, $h'(0) = h'(-0) = 0$, as these points are fixed.

□

Lemma 0.3. *Let $f \in C^2([0,L])$, such that $f(0) = f(L) = 0$ and $f''_+(0) = f''_+(L) = 0$, $(*)$, then there exists $h \in C^2([-L,L])$, (with endpoints identified), such that $h|_{[0,L]} = f$, h is asymmetric about 0, and h' is symmetric about 0. Let $f \in C^2([0,L])$, such that $f(0) = f(L) = 0$ and $f'_+(0) = f'_-(L) = 0$, $(**)$, then there exists $h \in C^2([-L,L])$, (with endpoints identified), such that $h|_{[0,L]} = f$, h is symmetric about 0, and h' is asymmetric about 0.*

Proof. Suppose that f satisfies $(*)$ and let $h(x) = f(x)$, for $x \in [0,L]$, and $h(x) = -f(-x)$, for $x \in [-L,0)$. Then clearly h is asymmetric about 0, $h(0) = h(L) = h(-L) = 0$, and $h \in C([-L,L])$. Moreover, $h|_{(-L,0)\cup(0,L)} \in C^1((-L,0)\cup(0,L))$, as $f \in C^1([0,L])$. By Lemma 0.2, we have that $h' \in C([-L,L])$, and h' is symmetric. Moreover, $h'|_{(-L,0)\cup(0,L)} \in C^1((-L,0)\cup(0,L))$, as $f \in C^2([0,L])$ and $f' \in$

$C^1([0,L])$. We have that;

$(h')'_+(-L) = lim_{s\rightarrow 0^+}\frac{h'(-L+s)-h'(-L)}{s}$

$= lim_{s\rightarrow 0}\frac{h'(L-s)-h'(L)}{s}$ (by asymmetry)

$= -lim_{s\rightarrow 0}\frac{h'(L)-h'(L-s)}{s}$

$= -lim_{s\rightarrow 0}\frac{f'(L)-f'(L-s)}{s} = -f''_+(L) = 0$

and;

$(h')'_-(L) = lim_{s\rightarrow 0^+}\frac{h'(L)-h'(L-s)}{s}$

$= lim_{s\rightarrow 0}\frac{h'_-(L)-f'(L-s)}{s}$

$= lim_{s\rightarrow 0}\frac{f'_-(L)-f'(L-s)}{s}$

$= lim_{s\rightarrow 0}\frac{f'(L)-f'(L-s)}{s} = f''_+(L) = 0$

Similarly, $(h')'_+(0) = f''_+(0) = 0$, $(h')'_-(0) = -f''_+(0) = 0$

Applying Lemma 0.2 again, we obtain that $(h')' \in C[-L,L]$, hence $h \in C^2([-L,L])$.

Suppose that f satisfies $(**)$ and let $h(x) = f(x)$, for $x \in [0,L]$, $h(x) = f(-x)$, for $x \in [-L,0)$. Then h is symmetric and $h|_{(-L,0)\cup(0,L)} \in C^1((-L,0)\cup(0,L))$. Moreover;

$h'_+(-L) = lim_{s\rightarrow 0}\frac{h(-L+s)-h(-L)}{s}$

$= lim_{s\rightarrow 0}\frac{h(L-s)-h(L)}{s}$

$= lim_{s\rightarrow 0}\frac{f(L-s)-f(L)}{s}$

$= -lim_{s\rightarrow 0}\frac{f(L)-f(L-s)}{s}$

$= -f'_-(L) = 0$

Similarly, $h'_-(L) = f'_-(L) = 0$, $h'_+(0) = f'_+(0) = 0$, and $h'_-(0) = -f'_+(0) = 0$. Again, applying Lemma 0.2, we obtain that $h' \in C([-L,L])$

is asymmetric. We have that $h'|_{(-L,0)\cup(0,L)} \in C^1((-L,0)\cup(0,L))$, as $f \in C^2([0,L])$ and $f' \in C^1([0,L])$. Applying Lemma 0.2, we obtain that $(h')' \in C[-L,L]$, hence $h \in C^2([-L,L])$.

□

Lemma 0.4. *Let $f \in C^4([0,L])$, such that $f(0) = f(L) = 0$ and $f_+^{(2)}(0) = f_+^{(2)}(L) = 0$, $f_+^{(4)}(0) = f_+^{(4)}(L) = 0$, $(*)$, then there exists $h \in C^4([-L,L])$, (with endpoints identified), such that $h|_{[0,L]} = f$, $\{h, h^{(2)}\}$ are asymmetric about 0, and $\{h^{(1)}, h^{(3)}\}$ are symmetric about 0. Let $f \in C^4([0,L])$, such that $f(0) = f(L) = 0$ and $f_+^{(1)}(0) = f_+^{(1)}(L) = 0$, $f_+^{(3)}(0) = f_+^{(3)}(L) = 0$, $(**)$, then there exists $h \in C^4([-L,L])$, (with endpoints identified), such that $h|_{[0,L]} = f$, $\{h, h^{(2)}\}$ are symmetric about 0, and $\{h^{(1)}, h^{(3)}\}$ are asymmetric about 0.*

Proof. For the first part, let h be defined as in 0.3, then $h \in C^2([-L,L])$, (with endpoints identified), $h|_{[0,L]} = f$, h is asymmetric about 0 and $h^{(1)}$ is symmetric about 0. We have that $f^{(2)} \in C^2([0,L])$, $f_+^{(2)}(0) = f_+^{(2)}(L) = 0$, and $f_+^{(4)}(0) = f_+^{(4)}(L) = 0$, so $f^{(2)}$ satisfies the hypotheses of Lemma 0.3. Moreover, $h^{(2)}(x) = f^{(2)}(x)$, for $x \in [0,L]$, and $h^{(2)}(-x) = -f^{(2)}(-x)$, for $x \in [-L,0)$. Then, by the result of 0.3, we have that $h^{(2)} \in C^2([-L,L])$, (with endpoints identified), $h^{(2)}$ is asymmetric about 0 and $h^{(3)}$ is symmetric about 0. Hence $h \in C^4([-L,L])$, and the remaining claims are clear. The proof of the second part of the lemma follows the same strategy.

□

Lemma 0.5. *Let $f \in C_0^\infty([0,L])$, then there exists $\{f_1, f_2\} \subset C_0^\infty([0,L])$, with $f'_{1,+}(0) = f'_{1,+}(L) = 0$, $f''_{2,+}(0) = f''_{2,+}(L) = 0$, such that $f = f_1 + f_2$.*

Proof. Consider the equations $g(0) = g(L) = 0$, $g'(0) = g'(L) = 0$, $g''(0) = f''_+(0)$ and $g''(L) = f''_-(L)$, $(*)$ on the space $V_6 = \{g \in \mathcal{R}[x] : deg(g) = 5\}$. Let $T : V_6 \to \mathcal{R}^6$ be given by;

$$T(g) = (g(0), g(L), g'(0), g'(L), g''(0), g''(L))$$

We have that $Ker(T) = 0$, as if $T(g) = 0$, then, clearly $g(x) = dx^3 + ex^4 + fx^5$, with $\{d, e, f\} \subset \mathcal{R}$, then, $g'(x) = 3dx^2 + 4ex^3 + 5fx^4$, $g''(x) = 6dx + 12ex^2 + 20fx^3$, and we have that $g(L) = g'(L) = g''(L) = 0$, iff;

$$A\centerdot\begin{pmatrix}d\\e\\f\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix},A=\begin{pmatrix}1&L&L^2\\3&4L&5L^2\\6&12L&20L^2\end{pmatrix}$$

We have that $det(A)=2L^3\neq 0$, hence, $d=e=f=0$, as required. Then, T is onto, by the rank-nullity theorem, hence, we can find a solution to $(*)$, corresponding to $T(g)=v_1$, where $v_1=(0,0,0,0,f''(0),f''(L))$. Let f_1 be the unique polynomial in V_5, satisfying these conditions, and let $f_2=f-f_1$. It is now a simple calculation to see that $\{f_1,f_2\}$ satisfy the required conditions. □

Lemma 0.6. *Let $f\in C_0^\infty([0,L])$, and $n\in\mathcal{Z}_{\geq 1}$, then there exists $\{f_1,f_2\}\subset C_0^\infty([0,L])$, with $f_{1,+}^{(2j-1)}(0)=f_{1,-}^{(2j-1)}(L)=0$, $f_{1,+}^{(2j)}(0)=f_{1,-}^{(2j)}(L)=0$, for $1\leq j\leq n$, such that $f=f_1+f_2$.*

Proof. Consider the equations $g(0)=g(L)=0$, $g^{(2j-1)}(0)=g^{(2j-1)}(L)=0$, and $g^{(2j)}(0)=f_+^{(2j)}(0)$, $g^{(2j)}(L)=f_-^{(2j)}(L)$, for $1\leq j\leq n$, $(*)$, on the space $V_{2(2n+1)}=\{g\in\mathcal{R}[x]:deg(g)=4n+1\}$. Let $T:V_{2(2n+1)}\rightarrow\mathcal{R}^{2(2n+1)}$ be given by;

$(T(g))_1=g(0)$

$(T(g))_2=g(L)$

$(T(g))_{1+2j}=g^{(j)}(0)$

$(T(g))_{2+2j}=g^{((j))}(L)$ $(1\leq j\leq 2n)$

We have that $Ker(T)=0$, as if $T(g)=0$, then, using the fact that $g(0)=0$, $g^{(j)}(0)=0$, for $1\leq j\leq 2n$, we have $g(x)=\sum_{i=2n+1}^{4n+1}a_ix^i$, with $a_i\in\mathcal{R}$, for $2n+1\leq i\leq 4n+1$. Then, for $1\leq j\leq 2n$;

$g^{(j)}(x)=\sum_{i=2n+1}^{4n+1}\frac{i!}{(i-j)!}a_ix^{i-j}$

and we have that $g(L)=0$, $g^{(j)}(L)=0$, for $1\leq j\leq 2n$ iff;

$$A.\begin{pmatrix} a_{2n+1} \\ . \\ . \\ a_{2n+i} \\ . \\ . \\ a_{4n+1} \end{pmatrix} = \begin{pmatrix} 0 \\ . \\ . \\ 0 \\ . \\ . \\ 0 \end{pmatrix}, A = \begin{pmatrix} 1 & . & . & L^{i-1} & . & . & L^{2n-1} \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ \frac{(2n+1)!}{(2n+2-j)!} & . & . & \frac{(2n+i)!L^{i-1}}{(2n+1+i-j)!} & . & . & \frac{(4n+1)!L^{2n-1}}{(4n+1-j)!} \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ \frac{(2n+1)!}{2!} & . & . & \frac{(2n+i)!L^{i-1}}{(i+1)!} & . & . & \frac{(4n)!L^{2n-1}}{(2n+1)!} \end{pmatrix}$$

for $1 \leq i, j \leq 2n$.

We have that $det(A) = cL^{n(2n-1)} \neq 0$, (work out c) hence, $a_i = 0$, for $2n+1 \leq i \leq 4n+1$, as required. Then, T is onto, by the rank-nullity theorem, hence, we can find a solution to $(*)$, corresponding to $T(g) = v_1$, where;

$(v_1)_j = 0$, $1 \leq j \leq 2$

$(v_1)_j = 0$, $(j = 4k-1, j = 4k, 1 \leq k \leq n)$

$(v_1)_j = f_-^{(2k)}(0)$, $(j = 4k+1, 1 \leq k \leq n)$

$(v_1)_j = f_-^{(2k)}(L)$, $(j = 4k+2, 1 \leq k \leq n)$

Let f_1 be the unique polynomial in $V_{2(2n+1)}$, satisfying these conditions, and let $f_2 = f - f_1$. It is now a simple calculation to see that $\{f_1, f_2\}$ satisfy the required conditions. □

Lemma 0.7. *Let $f \in C_0^\infty([0,L])$, then, for all $\epsilon > 0$, there exists $g \in C^2([-L,L])$, such that;*

$g|_{[\epsilon,L-\epsilon)} = f|_{[\epsilon,L-\epsilon)}$.

Proof. By Lemma 0.5, we can find $\{f_1, f_2\} \subset C_0^\infty([0,L])$, with $f'_{1,+}(0) = f'_{1,+}(L) = 0$, $f''_{2,+}(0) = f''_{2,+}(L) = 0$, such that $f = f_1 + f_2$. By Lemma 0.3, we can find $\{g_1, g_2\} \subset C_0^2([0,L])$, with $g_1|_{[0,L]} = f_1$, $g_2|_{[0,L]} = f_2$ and g_1 symmetric, g_2 asymmetric. Let $g = g_1 + g_2$, then $g \in C_0^2([0,L])$, and $g|_{[\epsilon,L-\epsilon)} = f|_{[\epsilon,L-\epsilon)}$. □

Lemma 0.8. *Let $f \in C_0^\infty([0,L])$, then, for all $\epsilon > 0$, there exists $g \in C^4([-L,L])$, such that;*

$g|_{[\epsilon,L-\epsilon)} = f|_{[\epsilon,L-\epsilon)}$.

Proof. By Lemma 0.6, we can find $\{f_1, f_2\} \subset C_0^\infty([0, L])$, with $f_{1,+}^{(1)}(0) = f_{1,-}^{(1)}(L) = 0$, $f_{1,+}^{(3)}(0) = f_{1,-}^{(3)}(L) = 0$, $f_{2,+}^{(2)}(0) = f_{2,-}^{(2)}(L) = 0$, $f_{2,+}^{(4)}(0) = f_{2,-}^{(4)}(L) = 0$ such that $f = f_1 + f_2$. By Lemma 0.4, we can find $\{g_1, g_2\} \subset C_0^4([-L, L])$, with $g_1|_{[0,L]} = f_1$, $g_2|_{[0,L]} = f_2$ and g_1 symmetric, g_2 asymmetric. Let $g = g_1 + g_2$, then $g \in C_0^4([-L, L])$, and $g|_{[\epsilon,L-\epsilon)} = f|_{[\epsilon,L-\epsilon)}$. □

Lemma 0.9. *Let $F \in C^2([0, L] \times \mathcal{R})$, such that $F(0, t) = F(L, t) = 0$, for all $t \in \mathcal{R}$, and let $F''_{t,+}(0) = F''_{t,+}(L) = 0$, $(*)$, then there exists $H \in C^2([-L, L] \times \mathcal{R})$, (with endfaces identified), such that $H|_{[0,L]\times\mathcal{R}} = F$, H is asymmetric about 0, and $\frac{\partial H}{\partial x}$ is symmetric about 0. Let $F \in C^2([0, L] \times \mathcal{R})$, such that $F(0) = F(L) = 0$ and $F'_{t,+}(0) = f'_{t,-}(L) = 0$, $(**)$, then there exists $H \in C^2([-L, L] \times \mathcal{R})$, (with endfaces identified), such that $H|_{[0,L]\times\mathcal{R}} = F$, H is symmetric about 0, and $\frac{\partial H}{\partial x}$ is asymmetric about 0.*

Proof. Suppose that F satisfies $(*)$ and let $H(x, t) = F(x, t)$, for $(x, t) \in [0, L] \times \mathcal{R}$, and $H(x, t) = -F(-x, t)$, for $(x, t) \in [-L, 0) \times \mathcal{R}$, $(***)$. Using the result of Lemma 0.3, we have, for $t \in \mathcal{R}$, that $H_t \in C^2([-L, L])$, $(****)$, $H_t|_{[0,L]} = F_t$, $(*******)$, H_t is asymmetric about 0, $(\dagger)$, and H'_t is symmetric about 0, $(\dagger\dagger)$. Let $r_2 \in C([0, L]\times\mathcal{R})$ be given, as in Definition 0.1, for F, so that $r_2|_{(0,L)\times\mathcal{R}} = \frac{\partial^2 H}{\partial x^2}|_{(0,L)\times\mathcal{R}}$, $(******)$, and let $r_{2,l} \in C([-L, 0] \times \mathcal{R})$ be given by $r_{2,l}(x, t) = -r_2(-x, t)$, for $(x, t) \in [-L, 0] \times \mathcal{R}$, so that $r_{2,l}|_{(-L,0)\times\mathcal{R}} = \frac{\partial^2 H}{\partial x^2}|_{(-L,0)\times\mathcal{R}}$, $(*****)$. Let R_2 be defined by $R_2(x, t) = r_2(x, t)$, if $(x, t) \in [0, L]\times\mathcal{R}$, and $R_2(x, t) = r_{2,l}(x, t)$, if $(x, t) \in [-L, 0]\times\mathcal{R}$. Then $R_{2,t}|_{[-L,L]} = H_t$, hence, by $(****)$, in fact, $R_2 \in C([-L, L] \times \mathcal{R})$, and, by $(***)$, $(*****)$, $(******)$, $R_2|_{((-L,0)\cup(0,L))\times\mathcal{R}} = \frac{\partial^2 H}{\partial x^2}$. It follows that $H \in C^2([-L, L] \times \mathcal{R})$ (with endpoints identified). By $(*******)$, we obtain immediately that $H|_{[0,L]\times\mathcal{R}} = F$. The fact that H is asymmetric about 0, is obvious, from $(\dagger)$. In order to see the final claim, let $r_1 \in C([0, L] \times \mathcal{R})$ be given, as above, $r_{1,l} \in C([-L, 0] \times \mathcal{R})$, be given by, $r_{1,l}(x, t) = r_1(-x, t)$, and $R_1 \in C([0, L]\times\mathcal{R})$ be defined by $R_1(x, t) = r_1(x, t)$, if $(x, t) \in [0, L]\times\mathcal{R}$, and $R_1(x, t) = r_{1,l}(x, t)$, if $(x, t) \in [-L, 0] \times \mathcal{R}$. It is easy to see, as above, that $R_1 \in C([-L, L] \times \mathcal{R})$ and $R_1|_{(-L,L)\times\mathcal{R}} = \frac{\partial H}{\partial x}$. Then, for $t \in \mathcal{R}$, $R_{1,t}|_{(-L,L)} = (H_t)'$, so that, for $t \in \mathcal{R}$, $R_{1,t} = r_{1,t}$, $(\dagger\dagger\dagger)$, where $r_{1,t}$ is given, as in Definition 0.1, for each H_t. Then, the fact that $\frac{\partial H}{\partial x}$ is symmetric about 0, follows from the pointwise property $(\dagger\dagger)$, and,

(†††). The second part of the lemma is the similar, following the proof above □

Lemma 0.10. *Let $F \in C^4([0,L]\times\mathcal{R})$, such that $F(0,t) = F(L,t) = 0$, for all $t \in \mathcal{R}$, and let $F^{(2)}_{t,+}(0) = F^{(2)}_{t,+}(L) = 0$, $F^{(4)}_{t,+}(0) = F^{(4)}_{t,+}(L) = 0$ $(*)$, then there exists $H \in C^4([-L,L]\times\mathcal{R})$, (with endfaces identified), such that $H|_{[0,L]\times\mathcal{R}} = F$, H, $\frac{\partial^2 H}{\partial x^2}$ are asymmetric about 0, and $\frac{\partial H}{\partial x}$, $\frac{\partial^3 H}{\partial x^3}$ are symmetric about 0. Let $F \in C^4([0,L]\times\mathcal{R})$, such that $F(0) = F(L) = 0$ and $F^{(1)}_{t,+}(0) = F^{(1)}_{t,-}(L) = 0$, $F^{(3)}_{t,+}(0) = F^{(3)}_{t,-}(L) = 0$ $(**)$, then there exists $H \in C^4([-L,L]\times\mathcal{R})$, (with endfaces identified), such that $H|_{[0,L]\times\mathcal{R}} = F$, H, $\frac{\partial^2 H}{\partial x^2}$ are symmetric about 0, and $\frac{\partial H}{\partial x}$, $\frac{\partial^3 H}{\partial x^3}$ are asymmetric about 0.*

Proof. For the first part, by Lemma 0.9, we can find $H \in C^2([-L,L]\times\mathcal{R})$, with $H|_{[0,L]\times\mathcal{R}} = F$, H asymmetric about 0, and $\frac{\partial H}{\partial x}$ symmetric about 0. We have that $\frac{\partial^2 F}{\partial x^2}$ satisfies the conditions of Lemma 0.9, as, by the assumptions, $\frac{\partial^2 F}{\partial x^2} \in C^2([0,L]\times\mathcal{R})$, $\frac{\partial^2 F}{\partial x^2}(0,t) = \frac{\partial^2 F}{\partial x^2}(L,t) = 0$ and $(\frac{\partial^2 F}{\partial x^2})^{(2)}_{t,+}(0) = F^{(4)}_{t,+}(0) = F^{(4)}_{t,-}(L) = (\frac{\partial^2 F}{\partial x^2})^{(2)}_{t,-}(L) = 0$, for all $t \in \mathcal{R}$, ([3]). Moreover, by definition of H, we have that $\frac{\partial^2 H}{\partial x^2}(x',t) = \frac{\partial^2 F}{\partial x^2}(x',t)$, for $(x',t) \in ([0,L]\times\mathcal{R})$, and $\frac{\partial^2 H}{\partial x^2}(x',t) = -\frac{\partial^2 F}{\partial x^2}(-x',t)$, for $(x',t) \in ((-L,0)\times\mathcal{R})$. Hence, by the conclusion of Lemma 0.9, we have that $\frac{\partial^2 H}{\partial x^2} \in C^2([-L,L])$, (with endfaces identified) $\frac{\partial^2 H}{\partial x^2}$ is symmetric about 0, and $\frac{\partial^3 H}{\partial x^3}$ is asymmetric about 0, as required.

□

Lemma 0.11. *Let $F \in C^\infty_0([0,L]\times\mathcal{R})$, then there exists $\{F_1, F_2\} \subset C^\infty_0([0,L]\times\mathcal{R})$, with $F'_{1,t,+}(0) = F'_{1,t,-}(L) = 0$, $F''_{2,t,+}(0) = F''_{2,t,-}(L) = 0$, such that $F = F_1 + F_2$.*

Proof. This is just a uniform version of Lemma 0.5. Let;

$v_{1,t} = (0,0,0,0,F''_{t,+}(0),F''_{t,+}(L))$, $p_{1,t} = T^{-1}(v_{1,t})$

Then;

$p_{1,t} = \sum_{i=0}^{5} d_i(t)x^i$

where the coefficients $d_i(t) = \lambda_i F''_{t,+}(0) + \mu_i F''_{t,+}(L)$

[3]Here, we use the fact that, for $t \in \mathcal{R}$, $((\frac{\partial^2 F}{\partial x^2})_t)|_{(0,L)} = (F_t)^{(2)}|_{(0,L)}$, so $((\frac{\partial^2 F}{\partial x^2})_t)^{(2)}|_{(0,L)} = (F_t)^{(4)}|_{(0,L)}$, $(*)$, and, using Definition **??**, the limits $(\frac{\partial^2 F}{\partial x^2})^{(2)}_{t,+}(0) = F^{(4)}_{t,+}(0)$ are recovered uniquely from the relation $(*)$

for fixed constants $\{\lambda_i, \mu_i\} \subset \mathcal{R}$, $0 \leq i \leq 5$. Let $r_2 \in C_0^\infty([0,L] \times \mathcal{R})$ be given, as in Definition 0.1, and $\phi_0(t) = r_2(t,0), \phi_L(t) = r_2(t,L)$, then, clearly, $\{\phi_0, \phi_L\} \subset C^\infty(\mathcal{R})$, so clearly, we have that;

$p_{1,t} = \sum_{i=0}^{5}(\lambda_i\phi_0(t) + \mu_i\phi_L(t))x^i$

and $p_{1,t} \in C_0^\infty([0,L] \times \mathcal{R})$. Letting $F_1 = p_{1,t}$, and $F_2 = F - F_1$, we obtain the result.

□

Lemma 0.12. *Let $F \in C_0^\infty([0,L] \times \mathcal{R})$, then there exists $\{F_1, F_2\} \subset C_0^\infty([0,L]\times\mathcal{R})$, with $F_{1,t,+}^{(2j-1)}(0) = F_{1,t,-}^{(2j-1)}(L) = 0$, $F_{2,t,+}^{(2j)}(0) = F_{2,t,-}^{(2j)}(L) = 0$, for $1 \leq j \leq n$, such that $F = F_1 + F_2$.*

Proof. This is just a uniform version of Lemma 0.6. Let $v_{1,t}$ be defined as in Lemma 0.6, replacing $\{f_+^{(2k)}(0), f_-^{(2k)}(L) : 1 \leq k \leq n\}$ by $\{F_{t,+}^{(2k)}(0), F_{t,-}^{(2k)}(L) : 1 \leq k \leq n\}$, and, let $p_{1,t} = T^{-1}(v_{1,t})$.

Then;

$p_{1,t} = \sum_{i=0}^{4n+1} d_i(t)x^i$

where the coefficients $d_i(t) = \sum_{k=1}^{n}(\lambda_{ik}F_{t,+}^{(2k)}(0) + \mu_{ik}F_{t,-}^{(2k)}(L)$

for fixed constants $\{\lambda_{ik}, \mu_{ik} : 0 \leq i \leq 4n+1, 1 \leq k \leq n\} \subset \mathcal{R}$. Let $\{r_{2k} : 1 \leq k \leq n\} \subset C_0^\infty([0,L] \times \mathcal{R})$ be given, as in Definition 0.1, and $\phi_{0,k}(t) = r_{2k}(t,0), \phi_{L,k}(t) = r_{2k}(t,L)$, then, $\{\phi_{0,k}, \phi_{L,k} : 1 \leq k \leq n\} \subset$

$C^{\infty}(\mathcal{R})$, ([4]). We have that;

$p_{1,t} = \sum_{i=0}^{4n+1}(\sum_{k=1}^{n}(\lambda_{ik}\phi_{0,k}(t) + \mu_{ik}\phi_{L,k})x^i$

and $p_{1,t} \in C_0^{\infty}([0,L] \times \mathcal{R})$. Letting $F_1 = p_{1,t}$, and $F_2 = F - F_1$, we obtain the result.

□

Lemma 0.13. *Let $F \in C_0^{\infty}([0,L]\times\mathcal{R})$, then, there exist $\{G_1, G_2, G\} \subset C^2([-L,L] \times \mathcal{R})$, such that, for all $\epsilon > 0$;*

(i).$G|_{[\epsilon,L-\epsilon)\times\mathcal{R}} = F|_{[\epsilon,L-\epsilon)\times\mathcal{R}}$.

(ii). G_1 is asymmetric and $\frac{\partial G_1}{\partial x}$ is symmetric about 0.

(iii). G_2 is symmetric and $\frac{\partial G_2}{\partial x}$ is asymmetric about 0.

Proof. By Lemma 0.11, we can find $\{F_1, F_2\} \subset C_0^{\infty}([0,L] \times \mathcal{R})$, with $F'_{1,+}(0) = F'_{1,+}(L) = 0$, $F''_{2,+}(0) = F''_{2,+}(L) = 0$, such that $F = F_1 + F_2$. By Lemma 0.9, we can find $\{G_1, G_2\} \subset C_0^2([-L,L])$, with $G_1|_{[0,L]} = G_1$, G_1 asymmetric and $\frac{\partial G_1}{\partial x}$ symmetric about 0, and with $G_2|_{[0,L]} = G_2$, G_2

[4]We have that;

$lim_{h\to 0}(\frac{r_{2k}(L,t+h)-r_{2k}(L,t)}{h}) = lim_{h\to 0}(lim_{x\to L}(\frac{r_{2k}(x,t+h)-r_{2k}(x,t)}{h}))$, $(*)$

As $r_{2k} \in C([-L,L] \times \mathcal{R})$, for fixed $h \neq 0$;

$lim_{x\to L}\frac{r_{2k}(x,t+h)-r_{2k}(x,t)}{h} = \frac{r_{2k}(L,t+h)-r_{2k}(L,t)}{h}$

For fixed $x' \neq L$;

$lim_{h\to 0}\frac{r_{2k}(x',t+h)-r_{2k}(x',t)}{h} = \frac{\partial^{2k+1}F}{\partial x^{2k+1}}(x',t)$

and, moreover, the convergence is uniform for $x' \in (0,L)$, as $\frac{\partial^{2k+1}F}{\partial x^{2k+1}}$ is bounded on $(0,L) \times (t-\epsilon, t+\epsilon)$, for any $\epsilon > 0$. It follows that we can interchange the limits in $(*)$, to obtain that;

$lim_{h\to 0}(\frac{r_{2k}(L,t+h)-r_{2k}(L,t)}{h})$

$= lim_{x\to L}(lim_{h\to 0}(\frac{r_{2k}(x,t+h)-r_{2k}(x,t)}{h}))$

$= lim_{x'\to L}\frac{\partial^{2k+1}F}{\partial x^{2k+1}}(x',t) = r_{2k+1}(L,t)$

symmetric and $\frac{\partial G_2}{\partial x}$ asymmetric about 0. Let $G = G_1 + G_2$, then $G \in C_0^2([-L,L] \times \mathcal{R})$, and $G|_{[\epsilon,L-\epsilon)\times\mathcal{R}} = F|_{[\epsilon,L-\epsilon)\times\mathcal{R}}$, as required. □

Lemma 0.14. *Let $F \in C_0^\infty([0,L]\times\mathcal{R})$, then, there exist $\{G_1, G_2, G\} \subset C^4([-L,L] \times \mathcal{R})$, such that, for all $0 \leq \epsilon < \frac{L}{2}$;*

(i).$G|_{[\epsilon,L-\epsilon)\times\mathcal{R}} = F|_{[\epsilon,L-\epsilon)\times\mathcal{R}}$.

(ii). G_1, $\frac{\partial^2 G_1}{\partial x^2}$ are asymmetric and $\frac{\partial G_1}{\partial x}$, $\frac{\partial^3 G_1}{\partial x^3}$ are symmetric about 0.

(iii). G_2, $\frac{\partial^2 G_2}{\partial x^2}$ are symmetric and $\frac{\partial G_2}{\partial x}$, $\frac{\partial^3 G_2}{\partial x^3}$ are asymmetric about 0.

Proof. By Lemma 0.12, we can find $\{F_1, F_2\} \subset C_0^\infty([0,L] \times \mathcal{R})$, with $F_{1,+}^{(1)}(0) = F_{1,-}^{(1)}(L) = 0$, $F_{1,+}^{(3)}(0) = F_{1,-}^{(3)}(L) = 0$, $F_{2,+}^{(2)}(0) = F_{2,-}^{(2)}(L) = 0$, $F_{2,+}^{(4)}(0) = F_{2,-}^{(4)}(L) = 0$, such that $F = F_1 + F_2$. By Lemma 0.10, we can find $\{G_1, G_2\} \subset C_0^4([-L,L])$, with $G_1|_{[0,L]} = F_1$, G_1, $\frac{\partial^2 G_1}{\partial x^2}$ asymmetric and $\frac{\partial G_1}{\partial x}$, $\frac{\partial^3 G_1}{\partial x^3}$ symmetric about 0, $G_2|_{[0,L]} = F_2$, G_2, $\frac{\partial^2 G_2}{\partial x^2}$ symmetric and $\frac{\partial G_2}{\partial x}$, $\frac{\partial^3 G_2}{\partial x^3}$ asymmetric about 0 Let $G = G_1 + G_2$, then $G \in C_0^4([-L,L] \times \mathcal{R})$, and $G|_{[\epsilon,L-\epsilon)\times\mathcal{R}} = F|_{[\epsilon,L-\epsilon)\times\mathcal{R}}$, as required. □

Lemma 0.15. *Let $F \in C_0^\infty([0,L]\times\mathcal{R})$ be a solution to the wave equation, then, for all $t \in \mathcal{R}$*

$$lim_{\epsilon\to 0}\frac{\partial^2 F}{\partial x^2}|_{(\epsilon,t)} = 0$$

$$lim_{\epsilon\to 0}\frac{\partial^2 F}{\partial x^2}(L-\epsilon,t) = 0$$

Proof. Let $\{G_1, G_2, G\}$ be given as in Lemma 0.14. Then, for all $t \in \mathcal{R}$, $G_t \in C^4([-L,L])$, and, using [2], the Fourier series expansion $\sum_{m\in\mathcal{Z}} c_m(t)e^{\frac{\pi i x m}{L}}$ of G_t converges uniformly to G_t on $[-L,L]$, ([5]). Similarly, as $G_t^{(n)} \in C^2([-L,L])$, for $0 \leq n \leq 2$, the Fourier series expansion $\sum_{m\in\mathcal{Z}} c_m(t)(\frac{\pi i m}{L})^n e^{\frac{\pi i x m}{L}}$ of $G_t^{(n)}$, converges uniformly to $G_t^{(n)}$ on $[-L,L]$, for $0 \leq n \leq 2$, $(*)$. We have that;

$$c_m(t) = \frac{1}{2L}\int_{-L}^{L} G(x,t)e^{-\frac{\pi i x m}{L}}dx$$

Hence, as, for $0 \leq n \leq 4$, $t_0 \in \mathcal{R}$, $\frac{\partial^n G}{\partial x^n}$ is bounded on $[-L,L] \times (t_0 - \delta, t_0 + \delta)$, by the DCT, we have that $c_{\epsilon,m} \in C^4(\mathcal{R})$. Moreover, we have,

[5] In fact, we only require that $G_t \in C^2([-L,L])$, see also [3]

for $0 \leq n \leq 4$;

$c_m^{(n)}(t) = \frac{1}{2L}\int_{-L}^{L} \frac{\partial^n G_\epsilon}{\partial t^n}(x,t)e^{-\frac{\pi i x m}{L}} dx$

Hence, again, as $\frac{\partial^n G_{\epsilon,t}}{\partial t^n} \in C^2([-L,L])$, for $0 \leq n \leq 2$, the Fourier series expansion $\sum_{m\in\mathcal{Z}} c_m^{(n)}(t)e^{\frac{\pi i x m}{L}}$ of $\frac{\partial^n G_t}{\partial t^n}$ converges uniformly to $\frac{\partial^n G_t}{\partial t^n}$ on $[-L,L]$, for $0 \leq n \leq 2$. Then;

$\frac{\partial^2 G_t}{\partial t^2} = \sum_{m\in\mathcal{Z}} c_m''(t)e^{\frac{\pi i x m}{L}}$

$\frac{\partial^2 G_t}{\partial x^2} = \sum_{m\in\mathcal{Z}} c_m(t)(\frac{\pi i m}{L})^2 e^{\frac{\pi i x m}{L}} = -\sum_{m\in\mathcal{Z}} c_m(t)(\frac{\pi^2 m^2}{L^2})e^{\frac{\pi i x m}{L}}$

Using the facts that $\frac{\partial^2 G_t}{\partial t^2} = \frac{T}{\mu}\frac{\partial^2 G_t}{\partial x^2}$, on $(0,L)$, the series $\sum_{m\in\mathcal{Z}}[c_m''(t) + c_m(t)(\frac{\pi^2 m^2 T}{\mu L^2})]e^{\frac{\pi i x m}{L}}$ is analytic on $[-L,L]$, and $\{e^{\frac{\pi i x m}{L}} : m \in \mathcal{Z}\}$ are orthogonal on $[-L,L]$, we obtain that;

$c_m''(t) + c_m(t)(\frac{\pi^2 m^2 T}{\mu L^2}) = 0 \ (t \in \mathcal{R})$

$c_m(t) = A_m e^{\frac{i\pi m\sqrt{T}t}{L\sqrt{\mu}}} + B_m e^{-\frac{i\pi m\sqrt{T}t}{L\sqrt{\mu}}}$

with $\{A_m, B_m\} \subset \mathcal{C}$, $A_m = a_m + ia_m'$, $B_m = b_m + ib_m'$ and;

$G = \sum_{m\in\mathcal{Z}} A_m e^{\frac{i\pi m\sqrt{T}t}{L\sqrt{\mu}}} e^{\frac{\pi i x m}{L}} + \sum_{m\in\mathcal{Z}} B_m e^{-\frac{i\pi m\sqrt{T}t}{L\sqrt{\mu}}} e^{\frac{\pi i x m}{L}}$

Then $\frac{\partial^2 G}{\partial x^2} = -[\sum_{m\in\mathcal{Z}} A_m \frac{\pi^2 m^2}{L^2} e^{\frac{i\pi m\sqrt{T}t}{L\sqrt{\mu}}} e^{\frac{\pi i x m}{L}} + \sum_{m\in\mathcal{Z}} B_m \frac{\pi^2 m^2}{L^2} e^{-\frac{i\pi m\sqrt{T}t}{L\sqrt{\mu}}} e^{\frac{\pi i x m}{L}}]$

$= -\sum_{m\in\mathcal{Z}_{\neq 0}}(a_m + b_m)\frac{\pi^2 m^2}{L^2} cos(\frac{\pi x m}{L})cos(\frac{\pi m\sqrt{T}t}{L\sqrt{\mu}}) + \theta(x,t) = S_t$

where $\theta(0,0) = \theta(L,0) = 0$

We have that;

$|(a_m + b_m)| = \frac{1}{2L}|\int_{-L}^{L} G_0(x)cos(\frac{\pi x m}{L})dx|$

$\leq \frac{L^{n-1}}{2\pi^n m^n}\int_{-L}^{L} |G_0^{(n)}|dx \leq \frac{C_{0,n}}{m^n}$, for $0 \leq n \leq 4$

where $C_{0,n} = \frac{L^{n-1}||G_0^{(n)}||_{L^1(-L,L)}}{2\pi^n}$.

Then;

$|\frac{\partial^2 G_0}{\partial x^2}|(0) = S_0 \leq \sum_{m\in\mathcal{Z}_{\neq 0}} |a_m + b_m|(\frac{\pi^2 m^2}{L^2})$

$\leq \sum_{1\leq |m|\leq k-1} |a_m + b_m|(\frac{\pi^2 m^2}{L^2}) + \sum_{|m|\geq k} \frac{C_{0,n}}{m^n}(\frac{\pi^2 m^2}{L^2})$

$= \sum_{1\leq |m|\leq k-1} |a_m + b_m|((\frac{\pi^2 m^2}{L^2}) + \sum_{|m|\geq k} \frac{L^{n-3}||G_0^{(n)}||_{L^1(-L,L)}}{2\pi^{n-2}m^{n-2}}$

Taking $n = 4$, we obtain;

$S_0 \leq \sum_{1\leq |m|\leq k-1} |a_m + b_m|(\frac{\pi^2 m^2}{L^2}) + \sum_{|m|\geq k} \frac{L}{2\pi^2 m^2}||G_0^{(4)}||_{L^1(-L,L)}$

$\leq \sum_{1\leq |m|\leq k-1} |a_m + b_m|(\frac{\pi^2 m^2}{L^2}) + \frac{L}{2\pi^2(k-1)}||G_0^{(4)}||_{L^1(-L,L)}$, (**)

We have, by conditions $(i), (ii)$ of Lemma 0.14 and the FTC, that, for all $0 < \epsilon < L$;

$|a_m + b_m| \leq \frac{1}{L}\int_{-\epsilon}^{\epsilon} |G_0(x)|dx + \int_{L-\epsilon}^{-L+\epsilon} |G_0(x)|dx$

$\leq \frac{1}{L}(|G_0(\epsilon)| + |G_0(-\epsilon)| + |G_0(L-\epsilon)| + |G_0(-L+\epsilon)|)$

$= \frac{1}{L}(|F_0(\epsilon)| + |G_{1,0}(-\epsilon)| + |G_{2,0}(-\epsilon)| + |F_0(L-\epsilon)| + |G_{1,0}(-L+\epsilon)|$

$+|G_{2,0}(-L+\epsilon)|)$

$\leq \frac{2L^2}{\pi^2(k-1)}(\frac{\delta'}{2})$

for sufficiently small $\epsilon(k,\delta')$, as $F_0 \in C_0([0,L])$ and $\{G_{1,0}, G_{2,0}\} \subset C_0([-L,L])$. Taking $k \geq \frac{4L||G_0^{(4)}||_{L^1(-L,L)}+1}{\pi^2\delta'}$, we then have that $|S_0| < \delta'$. Then, using condition (i) of Lemma 0.14, and the fact that $\frac{\partial^2 G_0}{\partial x^2}$ is continuous at 0, we obtain that $lim_{\epsilon\to 0}\frac{\partial^2 F_0}{\partial x^2}(\epsilon) = 0$ as required. In a similar way, using an expansion around an arbitrary $t_0 \in \mathcal{R}$, we obtain that $lim_{\epsilon\to 0}\frac{\partial^2 F_{t_0}}{\partial x^2}(\epsilon) = 0$, as required. By exactly the same method, we obtain that $lim_{\epsilon\to 0}\frac{\partial^2 F_{t_0}}{\partial x^2}(L-\epsilon) = 0$. □

Lemma 0.16. *Let $F \in C_0^\infty([0,L]\times\mathcal{R}$ be a solution to the wave equation. Then, the Fourier series expansion of F is given by;*

$\sum_{m\in\mathcal{Z}_{>0}} K_m cos(\frac{\pi m\sqrt{T}t}{L\sqrt{\mu}})sin(\frac{\pi xm}{L}) + L_m sin(\frac{\pi m\sqrt{T}t}{L\sqrt{\mu}})sin(\frac{\pi xm}{L})$

which converges uniformly to F on $[0,L]$.

Proof. By Lemma 0.15, we have that, for $t \in \mathcal{R}$, $lim_{\epsilon\rightarrow 0}\frac{\partial^2 F_t}{x^2}(\epsilon) = lim_{\epsilon\rightarrow 0}\frac{\partial^2 F_t}{x^2}(L-\epsilon) = 0$. Using the fact;

$lim_{\epsilon\rightarrow 0}\frac{\partial^2 G_{1,t}}{x^2}(\epsilon) = lim_{\epsilon\rightarrow 0}\frac{\partial^2 G_{1,t}}{x^2}(L-\epsilon) = 0$

from Lemma 0.14, we obtain that;

$lim_{\epsilon\rightarrow 0}\frac{\partial^2 G_{2,t}}{x^2}(\epsilon) = lim_{\epsilon\rightarrow 0}\frac{\partial^2 G_{2,t}}{x^2}(L-\epsilon) = 0$

Using Lemma 0.3 of [1], we obtain that;

$lim_{\epsilon\rightarrow 0}\frac{\partial^4 G_{2,t}}{x^4}(\epsilon) = lim_{\epsilon\rightarrow 0}\frac{\partial^4 G_{2,t}}{x^4}(L-\epsilon) = 0$

Hence, by Definition of G_2 in 0.12,0.14, we obtain that $G_2 = 0$. It follows that there exists $G_1 \in C_0^4([-L,L]\times\mathcal{R})$, with G_1 asymmetric about 0, such that $G_1|_{[0,L]} = F$.

Let $h \in C_0^4([-L,L])$ be an asymmetric function, and let;

$h(x) = \sum_{m\in\mathcal{Z}}\hat{h}(m)e^{\frac{\pi ixm}{L}}$ be the Fourier series expansion of h, with;

$\hat{h}(m) = \frac{1}{2L}\int_{-L}^{L} h(x)e^{\frac{-\pi ixm}{L}}$, for $m \in \mathcal{Z}$

We have that;

$\hat{h}(m) = \frac{1}{2L}\int_{-L}^{L} h(x)cos(\frac{\pi xm}{L})dx - \frac{i}{2L}\int_{-L}^{L} h(x)sin(\frac{\pi xm}{L})dx$

$= \frac{-i}{2L}\int_{-L}^{L} h(x)sin(\frac{\pi xm}{L})dx = \frac{-i}{L}\int_0^L f(x)sin(\frac{\pi xm}{L}) = ie_m$

with $e_m = -e_{-m}$, for $m \geq 0$, so $e_0 = 0$. Then;

$h(x) = -\sum_{m\in\mathcal{Z}_{>0}} 2e_m sin(\frac{\pi xm}{L})$

Then writing;

$G_1(t,x) = \sum_{m\in\mathcal{Z}_{>0}} f_m(t)sin(\frac{\pi xm}{L})$

and substituting into $(*)$ of Definition 0.1, justified by the method of Lemma 0.15 and the fact that $G_1 \in C^4([-L,L]\times\mathcal{R})$, we have that;

$\sum_{m\in\mathcal{Z}_{>0}} f_m''(t)sin(\frac{\pi xm}{L}) = -\frac{T}{\mu}(\sum_{m\in\mathcal{Z}} f_m(t)(\frac{\pi m}{L})^2 sin(\frac{\pi xm}{L}))$

Hence, $f_m''(t) = -\frac{T}{\mu}f_m(t)(\frac{\pi m}{L}^2) = -\frac{\pi^2 m^2 T}{L^2\mu}f_m(t)$

$f_m(t) = K_m cos(\frac{\pi m\sqrt{T}t}{L\sqrt{\mu}}) + L_m sin(\frac{\pi m\sqrt{T}t}{L\sqrt{\mu}})$

giving;

$G_1(t,x) = \sum_{m\in\mathcal{Z}_{>0}} K_m cos(\frac{\pi m\sqrt{T}t}{L\sqrt{\mu}})sin(\frac{\pi xm}{L}) + L_m sin(\frac{\pi m\sqrt{T}t}{L\sqrt{\mu}})sin(\frac{\pi xm}{L})$

where the convergence is uniform on $[-L, L]$. Using the fact that $G_1|_{[0,L]} = F$, by Lemma 0.11, we obtain that the series converges uniformly to F on $[0, L]$ as required.

□

References

[1] A Note on Convergence of Fourier Series, Tristram de Piro, (2013).

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[3] Fourier Analysis, An Introduction, Elias Stein and Rami Shakarchi, Princeton Lectures in Analysis, (2003).