NONSTANDARD ANALYSIS AND PHYSICS

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ABSTRACT. We make some observations on the the Navier-Stokes Equations, Laws of Thermodynamics and Electrodynamics, and Schrodinger's Equation, in relation to infinitesimal differentials and the nonstandard approach to Fourier Analysis.

1. Navier Stokes Equations

Definition 1.1. *Let $V \subset \mathcal{R}^3$ be a bounded open volume, in coordinates (x,y,z), with smooth boundary $S = \partial V$. We let $\rho(x,y,z,t)$, $p(x,y,z,t)$, and $\overline{v}(x,y,z,t)$, $\overline{a}$, with components $\{v_i(x,y,z,t), a_i(x,y,z,t) : 1 \leq i \leq 3\}$, denote the density, pressure, velocity and acceleration of a fluid enclosed in the volume V. We let $\overline{n} : S \rightarrow S^3(\mathcal{R})$, with components $\{n_i : 1 \leq i \leq 3\}$, denote the unit normal to the surface S, ([1]). We let $\overline{F}(x,y,z,t)$ denote the body force, with components $\{F_i(x,y,z,t) : 1 \leq i \leq 3\}$, and let $\sigma(x,y,z,t)$ the stress tensor, with components $\{\sigma_{ij}(x,y,z,t) : 1 \leq i,j \leq 3\}$ on V. We assume that $\{\rho, p, \overline{v}, \overline{F}, \sigma\}$ extend to smooth functions on $S = \partial V$.*

Remarks 1.2. *The classical Navier Stokes Equation states that;*

$$\rho \frac{D\overline{v}}{Dt} = \rho \overline{F} + \bigtriangledown'(\sigma)$$

where, for a vector field $\overline{v}$ on V;

$$\frac{D\overline{v}}{Dt} = \frac{\partial \overline{v}}{\partial t} + \overline{v} \cdot \bigtriangledown(\overline{v})$$

and;

[1]Fixing a smooth map $W : (B^3, S^3) \rightarrow (V, S)$, we obtain an open chart $P(r,\theta,\phi) = \lambda(rsin(\theta)cos(\phi), rsin(\theta)sin(\phi), rcos(\theta))$, $0 < r < 1, 0 < \theta \leq \pi, -\pi < \phi \leq \pi$. The normal $\overline{n} = \frac{\frac{\partial \lambda_1}{\partial \theta} \times \frac{\partial \lambda_1}{\partial \phi}}{|\frac{\partial \lambda_1}{\partial \theta} \times \frac{\partial \lambda_1}{\partial \phi}|}$ extends to a smooth field $\overline{n}'$ on $(V \setminus O)$, given by $\frac{r(\frac{\partial \lambda_r}{\partial \theta} \times \frac{\partial \lambda_r}{\partial \phi})}{|\frac{\partial \lambda_r}{\partial \theta} \times \frac{\partial \lambda_r}{\partial \phi}|}$, where $r \neq 0$. It is easy to see that $\overline{n}'$ extends to a smooth field on $\overline{V}$, by setting $\overline{n}'(O) = O$. We let $\{n_i' : 1 \leq i \leq 3\}$ denote the components of $\overline{n}'$

$$\bigtriangledown(\overline{v}) = (grad(v_1), \ldots, grad(v_i), \ldots, grad(v_n))$$

$$= \begin{pmatrix} \frac{\partial v_1}{\partial x_1} & \ldots & \frac{\partial v_i}{\partial x_1} & \ldots & \frac{\partial v_n}{\partial x_1} \\ . & & & & \\ . & & & & \\ \frac{\partial v_1}{\partial x_j} & \ldots & \frac{\partial v_i}{\partial x_j} & \ldots & \frac{\partial v_n}{\partial x_j} \\ . & & & & \\ . & & & & \\ \frac{\partial v_1}{\partial x_n} & \ldots & \frac{\partial v_i}{\partial x_n} & \ldots & \frac{\partial v_n}{\partial x_n} \end{pmatrix}$$

$$\overline{v} \centerdot (\overline{w}_1, \ldots, \overline{w}_i, \ldots, \overline{w}_n) = (\overline{v} \centerdot \overline{w}_1, \ldots, \overline{v} \centerdot \overline{w}_i, \ldots, \overline{v} \centerdot \overline{w}_n)$$

and, for a smooth matrix;

$$\sigma = \begin{pmatrix} \overline{\sigma}_1 \\ . \\ \overline{\sigma}_i \\ . \\ \overline{\sigma}_n \end{pmatrix}$$

on V. $\bigtriangledown'(\sigma) = (div(\overline{\sigma}_1), \ldots, div(\overline{\sigma}_i), \ldots, div(\overline{\sigma}_n))$

$$= (\sum_{j=1}^n \sigma_{1j}, \ldots, \sum_{j=1}^n \sigma_{ij}, \ldots, \sum_{j=1}^n \sigma_{nj})$$

Writing this in tensor notation, we obtain, for $1 \leq i \leq 3$, *(summing over* j*), that;*

$$\rho(\frac{\partial v_i}{\partial t} + v_j \frac{\partial v_i}{\partial x_j} - F_i) - (K - \frac{2\mu}{3})\frac{\partial^2 v_j}{\partial x_i \partial x_j} + \frac{\partial p}{\partial x_i} - \mu(\frac{\partial^2 v_i}{\partial x_j^2} + \frac{\partial^2 v_j}{\partial x_i \partial x_j}) = 0$$

where $\{K, \mu\}$ *are constants that vary with temperature and density. One of the practical problems, here, seems to be in computing the body force vector and stress tensor. The purpose of this note is to provide a new equation which depends only on* $\{\rho, p, \overline{v}\}$.

Lemma 1.3. *With hypotheses as in Definition 0.1, we have the following physically realistic equation version of the Navier-Stokes Equation;*

$$\rho(\frac{\partial v_i}{\partial t} + \frac{\partial v_i}{\partial x_j} v_j) - p\frac{\partial v_i}{\partial t}\frac{\partial n'_j}{\partial x_j} - p\frac{\partial v_i}{\partial t}|\frac{\partial \overline{v}}{\partial t}|^{-1} = 0$$

Proof. We use the equations;

$(i).\ p = |\frac{dF_I}{dV}| = |\frac{dF_B}{dS}|$;

$(ii).\ \frac{dF_I}{dV} = \rho\overline{a},\ \frac{dF_B}{dS} = \rho\overline{a}$

which implies that $\frac{dF_I}{dV} = p\frac{\overline{a}}{|\overline{a}|},\ \frac{dF_B}{dV} = p\frac{\overline{a}}{|\overline{a}|}$

$(\frac{dF_I}{dV} = \frac{\overline{a}}{|\overline{a}|}|\frac{dF_I}{dV}| = p\frac{\overline{a}}{|\overline{a}|},\ \frac{dF_B}{dS} = \frac{\overline{a}}{|\overline{a}|}|\frac{dF_B}{dS}| = p\frac{\overline{a}}{|\overline{a}|})$

Then the total boundary force;

$$F_B = \int_S dF_B = \int_S \frac{dF_B}{dS}dS = \int_S \rho\overline{a}dS = \int_S (p\frac{\overline{a}}{|\overline{a}|})dS$$

and internal force;

$$F_I = \int_V dF_I = \int_V \frac{dF_I}{dV}dV = \int_V \rho\overline{a}dS = \int_V (p\frac{\overline{a}}{|\overline{a}|})dV$$

The equation of motion for the fluid, compare [6], is then given, for a moving volume, using Reynolds Transport Theorem, for $1 \leq i \leq 3$, by;

$$\frac{d}{dt}\int_V (\rho v_i)dV = \int_V [\frac{\partial(\rho v_i)}{\partial t} + \bigtriangledown(\rho v_i\overline{v})]dV$$

$$= \int_S \frac{d(F_B)_i}{dS}dS + \int_V \frac{d(F_I)_i}{dV}dV$$

$$= \int_S (p\frac{a_i}{|\overline{a}|})dS + \int_V (p\frac{a_i}{|\overline{a}|})dV$$

We let;

$$N = \begin{pmatrix} pa_1(\overline{n}')^t \\ . \\ . \\ pa_i(\overline{n}')^t \\ . \\ . \\ pa_n(\overline{n}')^t \end{pmatrix}$$

and $M = \frac{N}{|\overline{a}|}$, so that, using the fact $|\overline{n}| = 1$, $M(\overline{n})|_S = p\frac{\overline{a}}{|\overline{a}|}$, $(*)$. We have, using $(*)$, the Divergence Theorem, and the assumption that $\overline{a}(O) \neq O$, that;

$$\int_S (p\frac{a_i}{|\overline{a}|})dS$$

$= \int_S (M_i \cdot \overline{n}) dS$

$= \int_S M_i \cdot d\overline{S}$

$= \int_V (\bigtriangledown \cdot M_i) dV$

Again, using the divergence theorem, see [1], we obtain;

$\int_V \{ \frac{\partial(\rho v_i)}{\partial t} + \bigtriangledown \cdot (\rho v_i \overline{v}) - \bigtriangledown \cdot (M_i) - \frac{p \frac{\partial v_i}{\partial t}}{|\frac{\partial \overline{v}}{\partial t}|} \} dV$

$= \int_V \frac{\partial \rho}{\partial t} v_i + \rho \frac{\partial v_i}{\partial t} + v_i \bigtriangledown \cdot (\rho \overline{v}) + \rho(\overline{v} \cdot \bigtriangledown) v_i - \bigtriangledown \cdot (M_i) - \frac{p \frac{\partial v_i}{\partial t}}{|\frac{\partial \overline{v}}{\partial t}|} dV = 0$

Writing this in non-component form, and, using the mass equation, and the definition of the material derivative, we obtain;

$\int_V \overline{v}(\frac{\partial \rho}{\partial t} + \bigtriangledown \cdot (\rho \overline{v})) + \rho(\frac{\partial \overline{v}}{\partial t} + (\overline{v} \cdot \bigtriangledown)\overline{v}) - \overline{\bigtriangledown}(M_1, \ldots M_n) - p \frac{\frac{\partial \overline{v}}{\partial t}}{|\frac{\partial \overline{v}}{\partial t}|} dV$

$= \int_V \rho(\frac{D\overline{v}}{Dt}) - \overline{\bigtriangledown}(M_1, \ldots M_n) - p \frac{\frac{\partial \overline{v}}{\partial t}}{|\frac{\partial \overline{v}}{\partial t}|} dV = 0$

$\rho(\frac{D\overline{v}}{Dt}) - \overline{\bigtriangledown}(M_1, \ldots M_n) - p \frac{\frac{\partial \overline{v}}{\partial t}}{|\frac{\partial \overline{v}}{\partial t}|} = 0$

where $\overline{\bigtriangledown}(M_1, \ldots M_n) = (\bigtriangledown \cdot (M_1), \ldots, \bigtriangledown \cdot (M_n))$

This gives, for $1 \leq i \leq 3$, (summing over j), that;

$\rho(\frac{\partial v_i}{\partial t} + \frac{\partial v_i}{\partial x_j} v_j) - p \frac{\partial v_i}{\partial t} \frac{\partial n'_j}{\partial x_j} - p \frac{\partial v_i}{\partial t} |\frac{\partial \overline{v}}{\partial t}|^{-1} = 0$

$\square$

Remarks 1.4. *In the analogous electrodynamic situation, we have that* $\rho = f_e \rho_q$*, where* ρ_q *is the charge density,* $f_e = \frac{m_e}{q_e}$*, and* m_e*,* q_e *are the electron masses and electron charges respectively. We have that* $\overline{v} = \frac{\overline{J}}{\rho_q}$*, and* $p = ||\rho_q \overline{E} + (\overline{J} \times \overline{B})||$*, where* $\{\overline{J}, \overline{E}, \overline{B}\}$ *are the volume current densities, electric and magnetic fields, see* [4]*. Using Maxwell's equations, we can eliminate the terms* $\{\rho_q, \overline{J}\}$*.*

2. Electrodynamics

Definition 2.1. *Maxwell's Equations*

Maxwell's equations are given by;

(i). $\bigtriangledown . \overline{E} = \frac{\rho}{\epsilon_0}$ *(Gauss's Law)*

(ii). $\bigtriangledown . \overline{B} = 0$ *(Gauss's Law for Magnetism)*

(iii). $(\bigtriangledown \times \overline{E}) = -\frac{\partial \overline{B}}{\partial t}$ *(Faraday's Law of Induction)*

(iv). $(\bigtriangledown \times \overline{B}) = \mu_0(\overline{J} + \epsilon_0 \frac{\partial \overline{E}}{\partial t})$ *(Ampere's Law with Maxwell's Correction)*

where $\{\overline{E}, \overline{B}, \overline{J}\} \subset C^\infty(\mathcal{R}^3 \times \mathcal{R}_{\geq 0}, \mathcal{C}^3)$ *denote the electric, magnetic fields, and volume current,* $\rho \in C^\infty(\mathcal{R}^3 \times \mathcal{R}_{\geq 0}, \mathcal{C})$ *is the charge density, and* $\{\epsilon_0, \mu_0\}$ *denote the permittivity and permeability of free space, ([2]). In regions where there is no charge and current, we obtain Maxwell's equations in free space;*

(i)′. $\bigtriangledown . \overline{E} = 0$

(ii)′. $\bigtriangledown . \overline{B} = 0$

(iii)′. $(\bigtriangledown \times \overline{E}) = -\frac{\partial \overline{B}}{\partial t}$

(iv)′. $\bigtriangledown \times \overline{B}) = \mu_0 \epsilon_0 \frac{\partial \overline{E}}{\partial t}$

Definition 2.2. *We define smoothly decaying solutions functions* $\mathcal{S}(\mathcal{R}^3, \mathcal{R}_{\geq 0}, \mathcal{C}) = \{f \in C^\infty(\mathcal{R}^3 \times \mathcal{R}_{\geq 0}, \mathcal{R}) : (\forall t \in \mathcal{R}_{\geq 0}) f_t \in \mathcal{S}(\mathcal{R}^3, \mathcal{C})\}$*, and, similarly, for smoothly decaying vector fields. We define smoothly decaying solutions of Maxwell's equations to be smoothly decaying* $\{\overline{E}, \overline{B}, \overline{J}\}$ *and* ρ *, which satisfy* $(i) - (iv)$*,* $(i)' - (iv)'$ *on* $\mathcal{R}^3 \times \mathcal{R}_{>0}$*.*

Lemma 2.3. *The smoothly decaying solutions of Maxwell's equations in free space, are given by;*

$\overline{E}(\overline{x}, t)$

$$= \int_{\overline{k} \in \mathcal{R}^3} \int_{S_{\overline{k}}} G(\overline{k}, \overline{n}) e^{i(\overline{k}.\overline{x} - \omega(\overline{k})t)} d\overline{S}_{\overline{k}}(\overline{n}) d\overline{k} + \int_{\overline{k} \in \mathcal{R}^3} \int_{S_{\overline{k}}} H(\overline{k}, \overline{n}) e^{i(\overline{k}.\overline{x} + \omega(\overline{k})t)} d\overline{S}_{\overline{k}}(\overline{n}) d\overline{k}$$

$(*)$

[2] Adopting the convention that $C^\infty(\mathcal{R}^3 \times \mathcal{R}_{\geq 0}, \mathcal{C}) = \{f \in C(\mathcal{R}^3 \times \mathcal{R}_{\geq 0}, \mathcal{C}) : f|_{\mathcal{R}^3 \times \mathcal{R}_{>0}} \in C^\infty(\mathcal{R}^3 \times \mathcal{R}_{>0}, \mathcal{C}), \exists g \in C(\mathcal{R}^3 \times \mathcal{R}_{\geq 0}, \mathcal{C}), g|_{\mathcal{R}^3 \times \mathcal{R}_{>0}} = (f|_{\mathcal{R}^3 \times \mathcal{R}_{>0}})^n, n \in \mathcal{Z}_{\geq 0}\}$, and, similarly, for $C^\infty(\mathcal{R}^3 \times \mathcal{R}_{\geq 0}, \mathcal{C}^3)$

where $k=|\overline{k}|$, $\omega(\overline{k})=c|\overline{k}|=\frac{k}{\sqrt{\mu_0\epsilon_0}}$ *and for* $\{G,H\}\subset\mathcal{S}(M)$ *and* $S_{\overline{k}}=(S^2(\overline{k},1)\cap P_{\overline{k}})$, $P_{\overline{k}}=\{\overline{n}:(\overline{n}-\overline{k})\centerdot\overline{k}=0\}$, $M=\{(\overline{k},\overline{n})\in\mathcal{R}^6:(\overline{n}-\overline{k})\centerdot\overline{k}=0,|\overline{n}-\overline{k}|=1\}$ *and* $\mathcal{S}(M)=\{f\in C(M):\int_{S_{\overline{k}}}fd\overline{S}_{\overline{k}}\in\mathcal{S}(\mathcal{R}^3,\mathcal{R}_{\geq 0},\mathcal{C}^3)\}$

$\overline{B}(\overline{x},t)$

$=\int_{\overline{k}\in\mathcal{R}^3}[\int_{S_{\overline{k}}}\overline{M}(\overline{k},\overline{n})e^{i(\overline{k}\centerdot\overline{x}-\omega(\overline{k})t)}dS_{\overline{k}}(\overline{n})]d\overline{k}$

$+\int_{\overline{k}\in\mathcal{R}^3}[\int_{S_{\overline{k}}}\overline{N}(\overline{k},\overline{n})e^{i(\overline{k}\centerdot\overline{x}+\omega(\overline{k})t)}dS_{\overline{k}}(\overline{n})]d\overline{k}$ $(**)$

where $k=|\overline{k}|$, $\omega(\overline{k})=c|\overline{k}|=\frac{k}{\sqrt{\mu_0\epsilon_0}}$ *and* $\overline{M}(\overline{k},\overline{n})=\frac{G(\overline{k},\overline{n})}{\omega(\overline{k})}(\overline{k}\times(\overline{n}-\overline{k}))$

$=\frac{G(\overline{k},\overline{n})}{c}(\overline{k}\times(\hat{\overline{n}-\overline{k}}))$

$\overline{N}(\overline{k},\overline{n})=\frac{-H(\overline{k},\overline{n})}{\omega(\overline{k})}(\overline{k}\times(\overline{n}-\overline{k}))$

$\frac{-H(\overline{k},\overline{n})}{c}(\overline{k}\times(\hat{\overline{n}-\overline{k}}))$, $(*)$

Proof. It is easily checked that the solutions in $(*)$ satisfy $(i)'-(iv)'$, Conversely, let $\{\overline{E},\overline{B}\}$ be smooth solutions of $(i)'-(iv)'$. We have, using $(i)',(iii)',(iv)'$, that;

$(\bigtriangledown\times\overline{E})=-\frac{\partial\overline{B}}{\partial t}$, and, hence;

$\bigtriangledown\times(\bigtriangledown\times\overline{E})=-(\bigtriangledown\times\frac{\partial\overline{B}}{\partial t})$

$-(\bigtriangledown)^2\overline{E}=-\frac{\partial}{\partial t}(\bigtriangledown\times\overline{B})=-\mu_0\epsilon_0\frac{\partial^2\overline{E}}{\partial t^2}$

$(\bigtriangledown)^2\overline{E}=\frac{1}{c^2}\frac{\partial^2\overline{E}}{\partial t^2}$ $(c=\frac{1}{\sqrt{\mu_0\epsilon_0}})$

If $E_i\in\mathcal{S}_{(\mathcal{R}^3,\mathcal{R})}(\mathcal{R}^4)$ solve the wave equation, $(\bigtriangledown)^2E_i=\frac{1}{c^2}\frac{\partial^2E_i}{\partial t^2}$, where $\mathcal{S}_{(\mathcal{R}^3,\mathcal{R})}(\mathcal{R}^4)=\{f\in C^\infty(\mathcal{R}^4):f_t\in\mathcal{S}(\mathcal{R}^3)\}$, for $t\in\mathcal{R}$, then;

$E_i(\overline{x},t)=(\frac{1}{2\pi})^3\int_{\mathcal{R}^3}\hat{E}_i(\overline{k},t)e^{i\overline{k}\centerdot\overline{x}}d\overline{k}$

by the Inversion theorem. Hence;

$(\bigtriangledown)^2E_i=-(\frac{1}{2\pi})^3\int_{\mathcal{R}^3}|\overline{k}|^2\hat{E}_i(\overline{k},t)e^{i\overline{k}\centerdot\overline{x}}d\overline{k}$

$\frac{\partial^2 E_i}{\partial t^2}=(\frac{1}{2\pi})^3\int_{\mathcal{R}^3}\frac{\partial^2 \hat{E}_i}{\partial t^2}(\overline{k},t)e^{i\overline{k}.\overline{x}}d\overline{k}$

$(\bigtriangledown)^2 E_i-\frac{1}{c^2}\frac{\partial^2 E_i}{\partial t^2}=(\frac{1}{2\pi})^3\int_{\mathcal{R}^3}(-|\overline{k}|^2\hat{E}_i-\frac{1}{c^2}\frac{\partial^2 \hat{E}_i}{\partial t^2})(\overline{k},t)e^{i\overline{k}.\overline{x}}d\overline{k}=0$

$|\overline{k}|^2\hat{E}_i+\frac{1}{c^2}\frac{\partial^2 \hat{E}_i}{\partial t^2}=0$, using the Inversion formula again.

$\hat{E}_i(\overline{k},t)=A_i(\overline{k})e^{-i|\overline{k}|ct}+B_i(\overline{k})e^{i|\overline{k}|ct}$, as $\hat{E}_i(\overline{k},t)\in\mathcal{S}_{(\mathcal{R}^3,\mathcal{R})}(\mathcal{R}^4)$

$E_i(\overline{x},t)=(\frac{1}{2\pi})^3(\int_{\mathcal{R}^3}(A_i(\overline{k})e^{-i|\overline{k}|ct})e^{i\overline{k}.\overline{x}}d\overline{k}+\int_{\mathcal{R}^3}(B_i(\overline{k})e^{i|\overline{k}|ct})e^{i\overline{k}.\overline{x}}d\overline{k})$

$=(\frac{1}{2\pi})^3(\int_{\mathcal{R}^3}A_i(\overline{k})e^{i(\overline{k}.\overline{x}-\omega(\overline{k})t)}d\overline{k}+\int_{\mathcal{R}^3}B_i(\overline{k})e^{i(\overline{k}.\overline{x}+\omega(\overline{k})t)}d\overline{k})$

$\overline{E}(\overline{x},t)=\int_{\mathcal{R}^3}\overline{A}(\overline{k})e^{i(\overline{k}.\overline{x}-\omega(\overline{k})t)}d\overline{k}+\int_{\mathcal{R}^3}\overline{B}(\overline{k})e^{i(\overline{k}.\overline{x}+\omega(\overline{k})t)}d\overline{k}$

Using $(i)'$, we have that;

$\int_{\mathcal{R}^3}(\overline{A}(\overline{k})\centerdot i\overline{k})e^{i(\overline{k}.\overline{x}-\omega(\overline{k})t)}d\overline{k}+\int_{\mathcal{R}^3}(\overline{B}(\overline{k})\centerdot\overline{k})e^{i(\overline{k}.\overline{x}+\omega(\overline{k})t)}d\overline{k}=0$

At $t=0$ and $t=1$, we obtain that;

$\int_{\mathcal{R}^3}(\overline{A}(\overline{k})\centerdot i\overline{k})e^{i(\overline{k}.\overline{x})}d\overline{k}+\int_{\mathcal{R}^3}(\overline{B}(\overline{k})\centerdot\overline{k})e^{i(\overline{k}.\overline{x})}d\overline{k}=0$

$\int_{\mathcal{R}^3}(\overline{A}(\overline{k})\centerdot i\overline{k})e^{i(\overline{k}.\overline{x}-ck)}d\overline{k}+\int_{\mathcal{R}^3}(\overline{B}(\overline{k})\centerdot\overline{k})e^{i(\overline{k}.\overline{x}+ck)}d\overline{k}=0$

As this holds for all $\overline{x}\in\overline{R}^3$, using the Inversion formula, we obtain that;

$(\overline{A}(\overline{k})+\overline{B}(\overline{k}))\centerdot\overline{k}=0$

$(\overline{A}(\overline{k})e^{-ck}+\overline{B}(\overline{k})e^{ck})\centerdot\overline{k}=0$

Assuming $k\neq 0$, we obtain that;

$\overline{A}(\overline{k})\centerdot\overline{k}=\overline{B}(\overline{k})\centerdot\overline{k}=0$

so that;

$A(\overline{k})=\int_{S_{\overline{k}}}G(\overline{k},\overline{n})d\overline{S}_{\overline{k}}(\overline{n})$, $B(\overline{k})=\int_{S_{\overline{k}}}H(\overline{k},\overline{n})d\overline{S}_{\overline{k}}(\overline{n})$ and the first part of the result $(*)$ follows.

Using $(iii)'$, we obtain, by the same argument, that;

$\overline{B}(\overline{x},t)$

$= \int_{\overline{k}\in\mathcal{R}^3}\int_{S_{\overline{k}}}\overline{K}(\overline{k},\overline{n})e^{i(\overline{k}\centerdot\overline{x}-\omega(\overline{k})t)}d\overline{S}_{\overline{k}}(\overline{n})d\overline{k}+\int_{\overline{k}\in\mathcal{R}^3}\int_{S_{\overline{k}}}\overline{L}(\overline{k},\overline{n})e^{i(\overline{k}\centerdot\overline{x}+\omega(\overline{k})t)}d\overline{S}_{\overline{k}}(\overline{n})d\overline{k}$ $(\dagger\dagger)$

Using $(iii)'$;

$\frac{\partial\overline{B}}{\partial t}(\overline{x},t)$

$= \int_{\overline{k}\in\mathcal{R}^3}\int_{S_{\overline{k}}}iG(\overline{k},\overline{n})e^{i(\overline{k}\centerdot\overline{x}-\omega(\overline{k})t)}(d\overline{S}_{\overline{k}}(\overline{n})\times\overline{k})d\overline{k}$

$+\int_{\overline{k}\in\mathcal{R}^3}\int_{S_{\overline{k}}}iH(\overline{k},\overline{n})e^{i(\overline{k}\centerdot\overline{x}-\omega(\overline{k})t)}(d\overline{S}_{\overline{k}}(\overline{n})\times\overline{k})d\overline{k}$

$= \int_{\overline{k}\in\mathcal{R}^3}\int_{S_{\overline{k}}}iG(\overline{k},\overline{n})((\overline{n}-\overline{k})\hat{}\times\overline{k})e^{i(\overline{k}\centerdot\overline{x}-\omega(\overline{k})t)}dS_{\overline{k}}(\overline{n})d\overline{k}$

$+\int_{\overline{k}\in\mathcal{R}^3}\int_{S_{\overline{k}}}iH(\overline{k},\overline{n})((\overline{n}-\overline{k})\hat{}\times\overline{k})e^{i(\overline{k}\centerdot\overline{x}+\omega(\overline{k})t)}dS_{\overline{k}}(\overline{n})d\overline{k}$

$= \int_{\overline{k}\in\mathcal{R}^3}\int_{S_{\overline{k}}}iG(\overline{k},\overline{n})\frac{(\overline{n}\times\overline{k})}{|\overline{n}-\overline{k}|}e^{i(\overline{k}\centerdot\overline{x}-\omega(\overline{k})t)}dS_{\overline{k}}(\overline{n})d\overline{k}$

$+\int_{\overline{k}\in\mathcal{R}^3}\int_{S_{\overline{k}}}iH(\overline{k},\overline{n})\frac{(\overline{n}\times\overline{k})}{|\overline{n}-\overline{k}|}e^{i(\overline{k}\centerdot\overline{x}+\omega(\overline{k})t)}dS_{\overline{k}}(\overline{n})]d\overline{k}$

$\overline{B}(\overline{x},t)$

$= \int_{\overline{k}\in\mathcal{R}^3}[\int_{S_{\overline{k}}}\frac{-G(\overline{k},\overline{n})}{\omega(\overline{k})}\frac{(\overline{n}\times\overline{k})}{|\overline{n}-\overline{k}|}e^{i(\overline{k}\centerdot\overline{x}-\omega(\overline{k})t)}dS_{\overline{k}}(\overline{n})]d\overline{k}$

$+\int_{\overline{k}\in\mathcal{R}^3}[\int_{S_{\overline{k}}}\frac{H(\overline{k},\overline{n})}{\omega(\overline{k})}\frac{(\overline{n}\times\overline{k})}{|\overline{n}-\overline{k}|}e^{i(\overline{k}\centerdot\overline{x}+\omega(\overline{k})t)}dS_{\overline{k}}(\overline{n})]d\overline{k}+\overline{\theta}(t)$

$= \int_{\overline{k}\in\mathcal{R}^3}[\int_{S_{\overline{k}}}\overline{M}(\overline{k},\overline{n})e^{i(\overline{k}\centerdot\overline{x}-\omega(\overline{k})t)}dS_{\overline{k}}(\overline{n})]d\overline{k}$

$+\int_{\overline{k}\in\mathcal{R}^3}[\int_{S_{\overline{k}}}\overline{N}(\overline{k},\overline{n})e^{i(\overline{k}\centerdot\overline{x}+\omega(\overline{k})t)}dS_{\overline{k}}(\overline{n})]d\overline{k}$ $(**)$

where $\overline{M}(\overline{k},\overline{n})=\frac{G(\overline{k},\overline{n})}{\omega(\overline{k})}\frac{(\overline{k}\times\overline{n})}{|\overline{n}-\overline{k}|}$, $\overline{N}(\overline{k},\overline{n})=\frac{-H(\overline{k},\overline{n})}{\omega(\overline{k})}\frac{(\overline{k}\times\overline{n})}{|\overline{n}-\overline{k}|}$, and $\overline{\theta}(t)=0$, using $(\dagger\dagger)$.

Using the fact that $\overline{k}\centerdot(\overline{k}\times\overline{n})=0$, we have that the expression $(**)$ for $\overline{B}(\overline{x},t)$, satisfies $(ii)'$. Finally, we obtain, using $(iv)'$;

$\int_{\overline{k}\in\mathcal{R}^3}[\int_{S_{\overline{k}}}-ik^2\frac{G(\overline{k},\overline{n})}{\omega(\overline{k})}\frac{\overline{n}-\overline{k}}{|\overline{n}-\overline{k}|}e^{i(\overline{k}\centerdot\overline{x}-\omega(\overline{k})t)}dS_{\overline{k}}(\overline{n})]d\overline{k}$

$+\int_{\overline{k}\in\mathcal{R}^3}[\int_{S_{\overline{k}}} -ik^2\frac{-H(\overline{k},\overline{n})}{\omega(\overline{k})}\frac{\overline{n}-\overline{k}}{|\overline{n}-\overline{k}|}e^{i(\overline{k}.\overline{x}+\omega(\overline{k})t)}dS_{\overline{k}}(\overline{n})]d\overline{k}$

$+\int_{\overline{k}\in\mathcal{R}^3}\int_{S_{\overline{k}}}\frac{ik}{c}G(\overline{k},\overline{n})e^{i(\overline{k}.\overline{x}-\omega(\overline{k})t)}d\overline{S}_{\overline{k}}(\overline{n})d\overline{k}$

$+\int_{\overline{k}\in\mathcal{R}^3}\int_{S_{\overline{k}}}\frac{-ik}{c}H(\overline{k},\overline{n})e^{i(\overline{k}.\overline{x}+\omega(\overline{k})t)}d\overline{S}_{\overline{k}}(\overline{n})d\overline{k}=0$

$\overline{E}=\overline{E}_0e^{i(\overline{k}.\overline{x}-\omega t)}$

where $c=\frac{\omega}{k}=\frac{1}{\sqrt{\epsilon_0\mu_0}}$. Using $(i)'$, we obtain that $\overline{E}_0=E_0\hat{n}$, where $\hat{n}.\overline{k}=0$, and $|\hat{n}|=1$;

$\overline{E}=E_0e^{i(\overline{k}.\overline{x}-\omega t)}\hat{n}$

Similarly, we obtain that $(\bigtriangledown)^2\overline{B}=\frac{1}{c^2}\frac{\partial^2\overline{B}}{\partial t^2}$;

$\overline{B}=\overline{B_0}e^{i(\overline{k}.\overline{x}-\omega t)}$, $(\dagger)$

Using $(iii)'$, we obtain that;

$\frac{\partial\overline{B}}{\partial t}=-ie^{i(\overline{k}.\overline{x}-\omega t)}(\overline{k}\times E_0\hat{n})$

$\overline{B}=\frac{1}{\omega}e^{i(\overline{k}.\overline{x}-\omega t)}(\overline{k}\times E_0\hat{n})+\overline{\theta}(t)$

$=B_0e^{i(\overline{k}.\overline{x}-\omega t)}(\overline{k}\times\hat{n})$

using $(\dagger)$, so that $\overline{B}_0=B_0(\overline{k}\times\hat{n})$, and $\overline{\theta}(t)=0$.

Using $(iv)'$, we obtain that;

$(\bigtriangledown\times\overline{B})-\frac{1}{c^2}\frac{\partial\overline{E}}{\partial t}$

$=((-ik^2B_0+\frac{i\omega}{c^2}E_0))e^{i(\overline{k}.\overline{x}-\omega t)}\hat{n}$

$=((\frac{-ik^2E_0}{\omega}+\frac{ikE_0}{c}))e^{i(\overline{k}.\overline{x}-\omega t)}\hat{n}=0$

$=((\frac{-ikE_0}{c}+\frac{ikE_0}{c}))e^{i(\overline{k}.\overline{x}-\omega t)}\hat{n}=0$

([3])

---

[3] Formulate a nonstandard version, for $\{\overline{E},\overline{B}\}\subset V(\mathcal{R}^4_{\pi\eta})$, ($\mathcal{R}_{\pi\eta}$ being the interval $[-\pi\eta,\pi\eta)$ with measure $\lambda_{\pi\eta}$, defined by $\lambda_{\pi\eta}([\frac{\pi i}{\eta},\frac{\pi(i+1)}{\eta}))=\frac{\pi}{\eta}$)

$(i)''.\ \bigtriangledown\centerdot\overline{E}=0$

$(ii)''.\ \bigtriangledown\centerdot\overline{B}=0$

$(iii)''.\ (\bigtriangledown\times\overline{E})=-\frac{\partial\overline{B}}{\partial t}$

$(iv)''.\ (\bigtriangledown\times\overline{B})=\mu_0\epsilon_0(\frac{\partial\overline{E}}{\partial t})$

Suppose $\{\overline{E},\overline{B}\}$ solve $(i)''-(iv)''$. As above, using $(i)'',(iii)'',(iv)''$, and the nonstandard version of [1], we obtain that;

$(\bigtriangledown)^2\overline{E}=\frac{1}{c^2}\frac{\partial^2\overline{E}}{\partial t^2}$

where, the operator $(\bigtriangledown)^2$ is defined, using nonstandard partial derivatives, see [7]. Generalising Lemma 0.16 of the same paper, we obtain;

$E_i(\overline{x},t)=(\frac{1}{2\pi})^3\int_{\mathcal{R}^3_{\pi\eta}}\hat{E}_i(\overline{k},t)exp_{\pi\eta}(i\overline{k}\centerdot\overline{x})(d\lambda^3_{\pi\eta})(\overline{k})$

$(\bigtriangledown)^2E_i-\frac{1}{c^2}\frac{\partial^2E_i}{\partial t^2}=(\frac{1}{2\pi})^3\int_{\mathcal{R}^3_{\pi\eta}}((T_\eta(\overline{k}))^2\hat{E}_i-\frac{1}{c^2}\frac{\partial^2\hat{E}_i}{\partial t^2})(\overline{k},t)e^{i\overline{k}\centerdot\overline{x}}d\overline{k}=0$

where $T_\eta(\overline{k})=(\sum_{j=1}^3\frac{\eta^2}{\pi^2}(exp_{\pi\eta}(\frac{i\pi k_j}{\eta})-1)^2)^{\frac{1}{2}}$.

Using Lemma 0,16 again;

$T_\eta(\overline{k})^2\hat{E}_i-\frac{1}{c^2}\frac{\partial^2\hat{E}_i}{\partial t^2}=0.$

$\frac{\partial^2\hat{E}_i}{\partial t^2}=(cT_\eta(\overline{k}))^2\hat{E}_i$

By Lemma 0.20 of [8], we can find $\{f_1,f_2\}$, with;

$\hat{E}_i(\overline{k},t)=A_i(\overline{k})f_1(\overline{k},t)+B_i(\overline{k})f_2(\overline{k},t)$

We have that, for finite $t\in\mathcal{R}_\eta,\overline{k}\in\mathcal{R}^3_\eta$;

$f_1(\overline{k},t)\simeq exp_\eta(icT_\eta(\overline{k})t)$

${}^\circ f_1(\overline{k},t)=exp(ic|{}^\circ\overline{k}|{}^\circ t)$

$f_2(\overline{k},t)\simeq exp_\eta(-icT_\eta(\overline{k})t)$

${}^\circ f_2(\overline{k},t)=exp(-ic|{}^\circ\overline{k}|{}^\circ t)$
$E_i(\overline{x},t)=(\frac{1}{2\pi})^3(\int_{\mathcal{R}^3_\eta}A_i(\overline{k})f_1(\overline{k},t)exp_\eta(i\overline{k}\centerdot\overline{x})(d\lambda^3_\eta)(\overline{k})$

□

$+\int_{\mathcal{R}^3_\eta} B_i(\overline{k})f_2(\overline{k},t)exp_\eta(i\overline{k}\centerdot\overline{x})(d\lambda^3_\eta)(\overline{k}))$

$\overline{E}(\overline{x},t)=(\int_{\mathcal{R}^3_\eta}\overline{A}(\overline{k})f_1(\overline{k},t)exp_\eta(i\overline{k}\centerdot\overline{x})(d\lambda^3_\eta)(\overline{k})$

$+\int_{\mathcal{R}^3_\eta}\overline{B}(\overline{k})f_2(\overline{k},t)exp_\eta(i\overline{k}\centerdot\overline{x})(d\lambda^3_\eta)(\overline{k}))$

Then, if $\overline{A}$ is S-integrable on $\mathcal{R}^3_\eta$, then, for $\overline{x}\in\mathcal{R}^3_\eta$, so is $\overline{A}(\overline{k})f_1(\overline{k},t)exp_\eta(i\overline{k}\centerdot\overline{x})$, with ${}^\circ\overline{A}(\overline{k})f_1(\overline{k},t)exp_\eta(i\overline{k}\centerdot\overline{x})=exp(i({}^\circ\overline{k}\centerdot{}^\circ\overline{x}-c|{}^\circ\overline{k}|{}^\circ t)$

In the same way;

$\overline{B}(\overline{x},t)=(\int_{\mathcal{R}^3_\eta}\overline{K}(\overline{k})f_1(\overline{k},t)exp_\eta(i\overline{k}\centerdot\overline{x})(d\lambda^3_\eta)(\overline{k})$

$+\int_{\mathcal{R}^3_\eta}\overline{L}(\overline{k})f_2(\overline{k},t)exp_\eta(i\overline{k}\centerdot\overline{x})(d\lambda^3_\eta)(\overline{k}))$

$\frac{\partial\overline{B}}{\partial t'}(\overline{x},t)=(\int_{\mathcal{R}^3_\eta}\overline{K}(\overline{k})\frac{\partial f_1}{\partial t'}(\overline{k},t)exp_\eta(i\overline{k}\centerdot\overline{x})(d\lambda^3_\eta)(\overline{k})$

$+\int_{\mathcal{R}^3_\eta}\overline{L}(\overline{k})\frac{\partial f_2}{\partial t'}(\overline{k},t)exp_\eta(i\overline{k}\centerdot\overline{x})(d\lambda^3_\eta)(\overline{k}))$ ($\sharp$)

We let $\overline{k}_\eta=\eta(exp_\eta(\frac{ik_1}{\eta})-1,exp_\eta(\frac{ik_2}{\eta})-1,exp_\eta(\frac{ik_3}{\eta})-1)$, $S_{\overline{k}_\eta}=(S^2(\overline{k}_\eta,1)\cap P_{\overline{k}_\eta})$, where $P_{\overline{k}_\eta}=\{\overline{n}:(\overline{n}-\overline{k}_\eta)\centerdot\overline{k}_\eta=0\}$. Then, again by $(i)'',(ii)''$;

$\overline{A}(\overline{k})=\int_{S_{\overline{k}_\eta}}G(\overline{k},\overline{n})d\overline{S_{\overline{k}_\eta}}$

$\overline{B}(\overline{k})=\int_{S_{\overline{k}_\eta}}H(\overline{k},\overline{n})d\overline{S_{\overline{k}_\eta}}$

$\overline{K}(\overline{k})=\int_{S_{\overline{k}_\eta}}K(\overline{k},\overline{n})d\overline{S_{\overline{k}_\eta}}$

$\overline{L}(\overline{k})=\int_{S_{\overline{k}_\eta}}L(\overline{k},\overline{n})d\overline{S_{\overline{k}_\eta}}$

where we adopt the convention that $d\overline{S_{\overline{k}_\eta}}(\overline{n})=(\overline{n}-\overline{k}_\eta)d\nu_{S_{\overline{k}_\eta}}$, where $\nu_{S_{\overline{k}_\eta}}=j^*_{\overline{k}_\eta}\nu_\eta$, for the internal map $j:S_{\overline{k}_\eta}\rightarrow S^1$ given by $j(\overline{n})=((\overline{n}-\overline{k}_\eta)\centerdot\overline{n}_1(\overline{k}_\eta),(\overline{n}-\overline{k}_\eta)\centerdot\overline{n}_2(\overline{k}_\eta))$, where $\{\overline{n}_1,\overline{n}_2\}$ are chosen so that $det(\overline{u},\overline{n}_1(\overline{u}),\overline{n}_2(\overline{u}))\neq 0$, for $\overline{u}\in{}^*\mathcal{R}^3$, hence, for $\overline{u}\in\mathcal{R}^3_\eta$. Hence;

$\overline{E}(\overline{x},t)=(\int_{\mathcal{R}^3_\eta}\int_{S_{\overline{k}_\eta}}G(\overline{k},\overline{n})f_1(\overline{k},t)exp_\eta(i\overline{k}\centerdot\overline{x})d\overline{S_{\overline{k}_\eta}}(d\lambda^3_\eta)(\overline{k})$

$+\int_{\mathcal{R}^3_\eta}\int_{S_{\overline{k}_\eta}}H(\overline{k},\overline{n})f_2(\overline{k},t)exp_\eta(i\overline{k}\centerdot\overline{x})d\overline{S_{\overline{k}_\eta}}(d\lambda^3_\eta)(\overline{k}))$

$\overline{B}(\overline{x},t)=(\int_{\mathcal{R}^3_\eta}\int_{S_{\overline{k}_\eta}}K(\overline{k},\overline{n})f_1(\overline{k},t)exp_\eta(i\overline{k}\centerdot\overline{x})d\overline{S_{\overline{k}_\eta}}(d\lambda^3_\eta)(\overline{k})$

$+\int_{\mathcal{R}^3_\eta}\int_{S_{\overline{k}_\eta}}L(\overline{k},\overline{n})f_2(\overline{k},t)exp_\eta(i\overline{k}\centerdot\overline{x})d\overline{S_{\overline{k}_\eta}}(d\lambda^3_\eta)(\overline{k}))$

$\frac{\partial\overline{B}}{\partial t'}(\overline{x},t)=(\int_{\mathcal{R}^3_\eta}\int_{S_{\overline{k}_\eta}}K(\overline{k},\overline{n})\frac{\partial f_1}{\partial t'}(\overline{k},t)exp_\eta(i\overline{k}\centerdot\overline{x})d\overline{S_{\overline{k}_\eta}}(d\lambda^3_\eta)(\overline{k})$

$+\int_{\mathcal{R}^3_\eta}\int_{S_{\overline{k}_\eta}}L(\overline{k},\overline{n})\frac{\partial f_1}{\partial t'}(\overline{k},t)exp_\eta(i\overline{k}\centerdot\overline{x})d\overline{S_{\overline{k}_\eta}}(d\lambda^3_\eta)(\overline{k}))$, ($\sharp\sharp$)

Using $(iii)'$;

$\frac{\partial \overline{B}}{\partial t}(\overline{x},t)$

$= i\int_{\overline{k}\in\mathcal{R}^3_\eta}\int_{S_{\overline{k}_\eta}}[G(\overline{k},\overline{n})f_1(\overline{k},t)exp_\eta(i(\overline{k}\centerdot\overline{x}))\times\overline{k}_\eta]d\overline{S_{\overline{k}_\eta}}(\overline{n})(d\lambda^3_\eta)(\overline{k})$

$+i\int_{\overline{k}\in\mathcal{R}^3_\eta}\int_{S_{\overline{k}_\eta}}[H(\overline{k},\overline{n})f_2(\overline{k},t)exp_\eta(i(\overline{k}\centerdot\overline{x}))\times\overline{k}_\eta]d\overline{S_{\overline{k}_\eta}}(\overline{n})(d\lambda^3_\eta)(\overline{k})$

$= \int_{\overline{k}\in\mathcal{R}^3_\eta}\int_{S_{\overline{k}_\eta}} iG(\overline{k},\overline{n})((\overline{n}-\overline{k}_\eta)\hat{}\times\overline{k}_\eta)f_1(\overline{k},t)exp_\eta(i(\overline{k}\centerdot\overline{x}))dS_{\overline{k}}(\overline{n})d\overline{k}$

$+\int_{\overline{k}\in\mathcal{R}^3_\eta}\int_{S_{\overline{k}}} iH(\overline{k},\overline{n})((\overline{n}-\overline{k}_\eta)\hat{}\times\overline{k}_\eta)f_2(\overline{k},t)exp_\eta(i(\overline{k}\centerdot\overline{x}))dS_{\overline{k}}(\overline{n})d\overline{k}$

$= \int_{\overline{k}\in\mathcal{R}^3_\eta}\int_{S_{\overline{k}}} iG(\overline{k},\overline{n})\frac{(\overline{n}\times\overline{k}_\eta)}{|\overline{n}-\overline{k}_\eta|}f_1(\overline{k},t)exp_\eta(i(\overline{k}\centerdot\overline{x}))dS_{\overline{k}}(\overline{n})d\overline{k}$

$+\int_{\overline{k}\in\mathcal{R}^3_\eta}\int_{S_{\overline{k}}} iH(\overline{k},\overline{n})\frac{(\overline{n}\times\overline{k}_\eta)}{|\overline{n}-\overline{k}_\eta|}f_2(\overline{k},t)exp_\eta(i\overline{k}\centerdot\overline{x})dS_{\overline{k}}(\overline{n})]d\overline{k}$ $(\sharp\sharp\sharp)$

Then;

$\overline{B}(\overline{x},t)$

$= \int_{\overline{k}\in\mathcal{R}^3}[\int_{S_{\overline{k}}}\frac{-G(\overline{k},\overline{n})}{\omega(\overline{k})}\frac{(\overline{n}\times\overline{k})}{|\overline{n}-\overline{k}|}e^{i(\overline{k}\centerdot\overline{x}-\omega(\overline{k})t)}dS_{\overline{k}}(\overline{n})]d\overline{k}$

$+\int_{\overline{k}\in\mathcal{R}^3}[\int_{S_{\overline{k}}}\frac{H(\overline{k},\overline{n})}{\omega(\overline{k})}\frac{(\overline{n}\times\overline{k})}{|\overline{n}-\overline{k}|}e^{i(\overline{k}\centerdot\overline{x}+\omega(\overline{k})t)}dS_{\overline{k}}(\overline{n})]d\overline{k}+\overline{\theta}(t)$

$= \int_{\overline{k}\in\mathcal{R}^3}[\int_{S_{\overline{k}}}\overline{M}(\overline{k},\overline{n})e^{i(\overline{k}\centerdot\overline{x}-\omega(\overline{k})t)}dS_{\overline{k}}(\overline{n})]d\overline{k}$

$+\int_{\overline{k}\in\mathcal{R}^3}[\int_{S_{\overline{k}}}\overline{N}(\overline{k},\overline{n})e^{i(\overline{k}\centerdot\overline{x}+\omega(\overline{k})t)}dS_{\overline{k}}(\overline{n})]d\overline{k}$ $(**)$

Determine general solution to nonstandard version;

$(i)'''$. $\bigtriangledown\centerdot\overline{E}=\frac{\rho''}{\epsilon_0}$

$(ii)'''$. $\bigtriangledown\centerdot\overline{B}=0$

$(iii)'''$. $(\bigtriangledown\times\overline{E})=-\frac{\partial\overline{B}}{\partial t}$

$(iv)'''$. $(\bigtriangledown\times\overline{B})=\mu_0(\overline{J''}+\epsilon_0\frac{\partial\overline{E}}{\partial t})$

for a surface charge ρ'' and current $\overline{J''}$, extended, as in remark 2.9, that is given $\{\rho',\overline{J'}\}$ as in remark 2.9, we let $\{\rho'',\overline{J''}\}$ be the measurable counterparts of the transfers $\{(\rho')^*,(\overline{J'})^*\}$.

Then, by $(i)'''$, $(iii)'''$, $(iv)'''$;

$(\bigtriangledown\times\overline{E})=-\frac{\partial\overline{B}}{\partial t}$

$\bigtriangledown\times(\bigtriangledown\times\overline{E})=-\bigtriangledown\times(\frac{\partial\overline{B}}{\partial t})$

$grad(div(\overline{E})) - \bigtriangledown^2(\overline{E}) = -\mu_0 \frac{\partial(\overline{J''}+\epsilon_0 \frac{\partial \overline{E}}{\partial t})}{\partial t}$

$grad(\frac{\rho''}{\epsilon_0}) - \bigtriangledown^2(\overline{E}) = -\mu_0 \frac{\partial \overline{J''}}{\partial t} - \epsilon_0 \mu_0 \frac{\partial^2 \overline{E}}{\partial^2 t}$

$\bigtriangledown^2(\overline{E}) - \frac{1}{c^2}\frac{\partial^2 \overline{E}}{\partial^2 t} = grad(\frac{\rho''}{\epsilon_0}) + \mu_0 \frac{\partial \overline{J''}}{\partial t}$, $(\sharp\sharp\sharp\sharp)$

and, by $(ii)'''$, $(iii)'''$, $(iv)'''$;

$(\bigtriangledown \times \overline{B}) = \mu_0(\overline{J''} + \epsilon_0 \frac{\partial \overline{E}}{\partial t})$

$\bigtriangledown \times (\bigtriangledown \times \overline{B}) = \mu_0(\bigtriangledown \times \overline{J''}) - \mu_0 \epsilon_0 \frac{\partial^2 \overline{B}}{\partial^2 t}$

$- \bigtriangledown^2 (\overline{B}) = \mu_0(\bigtriangledown \times \overline{J''}) - \frac{1}{c^2}\frac{\partial^2 \overline{B}}{\partial^2 t}$

$\bigtriangledown^2(\overline{B}) - \frac{1}{c^2}\frac{\partial^2 \overline{B}}{\partial^2 t} = -\mu_0(\bigtriangledown \times \overline{J''})$, $(\sharp\sharp\sharp\sharp\sharp)$

Let $\overline{\delta} = -\mu_0(\bigtriangledown \times \overline{J''})$, and let $\overline{\epsilon}'$ be defined by;

$\overline{\epsilon}'|_{(\overline{\mathcal{R}_{\pi\eta}} \setminus \{-\pi\eta, -\pi\eta+\frac{\pi}{\eta}, \pi\eta-\frac{\pi}{\eta}\})^3} = \overline{0}$

$\overline{\epsilon}'|_{(\overline{\mathcal{R}_{\pi\eta}})^3 \setminus (\overline{\mathcal{R}_{\pi\eta}} \setminus \{-\pi\eta, -\pi\eta+\frac{\pi}{\eta}, \pi\eta-\frac{\pi}{\eta}\})^3} = \overline{B}|_{(\overline{\mathcal{R}_{\pi\eta}})^3 \setminus (\overline{\mathcal{R}_{\pi\eta}} \setminus \{-\eta, -\eta+\frac{1}{\eta}, \eta-\frac{1}{\eta}\})^3}$

and, let $\overline{B}' = \overline{B} - \overline{\epsilon}'$, $\overline{\epsilon} = -(\bigtriangledown^2(\overline{\epsilon}') - \frac{1}{c^2}\frac{\partial^2 \overline{\epsilon}'}{\partial^2 t})$, then, we obtain, from $(\sharp\sharp\sharp\sharp\sharp)$;

$\bigtriangledown^2(\overline{B}') - \frac{1}{c^2}\frac{\partial^2 \overline{B}'}{\partial^2 t} = \overline{\delta} + \overline{\epsilon}$

Then, using Lemma 0.21 of [7],([4]);

$\frac{\partial^2 \hat{B}_i'}{\partial t^2} = (cT_{\pi\eta}(\overline{k}))^2 \hat{B}_i' - c^2(\hat{\delta}_i + \hat{\epsilon}_i)$

.... Compute $(\hat{\delta}_i + \hat{\epsilon}_i)$, with respect to $\hat{\delta}_i(\overline{t}) = \int_{\overline{\mathcal{R}_{\pi\eta}}} \delta_i(\overline{x}) exp_{\pi\eta}(-i\overline{x} \centerdot \overline{t})(d_{\lambda_{\pi\eta}})^3(\overline{x})$ and use footnote 6, Lemma 0.20 , from [8]. Assume $\overline{B}$ is in Schwartz class (nonstandard analogue) to get $\hat{\epsilon_i} \simeq 0$, in fact $\leq \frac{\pi^3}{\eta^3} 12\eta^2 (\frac{12C_{0,m}}{(\pi\eta-\frac{3\pi}{\eta})^{2m}} + \frac{3C_{0,m}}{c^2(\pi\eta-\frac{3\pi}{\eta})^{2m}})$, for $m \in \mathcal{Z}_{\geq 1}$. Assume $\overline{J}''$ is uniformly bounded in time, and approximate $\overline{J}''$ by $\overline{J}_\gamma(\overline{x}, t) = \overline{J}''(\overline{r}(\overline{x}), t)F(\overline{x})$, where $F(\overline{x}) = exp_{\pi\eta}(\frac{-|\overline{x}-\overline{r}(\overline{x})|^2}{\gamma})$, and $\overline{r} : (\mathcal{R}_{\pi\eta})^3 \rightarrow {}^*S$, for simple surfaces meeting each ray from O in a unique point. Then $\overline{J}_\gamma$ is in the Schwartz class (ns analogue), check so are ns derivatives, then $g_{0,1}$ corresponding to $-c^2 \hat{\delta_{i,\gamma}}$ is finite, not infinitesimal, $g_{0,2}$ corresponding to $-c^2\hat{\epsilon}_i$ is infinitesimal, $g_0 = g_{0,1} + g_{0,2}$. In footnote 6, can invert $\sum \frac{\pi}{\eta} g_{0,2}$ terms, without changing Schwartz condition, and $\sum \frac{\pi}{\eta} g_{0,1}$ terms are finite, for time step $t \in \mathcal{R}_{fin}$, and in ns Schwartz class. Check that inversion then gives ns Schwartz solution. Obtain usual wave equation solution$+^\vee({}^*\sum_{s=1}^{j} -c^2 Q^1_{j-s}((cT_{\pi\eta}(\overline{k}))^2) \frac{\pi}{\eta} \hat{\delta_{i,\gamma}}(\overline{k}, s-1))$, obtain a double integral of wave packets over time and space. verify what happens as $\gamma \rightarrow 0$ or ${}^\circ\gamma = 0$. ...

Lemma 2.4. *Using the notation above, the smoothly decaying real solutions of Maxwell's equations in free space, are given by;*

$\overline{E}(\overline{x},t)$

$= \int_{\overline{k}\in\mathcal{R}^3}\int_{S_{\overline{k}}} Re(G)(\overline{k},\overline{n})cos(\overline{k}\centerdot\overline{x}-ckt)d\overline{S}_{\overline{k}}(\overline{n})d\overline{k}$

$-\int_{\overline{k}\in\mathcal{R}^3}\int_{S_{\overline{k}}} Im(G)(\overline{k},\overline{n})sin(\overline{k}\centerdot\overline{x}-ckt))d\overline{S}_{\overline{k}}(\overline{n})d\overline{k}$

$+\int_{\overline{k}\in\mathcal{R}^3}\int_{S_{\overline{k}}} Re(H)(\overline{k},\overline{n})cos(\overline{k}\centerdot\overline{x}+ckt)d\overline{S}_{\overline{k}}(\overline{n})d\overline{k}$

$-\int_{\overline{k}\in\mathcal{R}^3}\int_{S_{\overline{k}}} Im(H)(\overline{k},\overline{n})sin(\overline{k}\centerdot\overline{x}+ckt)d\overline{S}_{\overline{k}}(\overline{n})d\overline{k}$

$(*)$

and;

$\overline{B}(\overline{x},t)$

$= \int_{\overline{k}\in\mathcal{R}^3}[\int_{S_{\overline{k}}} Re(\overline{M})(\overline{k},\overline{n})cos(\overline{k}\centerdot\overline{x}-ckt)dS_{\overline{k}}(\overline{n})]d\overline{k}$

$-\int_{\overline{k}\in\mathcal{R}^3}[\int_{S_{\overline{k}}} Im(\overline{M})(\overline{k},\overline{n})sin(\overline{k}\centerdot\overline{x}-ckt)dS_{\overline{k}}(\overline{n})]d\overline{k}$

$+\int_{\overline{k}\in\mathcal{R}^3}[\int_{S_{\overline{k}}} Re(\overline{N})(\overline{k},\overline{n})cos(\overline{k}\centerdot\overline{x}+ckt)dS_{\overline{k}}(\overline{n})]d\overline{k}$

$-\int_{\overline{k}\in\mathcal{R}^3}[\int_{S_{\overline{k}}} Im(\overline{N})(\overline{k},\overline{n})sin(\overline{k}\centerdot\overline{x}+ckt)dS_{\overline{k}}(\overline{n})]d\overline{k}$

$(**)$

Proof. The proof follows by taking real and imaginary parts from the previous lemma.

□

Lemma 2.5. *For the real smooth solutions of Maxwell's equations in free space, given, for fixed $\overline{k}_0$ by;*

$\overline{E}(\overline{x},t)$

$= \int_{-\pi}^{\pi} f(\theta)n_{\theta,\hat{\overline{k}}_0}cos(\overline{k}_0\centerdot\overline{x}-ck_0t)d\theta$

$-\int_{-\pi}^{\pi} g(\theta)n_{\theta,\hat{\overline{k}}_0}sin(\overline{k}_0\centerdot\overline{x}-ck_0t)d\theta$

$+\int_{-\pi}^{\pi} h(\theta) n_{\theta}, \hat{\overline{k}_0} cos(\overline{k}_0 \cdot \overline{x} + ck_0 t) d\theta$

$-\int_{-\pi}^{\pi} r(\theta) n_{\theta}, \hat{\overline{k}_0} sin(\overline{k}_0 \cdot \overline{x} + ck_0 t) d\theta$

where $n_{\theta}, \hat{\overline{k}_0} = M_{\overline{k}_0}(0, cos(\theta), sin(\theta))$, *for a real special orthogonal matrix with* $M_{\overline{k}_0}(1,0,0) = \hat{\overline{k}_0}$, $\{f, g\} \subset C^{\infty}(S^1, \mathcal{R})$,

and;

$\overline{B}(\overline{x}, t)$

$= \frac{1}{c} \int_{-\pi}^{\pi} f(\theta) n_{\theta} \hat{+ \frac{\pi}{2}}, \overline{k}_0 cos(\overline{k}_0 \cdot \overline{x} - ck_0 t) d\theta$

$-\frac{1}{c} \int_{-\pi}^{\pi} g(\theta) n_{\theta} \hat{+ \frac{\pi}{2}}, \overline{k}_0 sin(\overline{k}_0 \cdot \overline{x} - ck_0 t) d\theta$

$+\frac{1}{c} \int_{-\pi}^{\pi} h(\theta) n_{\theta - \hat{\frac{\pi}{2}}, \overline{k}_0} cos(\overline{k}_0 \cdot \overline{x} + ck_0 t) d\theta$

$-\frac{1}{c} \int_{-\pi}^{\pi} r(\theta) n_{\theta - \hat{\frac{\pi}{2}}}, \overline{k}_0 sin(\overline{k}_0 \cdot \overline{x} + ck_0 t) d\theta$

Then for $t \in \mathcal{R}_{\geq 0}$, *the Poynting vector* $\overline{P} = \overline{E} \times \overline{B} = \overline{0}$ *almost everywhere iff* $\overline{E} = \overline{B} = \overline{0}$

Proof. Letting $\overline{E} = \overline{E}_1 - \overline{E}_2 + \overline{E}_3 - \overline{E}_4$ and $\overline{B} = \overline{B}_1 - \overline{B}_2 + \overline{B}_3 - \overline{B}_4$, with the ordering above we have that;

$\overline{E} \times \overline{B} = \sum_{i,j=1}^{4} (-1)^{i+j} \overline{E}_i \times \overline{B}_j$

We have;

$\overline{E}_1 \times \overline{B}_1(\overline{x}, t)$

$= (\int_{-\pi}^{\pi} f(\theta) \hat{n_{\theta}}, \overline{k}_0 cos(\overline{k}_0 \cdot \overline{x} - ck_0 t) d\theta) \times \frac{1}{c} \int_{-\pi}^{\pi} f(\theta) n_{\theta \hat{+} \frac{\pi}{2}}, \overline{k}_0 cos(\overline{k}_0 \cdot \overline{x} - ck_0 t) d\theta$

$= \frac{1}{c} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} f(\theta) f(\phi) cos^2(\overline{k}_0 \cdot \overline{x} - ck_0 t)(\hat{n_{\theta}}, \overline{k}_0 \times n_{\theta} \hat{+} \frac{\pi}{2}) d\theta d\phi$

$= \frac{1}{c} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} f(\theta) f(\phi) cos^2(\overline{k}_0 \cdot \overline{x} - ck_0 t)(\hat{n_{\theta}}, \overline{k}_0 \times \hat{\overline{k}_0} d\theta d\phi$

Suppose that $\overline{E}_1 \times \overline{B}_1 = \overline{0}$ then, for fixed $t_0 \in \mathcal{R}$, as there exists $\overline{x}_0$ with $cos^2(\overline{k}_0 \centerdot \overline{x_0} - ck_0t_0) \neq 0$, that;

$\int_{-\pi}^{\pi} \int_{-\pi}^{\pi} f(\theta)f(\phi)d\theta d\phi = 0$

so that;

$\int_{-\pi}^{\pi} f(\theta)d\theta = 0$

As $f \in C^\infty(S^1)$, writing;

$f(\theta) = \sum_{m \geq 0} a_m cos(m\theta) + \sum_{m \geq 1} b_m sin(m\theta)$

we have that $a_0 = 0$, so;

$f(\theta) = \sum_{m \geq 1} a_m cos(m\theta) + \sum_{m \geq 1} b_m sin(m\theta)$

Then;

$\overline{E}_1(\overline{x}, t) = \int_{-\pi}^{\pi} f(\theta)\hat{n_\theta}, \overline{k}_0 cos(\overline{k}_0 \centerdot \overline{x} - ck_0t)d\theta$

$= \int_{-\pi}^{\pi} f(\theta)(M_{\overline{k}_0}(0, cos(\theta), sin(\theta)))cos(\overline{k}_0 \centerdot \overline{x} - ck_0t)d\theta$

$= cos(\overline{k}_0 \centerdot \overline{x} - ck_0t)M_{\overline{k}_0}(\int_{-\pi}^{\pi} f(\theta)(0, cos(\theta), sin(\theta)))d\theta)$, $(\dagger)$

We have that;

$\int_{-\pi}^{\pi} f(\theta)cos(\theta)d\theta$

$= \int_{-\pi}^{\pi}(\sum_{m \geq 1} a_m cos(m\theta) + \sum_{m \geq 1} b_m sin(m\theta))cos(\theta)d\theta$

$= \sum_{m \geq 1} a_m \int_{-\pi}^{\pi}(cos(\theta)cos(m\theta)d\theta$

$+ \sum_{m \geq 1} b_m cos(\theta)sin(m\theta)d\theta$

$= a_1 \int_{-\pi}^{\pi}(cos^2(\theta)d\theta$

$= \pi a_1$

and;

$\int_{-\pi}^{\pi} f(\theta)sin(\theta)d\theta$

$= \int_{-\pi}^{\pi}(\sum_{m\geq 1} a_m cos(m\theta) + \sum_{m\geq 1} b_m sin(m\theta))sin(\theta)d\theta$

$= \sum_{m\geq 1} a_m \int_{-\pi}^{\pi} sin(\theta)cos(m\theta)d\theta$

$+ \sum_{m\geq 1} b_m sin(\theta)sin(m\theta)d\theta$

$= b_1 \int_{-\pi}^{\pi}(sin^2(\theta)d\theta$

$= \pi b_1$

It follows that;

$\overline{E}_1(\overline{x}, t)$

$= cos(\overline{k}_0 \centerdot \overline{x} - ck_0 t)M_{\overline{k}_0}(0, \pi a_1, \pi b_1)$, $(\dagger\dagger)$

Similarly;

$\overline{B}_1(\overline{x}, t)$

$= \frac{1}{c}\int_{-\pi}^{\pi} f(\theta)n_\theta \hat{+} \frac{\pi}{2}, \overline{k}_0 cos(\overline{k}_0 \centerdot \overline{x} - ck_0 t)d\theta$

$= \frac{1}{c}\int_{-\pi}^{\pi} f(\theta)(M_{\overline{k}_0}(0, cos(\theta + \frac{\pi}{2}), sin(\theta + \frac{\pi}{2})))cos(\overline{k}_0 \centerdot \overline{x} - ck_0 t)d\theta$

$= \frac{1}{c}cos(\overline{k}_0 \centerdot \overline{x} - ck_0 t)M_{\overline{k}_0}(\int_{-\pi}^{\pi} f(\theta)(0, cos(\theta + \frac{\pi}{2}), sin(\theta) + \frac{\pi}{2}))d\theta$, $(\dagger\dagger)$

$= \frac{1}{c}cos(\overline{k}_0 \centerdot \overline{x} - ck_0 t)M_{\overline{k}_0}(\int_{-\pi}^{\pi} f(\theta)(0, -sin(\theta), cos(\theta)))d\theta$

$= \frac{1}{c}cos(\overline{k}_0 \centerdot \overline{x} - ck_0 t)M_{\overline{k}_0}(0, -\pi b_1, \pi a_1)$

$(\dagger\dagger\dagger)$

As this holds for arbitrary $t_0 \in \mathcal{R}$, we have;

$\overline{E}_1 \times \overline{B}_1(\overline{x}, \overline{t})$

$= cos(\overline{k}_0 \centerdot \overline{x} - ck_0 t)M_{\overline{k}_0}(0, \pi a_1, \pi b_1) \times \frac{1}{c}cos(\overline{k}_0 \centerdot \overline{x} - ck_0 t)M_{\overline{k}_0}(0, -\pi b_1, \pi a_1)$

$= cos^2(\overline{k}_0 \centerdot \overline{x} - ck_0 t)M_{\overline{k}_0}((0, \pi a_1, \pi b_1) \times (0, -\pi b_1, \pi a_1))$

$= cos^2(\overline{k}_0 \centerdot \overline{x} - ck_0 t)M_{\overline{k}_0}(\pi^2(a_1^2 + b_1^2), 0, 0)$

so that $a_1 = b_1 = 0$, and $\overline{E}_1 = \overline{B}_1 = \overline{0}$

□

Lemma 2.6. *Maxwell's Equations in free space with a constant Potential*

Let $\overline{E_0}$ be the static electric field generated by a single proton situated at the origin $O \in \mathcal{R}^3$, with potential V_0. Then $\overline{E} + \overline{E_0}$ is a solution to Maxwell's equations in free space, for this situation, iff $\overline{E}$ is a solution to Maxwell's equations in vacuum.

Proof. We have that the electric field $\overline{E_0} : \mathcal{R}^3 \rightarrow \mathcal{R}^3$ is given by;

$\overline{E_0}(\overline{r}) = \frac{-q_p}{4\pi\epsilon_0|\overline{r}|^2}\hat{r}$

A simple calculation gives that $\bigtriangledown \centerdot \overline{E_0} = 0$, and $\bigtriangledown \times E_0 = 0$, and as $\overline{E_0}$ is time invariant, that $\frac{\partial E_0}{\partial t} = 0$. Maxwell's equations, in the absence of current and charge, then become;

(i). $\bigtriangledown \centerdot (\overline{E} + \overline{E_0}) = \bigtriangledown \centerdot \overline{E} = 0$

(ii). $\bigtriangledown \centerdot \overline{B} = 0$

(iii). $\bigtriangledown \times (\overline{E} + \overline{E_0}) = \bigtriangledown \times \overline{E} = -\frac{\partial B}{\partial t}$

(iv). $\bigtriangledown \times \overline{B} = \mu_0\epsilon_0 \frac{\partial(\overline{E} + \overline{E}_0)}{\partial t} = \mu_0\epsilon_0 \frac{\partial \overline{E}}{\partial t}$

as required. □

Lemma 2.7. *Alfven's Theorem*

The magnetic flux through a loop $\mathcal{P}$, moving with a perfectly conducting fluid, for example, a gas of free electrons, is constant in time.

Proof. By a loop $\mathcal{P}$, moving with a fluid, we mean that there exists a map $\psi : D^1 \times [0,1] \rightarrow \mathcal{R}^3$, with $\psi \in C^\infty(D^1 \times [0,1])$, (in the sense that $\psi \in C^\infty(D^{1,int} \times (0,1)) \cap C(D^1 \times [0,1])$), and, for all $t \in [0,1]$, and $\theta \in \partial D^1$, $\frac{\partial \psi}{\partial \theta}|_{(\theta,t)} \neq 0$, and $\frac{\partial \psi}{\partial t}|_{(x_0,t_0)} = \overline{v}|_{(\psi(x_0,t_0),t_0)}$. We let $\mathcal{P}_t = Im(\psi|_{\partial D^1 \times \{t\}})$, $\mathcal{Q}_t = Im(\psi|_{D^1 \times \{t\}})$ and, for $h > 0$, $\mathcal{R}_{t,t+h} = Im(\psi|_{\partial D^1 \times [t,t+h]})$ and $\mathcal{F}_{t,t+h} = Im(\psi|_{D^1 \times [t,t+h]})$. By Ohm's Law, we have that $\overline{J} = \sigma(\overline{E} + \overline{v} \times \overline{B})$, hence, as $\sigma = \infty$, and $\overline{J}$ is finite, $\overline{E} = -(\overline{v} \times \overline{B})$. By Faraday's Law, Maxwell's Equation (iii), $\frac{\partial \overline{B}}{\partial t} = -(\bigtriangledown \times \overline{E}) = \bigtriangledown \times (\overline{v} \times \overline{B})$. Fix $h > 0$, and let $\Phi : \mathcal{R} \rightarrow \mathcal{R}$ denote the magnetic flux through the loop $\mathcal{P}_t$ at time t. Then;

$$\Phi(t+h) - \Phi(t) = \int_{\mathcal{Q}_{t+h}} \overline{B}(t+h).dS_{t+h} - \int_{\mathcal{Q}_t} \overline{B}(t).dS_t$$

and, by the divergence theorem, using the fact that $\partial \mathcal{F}_{t,t+h} = \mathcal{Q}_{t+h} \cup \mathcal{Q}_t \cup \mathcal{R}_{t,t+h}$, and Maxwell's equation (ii);

$$\int_{\mathcal{Q}_{t+h}} \overline{B}(t+h).dS_{t+h} - \int_{\mathcal{Q}_t} \overline{B}(t).dS_t + \int_{\mathcal{R}_{t,t+h}} \overline{B}(t+h).dS_{t,t+h}$$

$$= \int_{\mathcal{F}_{t,t+h}} div(\overline{B}) dvol_{t,t+h} = 0$$

$$\Phi(t+h) - \Phi(t) = \int_{\mathcal{Q}_t} \overline{B}(t+h).dS_t - \int_{\mathcal{R}_{t,t+h}} \overline{B}(t+h).dS_{t,t+h} - \int_{\mathcal{Q}_t} \overline{B}(t).dS_t$$

We have, for $t_0 \in [t, t+h]$, $\theta_0 \in S^1$, that $dS_{t,t+h}|_{(\theta_0,t_0)} = \frac{\partial \psi}{\partial \theta}|_{(\theta_0,t_0)} \times \frac{\partial \psi}{\partial t}_{(\theta_0,t_0)} = \frac{\partial \psi}{\partial \theta}|_{(\theta_0,t_0)} \times \overline{v}|_{(\psi(\theta_0,t_0),t_0)}$, and;

$$(\overline{B}(t+h).dS_{t,t+h})|_{(\psi(\theta_0,t_0),t_0)}$$

$$= (\overline{B}(t+h)|_{(\psi(\theta_0,t_0),t_0)}.(\frac{\partial \psi}{\partial \theta}|_{(\theta_0,t_0)} \times \overline{v}|_{(\psi(\theta_0,t_0),t_0)}))$$

$$= -(\overline{v} \times \overline{B}(t+h))|_{(\psi(\theta_0,t_0),t_0)}.\frac{\partial \psi}{\partial \theta}|_{(\theta_0,t_0)}$$

$$\int_{\mathcal{R}_{t,t+h}} \overline{B}(t+h).dS_{t,t+h}$$

$$= -\int_t^{t+h} \int_0^{2\pi} (\overline{v} \times \overline{B}(t+h))|_{(\psi(\theta_0,t_0),t_0)}.\frac{\partial \psi}{\partial \theta}|_{(\theta_0,t_0)} dt_0 d\theta_0$$

Hence;

$$\frac{d\Phi}{ds}|_t = lim_{h\rightarrow 0} \frac{\Phi(t+h)-\Phi(t)}{h}$$

$$= lim_{h\rightarrow 0}(\int_{\mathcal{Q}_t} \frac{(\overline{B}(t+h)-\overline{B}(t))}{h}.dS_t$$

$$-lim_{h\rightarrow 0} \int_{\mathcal{R}_{t,t+h}} \frac{\overline{B}(t+h)}{h}.dS_{t,t+h}$$

$$= \int_{\mathcal{Q}_t} \frac{\partial \overline{B}}{\partial t}.dS_t - \int_0^{2\pi} (\overline{v} \times \overline{B}(t))_{(\psi(\theta_0,t_0),t_0)}.\frac{\partial \psi}{\partial \theta}|_{(\theta_0,t_0)} d\theta_0$$

$$= \int_{\mathcal{Q}_t} \frac{\partial \overline{B}}{\partial t}.dS_t - \int_{\mathcal{P}_t} (\overline{v} \times \overline{B}(t)).d\bar{l}_t$$

Hence, by Stokes' Theorem;

$\frac{d\Phi}{ds}|_t = \int_{\mathcal{Q}_t}(\frac{\partial\overline{B}}{\partial t} - \bigtriangledown \times (\overline{v} \times \overline{B}(t))).d S_t = 0$

as required.

□

Lemma 2.8. *The charge density and volume current $\{\rho, \overline{J}\}$ in a solution to Maxwell's Equations, satisfy the continuity equation for fluids;*

$\frac{\partial\rho}{\partial t} + div(\overline{J}) = 0$

and, therefore, so do the fluid densities and currents $\{f_q\rho, f_q\overline{J}\}$.

Proof. We have, by $(i), (iv)$;

$\rho = \epsilon_0(\bigtriangledown \cdot \overline{E})$

$\frac{\partial\rho}{\partial t} = \epsilon_0(\bigtriangledown \cdot \frac{\partial\overline{E}}{\partial t})$

$= \epsilon_0(\bigtriangledown \cdot (\frac{1}{\mu_0\epsilon_0}(\bigtriangledown \times \overline{B} - \mu_0\overline{J})))$

$= \frac{1}{\mu_0}(\bigtriangledown \cdot (\bigtriangledown \times \overline{B}) - \mu_0\overline{J})$

$= - \bigtriangledown \cdot \overline{J} = -div(\overline{J})$

□

Remarks 2.9. *It is a straightforward exercise to show that for a steady* surface *current $\overline{J}$, with $\overline{B}$ determined by the Biot-Savart Law, (ii) and (iv) of Definition 2.1 hold, that is $\bigtriangledown \cdot \overline{B} = 0$ and $(\bigtriangledown \times \overline{B}) = \mu_0\overline{J'}$, where $\overline{J'}|_S = \overline{J}$, and $\overline{J'}|_{\mathcal{R}^3\setminus S} = \overline{0}$. Similarly, for a static surface charge ρ, with $\overline{B}$ determined by Coulomb's Law, (i) and (iii) of Definition 2.1 hold, that is $\bigtriangledown \cdot \overline{E} = \frac{\rho'}{\epsilon_0}$, $(\bigtriangledown \times \overline{E}) = \overline{0}$. Hence, one can use Maxwell's equations to determine $\{\overline{E}, \overline{B}\}$ in $\mathcal{R}^3$, for a static surface charge and steady surface current $\{\overline{J}, \rho\}$, from $\{\overline{J'}, \rho'\}$. It seems reasonable to generalise this result to electrodynamics for a time-dependent surface charge and current.*

3. Schrodinger's Equation

Definition 3.1. *We define a wave packet, centred at $O \in \mathcal{R}^3$ to be a smooth superposition of plane waves, with vector $\overline{k}$, and angular frequency $\omega = \frac{hk^2}{m}$, energy $E = h\omega = \frac{\rho^2}{2m} = \frac{h^2k^2}{m} = \frac{mv^2}{2}$, velocity*

$v=\frac{\omega}{k}=\nu\lambda$, $\overline{v}=\frac{\omega}{k^2}\overline{k}=\frac{h}{m}\overline{k}$, and momentum $\overline{p}=m\overline{v}=h\overline{k}$, see notation in 2.1, where $h=2\pi h'$ and h' denotes Planck's constant;

$$\Psi_f(\overline{x},t)=\int_{\mathcal{R}^3}f(\overline{k})e^{i(\overline{k}.\overline{x}-\omega t)}d\overline{k}=\int_{\mathcal{R}^3}f(\overline{k})e^{i(\overline{k}.\overline{x}-\frac{E(\overline{k})}{h}t)}d\overline{k}$$

where $f\in C^\infty(\mathcal{R}^3)$. Writing this as a superposition of plane waves, with momentum $\overline{p}$;

$$\Psi_f(\overline{x},t)=(\tfrac{m}{h})^3\int_{\mathcal{R}^3}f(\tfrac{m}{h}\overline{v})e^{\frac{i}{h}(m\overline{v}.\overline{x}-E(\frac{m}{h}\overline{v})t)}d\overline{v}$$

$$=(\tfrac{1}{h})^3\int_{\mathcal{R}^3}f(\tfrac{1}{h}\overline{p})e^{\frac{i}{h}(\overline{p}.\overline{x}-E(\frac{\overline{p}}{h})t)}d\overline{p}$$

$$=\int_{\mathcal{R}^3}F(\overline{p})e^{\frac{i}{h}(\overline{p}.\overline{x}-E'(\overline{p})t)}d\overline{p}$$

where $F(\overline{z})=f(\frac{\overline{z}}{h})$, and $E'(\overline{z})=E(\frac{\overline{z}}{h})=\frac{|\overline{z}|^2}{2m}$.

Remarks 3.2. *Observe that when $\omega=\frac{k}{\sqrt{\mu_0\epsilon_0}}$, a plane wave $\Psi_{\overline{k}}(\overline{x},t)=e^{i(\overline{k}.\overline{x}-\omega t)}$, with vector $\overline{k}$, determines a solution $\Psi_{\overline{k}}(\overline{x},t)\hat{n}$ to Maxwell's equations in free space. We then have that $v=\frac{\omega}{k}=c$, light speed, $\omega=ck$, and $m=\frac{hk^2}{\omega}=\frac{hk}{c}$, if the wave is associated to a particle. We have that $\overline{p}=h\overline{k}$, so the association to a solution of Maxwell's equation, places no restriction on $\overline{p}$, but alters m. However, for the definition of the energy term E of the wave packet, mass is assumed to be constant.*

Lemma 3.3. *Schrodinger's equation*

Using the notation above, we have, if $\Psi_f(\overline{x},t)$ defines a wave packet, then;

$$ih\frac{\partial\Psi_f}{\partial t}=-\frac{h^2}{2m}(\bigtriangledown\Psi_f)$$

Proof. We have, differentiating under the integral sign, and, using the fact that, for any $\overline{p}\in\mathcal{R}^3$, $F(\overline{p})e^{\frac{i}{h}(\overline{p}.\overline{x}-E'(\overline{p})t)}\in C^\infty(\mathcal{R}^3\times\mathcal{R})$;

$$\frac{\partial\Psi_f}{\partial t}=\int_{\mathcal{R}^3}\frac{\partial F(\overline{p})e^{\frac{i}{h}(\overline{p}.\overline{x}-E'(\overline{p})t)}}{\partial t}d\overline{p}$$

$$=\frac{1}{ih}\int_{\mathcal{R}^3}E'(\overline{p})F(\overline{p})e^{\frac{i}{h}(\overline{p}.\overline{x}-E'(\overline{p})t)}d\overline{p}$$

$= \frac{1}{2mih}\int_{\mathcal{R}^3}|\overline{p}|^2 F(\overline{\overline{p}})e^{\frac{i}{h}(\overline{p}.\overline{x}-E'(\overline{p})t)}d\overline{p}$

$\bigtriangledown(\Psi_f) = \int_{\mathcal{R}^3}\bigtriangledown(F(\overline{p})e^{\frac{i}{h}(\overline{p}.\overline{x}-E'(\overline{p})t)})d\overline{p}$

$= -\frac{1}{h^2}\int_{\mathcal{R}^3}|\overline{p}|^2 F(\overline{p})e^{\frac{i}{h}(\overline{p}.\overline{x}-E'(\overline{p})t)}d\overline{p}$

$= -\frac{1}{h^2}2mih\frac{\partial\Psi_f}{\partial t}$

$= -\frac{2mi}{h}\frac{\partial\Psi_f}{\partial t}$

as required. □

Definition 3.4. *For $F \in C^\infty(T(\mathcal{R}^3))$, we define a free space wave packet $\Psi_f \in C^\infty(\mathcal{R}^4)$;*

$\Psi_{free,F}(\overline{x},t) = \int_{(\overline{x}',\overline{p})\in T(\mathcal{R}^3)} F(\overline{x}',\overline{p})e^{\frac{i}{h}(\overline{p}.(\overline{x}-\overline{x}')-E'(\overline{p})t)}d\overline{x}'d\overline{p}$

For $F \in C^\infty((0,\pi)\times(-\pi,\pi)\times\mathcal{R}^2)$, we define a restricted space wave packet $\Psi^1_{res,F} \in C^\infty((0,\pi)\times(-\pi,\pi)\times\mathcal{R})$;

$\Psi^1_{res,F}(\overline{x},t) = \int_{(\overline{x}',\overline{p})\in(0,\pi)\times(-\pi,\pi)\times\mathcal{R}^2} F(\overline{x}',\overline{p})e^{\frac{i}{h}(\overline{p}.(\overline{x}-\overline{x}')-E'(\overline{p})t)}d\overline{x}'d\overline{p}$

$\Psi^1_{res,F'}(\overline{y},t) = \int_{(\overline{y}',\overline{p})\in TS^2}\frac{F'(\overline{y}',\overline{p})}{r^2 y_2'}\sqrt{r^2-(y_3')^2}e^{\frac{i}{h}(|\overline{p}|cos^{-1}(\overline{y}.\overline{y}')cos(\lambda(\overline{y},\overline{y}',\overline{p}))-E'(\overline{p})t)}dSd\overline{p}$

where $dLeb = \Phi^* dS$, $\Phi : (0,\pi)\times(-\pi,\pi)\to S^2(r)$;

$\Phi : (\theta,\phi) = (y_1,y_2,y_3) = (rcos(\theta)cos(\phi), rsin(\theta)cos(\phi), rsin(\phi))$: $0\leq\theta\leq\pi$, $-\pi\leq\phi\leq\pi$

$F' = (F\circ\Phi^{-1})$, and, for two great circle trajectories $\{\gamma_1,\gamma_2\} : [0,1]\to S^2(r)$ with $\gamma_1(0) = \gamma_2(0) = \overline{y}'$, $\gamma_1(1) = \overline{y}$, and $\gamma_2'(0)\frac{|\overline{p}|}{|\overline{r}|} = \overline{p}$, ([5]);

[5] A great circle trajectory γ_1, through $\{\overline{y}_0,\overline{y}_1\}\subset S^2(r)$ is defined by;

$\{((AR_\psi A^{-1})\overline{y}_0) : 0\leq\psi\leq 2\pi\}$

where

$$R_\psi = \begin{pmatrix} cos(\psi) & -sin(\psi) & 0\\ sin(\psi) & cos(\psi) & 0\\ 0 & 0 & 1\end{pmatrix}$$

and $A = BC^{-1}$;

$cos(\lambda(\overline{y},\overline{y}',\overline{p})) = \frac{\overline{p}.(\frac{|\overline{y}'|^2\overline{y}}{\overline{y}'.\overline{y}}-\overline{y}')}{|\overline{p}||(\frac{|\overline{y}'|^2\overline{y}}{\overline{y}'.\overline{y}}-\overline{y}')|} = \frac{((\gamma_1)'(0).(\gamma_2)'(0))}{|(\gamma_1)'(0)|.|(\gamma_2)'(0)|}$

We define a spherical wave packet to be a function $\Psi^1_{sphere,F} \in C^{\infty,sphere}([0,\pi]\times[-\pi,\pi]\times\mathcal{R})$, [6], *such that* $\Psi^1_{sphere,F}|_{(0,\pi)\times(-\pi,\pi)\times\mathcal{R}}$ *is a restricted wave packet, (corresponding to a sphere* $S^2(r)$*, of radius* $r=\frac{1}{2\pi}$*)*

For $G \in C^\infty((-\pi,\pi)\times(-\pi,\pi)\times\mathcal{R}^2)$*, we also define a restricted wave packet,* $\Psi^2_{res,G} \in C^\infty((-\pi,\pi)\times(-\pi,\pi)\times\mathcal{R})$*;*

$$\Psi^2_{res,G}(\overline{x},t) = \int_{(\overline{x}',\overline{p})\in(-\pi,\pi)\times(-\pi,\pi)\times\mathcal{R}^2} G(\overline{x}',\overline{p})e^{\frac{i}{\hbar}(\overline{p}.(\overline{x}-\overline{x}')-E'(\overline{p})t)}d\overline{x}'d\overline{p}$$

(toral coordinates;

$(y_1,y_2,y_3) = (cos(\theta)(R+2Scos(\phi)), sin(\theta)(R+2Scos(\phi)), Ssin(\phi))$, $-\pi\leq\theta\leq\pi$, $-\pi\leq\phi\leq\pi$

We define a toroidal wave packet to be a function $\Psi^2_{torus,G} \in C^{\infty,torus}([-\pi,\pi]\times[-\pi,\pi]\times\mathcal{R})$, *[7])*

$$B^t = \frac{1}{r^2}\begin{pmatrix} r\overline{y}_0 \\ . \\ r\overline{y}_1 \\ . \\ (\overline{y}_0\times\overline{y}_1)\end{pmatrix}$$

$$C = \begin{pmatrix} 1 & \frac{(\overline{y}_0.\overline{y}_1)}{r^2} & 0 \\ 0 & \sqrt{1-\frac{(\overline{y}_0.\overline{y}_1)^2}{r^4}} & 0 \\ 0 & 0 & 0\end{pmatrix}$$

A great circle trajectory γ_2, through $\overline{y}_0 \in S^2(r)$, with tangent $\gamma_2'(0)$ parallel to $\overline{p}_0$, is defined similarly, substituting $r\frac{(\overline{y}_0+\overline{p}_0)}{|\overline{y}_0+\overline{p}_0|}$, for $\overline{y}_1$ in the above definition.

[6]We define;

$C^{\infty,sphere}([0,\pi]\times[-\pi,\pi]\times\mathcal{R}) = \{f \in C([0,\pi]\times[-\pi,\pi]\times\mathcal{R}) : f|_{(0,\pi)\times(-\pi,\pi)\times\mathcal{R}} \in C^\infty([0,\pi]\times[-\pi,\pi]\times\mathcal{R}), f|_{[0,\pi]\times\{\pi\}\times\mathcal{R}} = f|_{[0,\pi]\times\{-\pi\}\times\mathcal{R}}, f(0,\phi,t) = f(\pi,\pi-\phi,t), f(0,-\phi,t) = f(\pi,\phi-\pi,t), 0\leq\phi\leq\pi, t\in\mathcal{R}\}$

[7]We define;

$C^{\infty,torus}([-\pi,\pi]\times[-\pi,\pi]\times\mathcal{R}) = \{f \in C([-\pi,\pi]\times[-\pi,\pi]\times\mathcal{R}) : f|_{(-\pi,\pi)\times(-\pi,\pi)\times\mathcal{R}} \in C^\infty([-\pi,\pi]\times[-\pi,\pi]\times\mathcal{R}), f|_{[-\pi,\pi]\times\{\pi\}} = f|_{[-\pi,\pi]\times\{-\pi\}}, f|_{\{-\pi\}\times[-\pi,\pi]} = f|_{\{\pi\}\times[-\pi,\pi]}\}$

such that $\Psi^2_{torus,G}|_{(-\pi,\pi)\times(-\pi,\pi)\times\mathcal{R}}$ *is a restricted wave packet.*

Lemma 3.5. *A free space wave packet* $\Psi_{free,F}$ *satisfies Schrodinger's equation on* $\mathcal{R}^4$ *and a restricted wave packet* $\Psi_{res,F}$ *satisfies Schrodinger's equation on* $(0,1)^2\times\mathcal{R}$.

Proof. Immediate from the above Definition 3.4, and following the proof of Lemma 3.3.

□

Lemma 3.6. *If* Ψ *defines a solution to Schrodinger's equation, in* $\mathcal{R}^3$ *or* $\mathcal{R}^4$*, with complex conjugate* Ψ^**, then, taking* $\rho=|\Psi|^2$*, and* $\overline{J}=Re[\Psi^*\frac{h'}{im}grad(\Psi)]$*, we again obtain the continuity equation, see Lemma 2.8;*

$$\frac{\partial\rho}{\partial t}+div(\overline{J})=0$$

Proof. We have that;

$$\frac{\partial\rho}{\partial t}=\frac{\partial|\Psi|^2}{\partial t}=\frac{\partial(\Psi\Psi^*)}{\partial t}=\Psi^*\frac{\partial\Psi}{\partial t}+\Psi\frac{\partial\Psi^*}{\partial t}$$

$$=\frac{i}{h'}(\Psi(H\Psi)^*-\Psi^*(H\Psi))$$

where H denotes the rescaled Laplacian operator $\frac{-(h')^2}{2m}\triangledown$. We have that;

$$\overline{J}=\frac{1}{2}([\Psi^*\frac{h'}{im}grad(\Psi)]+[\Psi^*\frac{h'}{im}grad(\Psi)])^*)$$

$$=([\Psi^*\frac{h'}{2im}grad(\Psi)]-[\Psi\frac{h'}{2im}grad(\Psi^*)])$$

$$div(\overline{J})=\frac{h'}{2im}div(\Psi^*grad(\Psi))-\frac{h'}{2im}div(\Psi grad(\Psi^*))$$

$$=\frac{h'}{2im}(grad(\Psi^*)\cdot grad(\Psi)+\Psi^*(\triangledown(\Psi)))$$

$$-\frac{h'}{2im}(grad(\Psi^*)\cdot grad(\Psi)+\Psi(\triangledown(\Psi^*)))$$

$$=\frac{h'}{2im}(\Psi^*(\triangledown(\Psi))-\Psi(\triangledown(\Psi^*)))$$

$$=\frac{-i}{h'}(\Psi(H\Psi^*)-\Psi^*(H\Psi))$$

as required.

□

Remarks 3.7. *This supports the idea of interpreting $|\Psi^2|$ as charge density, or, after renormalisation, the probability of finding an electron in a particular position. It seems reasonable to assign a probability of $\frac{1}{2}$ to a moving cat occupying the left or right hand side of a box. However, the analogy with a cat being alive or dead seems rather strange in this interpretation, as, clearly, a cat cannot switch between live and dead states. Some theoreticians interpret this probability in the sense of Liebniz, that is of possible or parallel worlds, however, in the author's opinion this is an unnecessary complication, and violates Ockham's razor, if one accepts one world with different possible positions on it. Similarly, one can interpret $Re[\Psi^*\frac{h'}{im}grad(\Psi)]$ as volume or surface current.*

Remarks 3.8. *For surface solutions to Schrodinger's equation, one can use the result of 3.6 to obtain a time-dependent surface charge and current, $\{\rho, \overline{J}\}$. The generalisation of the continuity equation to surfaces seems reasonable. Then, using Remark 2.9, we can determine the ambient fields $\{\overline{E}, \overline{B}\}$.*

Definition 3.9. *For $\psi \in \mathcal{S}(\mathcal{R}, \mathcal{R}_{\geq 0})$, we define the position to momentum operator by;*

$$(\mathcal{G}(\psi))(p,t) = \frac{1}{\sqrt{2\pi h'}}\int_{-\infty}^{\infty}\psi(x,t)e^{\frac{-ipx}{h'}}dx = \frac{1}{\sqrt{2\pi h'}}(\mathcal{F}(\psi))(\tfrac{p}{h'},t)$$

where $(\mathcal{F}(\psi))(p,t) = \int_{-\infty}^{\infty}\psi(x,t)e^{-ipx}dx$.

We define the position and momentum operators $\{X, P\}$ by;

$$X(\psi) = x\psi, P(\phi) = p\phi$$

and the transferred operators $\{X', P'\}$ by;

$$X'(\phi) = (\mathcal{G}(X(\mathcal{G}^{-1}(\phi)))), P'(\psi) = (\mathcal{G}^{-1}(P(\mathcal{G}(\psi))))$$

Remarks 3.10. *By the inversion theorem, we have that $(\mathcal{F}^2)(\psi)(x,t) = 2\pi\psi^r(x,t)$, where $\psi^r(x,t) = \psi(-x,t)$ and, by a simple change of variables, that $(\mathcal{F}(\psi))_{\frac{1}{h'}} = h'(\mathcal{F}(\psi_{h'}))$, where $\psi_{h'}(x,t) = \psi(h'x,t)$ and $\phi_{\frac{1}{h'}}(x,t) = \phi(\frac{x}{h'},t)$. It follows that;*

$(\mathcal{G})(\psi)=\frac{1}{\sqrt{2\pi h'}}(\mathcal{F}(\psi))_{\frac{1}{h'}}=\frac{1}{\sqrt{2\pi h'}}h'(\mathcal{F}(\psi_{h'}))=\sqrt{\frac{h'}{2\pi}}(\mathcal{F}(\psi_{h'}))$

$(\mathcal{G}^2)(\psi)=\frac{h'}{2\pi}\mathcal{F}((\mathcal{F}(\psi_{h'}))_{h'})$

$=\frac{h'}{2\pi}\frac{1}{h'}(\mathcal{F}(\mathcal{F}(\psi_{h'})))_{\frac{1}{h'}}$

$=\frac{1}{2\pi}(2\pi(\psi_{h'})^r)_{\frac{1}{h'}}=\psi^r$

It follows that the momentum to position operator is given by $(\mathcal{G})^{-1}(\phi)=(\mathcal{G})(\phi^r)$, $(*)$. *Differentiating under the integral sign, we have that;*

$\frac{\partial\mathcal{G}(\psi)}{\partial p}=\mathcal{G}(\frac{-ix\psi}{h'})$

and, using $(*)$*;*

$\frac{\partial\mathcal{G}^{-1}(\phi)}{\partial x}=\frac{\partial\mathcal{G}(\phi^r)}{\partial x}=\mathcal{G}(\frac{-ix\phi^r}{h'})=\mathcal{G}^{-1}(\frac{ip\phi}{h'})$

Hence;

$X'(\phi)=(\mathcal{G}(X(\mathcal{G}^{-1}(\phi))))=(\mathcal{G}(x\mathcal{G}^{-1}(\phi)))=\frac{\partial(\frac{-h'}{i}\mathcal{G})(\mathcal{G}^{-1}\phi)}{\partial p}=ih'\frac{\partial\phi}{\partial p}$

$P'(\psi)=(\mathcal{G}^{-1}(P(\mathcal{G}(\psi))))=(\mathcal{G}^{-1}(p\mathcal{G}(\psi)))=\frac{\partial(\frac{h'}{i}\mathcal{G}^{-1})(\mathcal{G}\psi)}{\partial x}=-ih'\frac{\partial\psi}{\partial x}$

4. Thermodynamic Equations

Let $W=(W_1,W_2)\in C^\infty(\mathcal{R}^2,\mathcal{R}^2)$, $W_i(T,p)$, $1\leq i\leq 2$ considered as functions of temperature T and pressure p. The second law of thermodynamics is equivalent to the fact that, for any smooth closed curve γ;

$\int_\gamma\frac{dQ}{T}=0$

where $dQ=\frac{\partial Q}{\partial T}dT+\frac{\partial Q}{\partial p}dp$ (i)

$=\frac{\partial Q}{\partial V}dV+\frac{\partial Q}{\partial p}dp$ (ii)

By Lemma 5.5, this holds iff;

$(i).$ $\frac{\partial}{\partial p}(\frac{1}{T}\frac{\partial Q}{\partial T})-\frac{\partial}{\partial T}(\frac{1}{T}\frac{\partial Q}{\partial p})=0$

iff $\frac{1}{T^2}(T\frac{\partial^2Q}{\partial T\partial p}-\frac{\partial T}{\partial p}\frac{\partial Q}{\partial T})-\frac{1}{T^2}(T\frac{\partial^2Q}{\partial p\partial T}-\frac{\partial T}{\partial T}\frac{\partial Q}{\partial p})=0$

iff $\frac{\partial Q}{\partial p}=0$, $(T\neq 0)$, $Q(T,p)$

$Q=f(T)$.

Similarly, (ii) $\frac{\partial Q}{\partial V}=0$, $Q(T,V)$

(iii). $\frac{\partial}{\partial p}(\frac{1}{T}\frac{\partial Q}{\partial V})-\frac{\partial}{\partial V}(\frac{1}{T}\frac{\partial Q}{\partial p})$

iff $\frac{\partial Q}{\partial V}\frac{\partial T}{\partial p}-\frac{\partial Q}{\partial p}\frac{\partial T}{\partial V}=0$

By the first law of thermodynamics, $dQ=dU+pdv$, we have;

$dQ=[\frac{\partial U}{\partial T}+p\frac{\partial V}{\partial T}]dT+[\frac{\partial U}{\partial p}+p(\frac{\partial V}{\partial p})]dp$ (iv)

$dQ=[\frac{\partial U}{\partial T}+p\frac{\partial V}{\partial T}]dT+[\frac{\partial U}{\partial V}+p]dV$ (v)

$dQ=[\frac{\partial U}{\partial p}+p\frac{\partial V}{\partial p}]dp+[\frac{\partial U}{\partial V}+p]dV$ (vi)

Hence, from $(i),(iv)$, (p,T);

$\frac{\partial U}{\partial p}+p(\frac{\partial V}{\partial p})=0$

from $(ii),(v)$, (V,T);

$\frac{\partial U}{\partial V}+p=0$

from $(iii),(vi)$, (p,V)

$\frac{\partial}{\partial V}[\frac{\partial U}{\partial V}+p]\frac{\partial T}{\partial p}=\frac{\partial}{\partial p}[\frac{\partial U}{\partial p}+p\frac{\partial V}{\partial p}]\frac{\partial T}{\partial V}$

giving;

$\frac{\partial^2 U}{\partial V^2}\frac{\partial T}{\partial p}=\frac{\partial^2 U}{\partial p^2}\frac{\partial T}{\partial V}$

We now make some observations on the probability distribution function g of molecular speeds in $\mathcal{R}^3$. It is reasonable to assume that;

(i). The probability distributions of each velocity component v_i, $1\leq i\leq 3$, are identical and independent.

(ii). The joint distribution g depends only on $|\overline{v}|$.

(iii) $g \in C^\infty(\overline{R}^3)$

see [5]. Interpreting (ii) in terms of probabilities, this suggests that for two given velocities $\{\overline{v}^1, \overline{v}^2\}$, with $|\overline{v}^1| = |\overline{v}^2|$, and an error $\overline{h}$ the probabilities $P_{\overline{v}^1,\overline{h}}$ and $P_{\overline{v}^2,\overline{h}}$ of finding a particle in the ranges $[v_1^1, v_1^1 + h_1] \times [v_2^1, v_2^1 + h_2] \times [v_3^1, v_3^1 + h_3]$ or $[v_1^2, v_1^2 + h_1] \times [v_2^2, v_2^2 + h_2] \times [v_3^2, v_3^2 + h_3]$ should be equal. This is obviously equivalent to the fact that for errors $\overline{h}, \overline{h}'$, the probabilities $P_{\overline{v}^1,\overline{h},\overline{h}'}$ and $P_{\overline{v}^2,\overline{h},\overline{h}'}$ of finding a particle in the ranges $[v_1^1 + h_1, v_1^1 + h_1'] \times [v_2^1 + h_2, v_2^1 + h_2'] \times [v_3^1 + h_3, v_3^1 + h_3']$ or $[v_1^2 + h_1, v_1^2 + h_1'] \times [v_2^2 + h_2', v_2^2 + h_2] \times [v_3^2 + h_3, v_3^2 + h_3']$ should be equal. By Lemma 4.4, in footnote 8, this implies that the distribution function g is $O(3)$-invariant, hence, using the assumption (i), we have that;

$$g(v_1, v_2, v_3) = G(|\overline{v}|) = f(v_1)f(v_2)f(v_3),\ (*)$$

where $G \in C^\infty(\mathcal{R}_{\geq 0})$, [8]

From $(*)$, we can now derive the expressions for $\{f, g\}$. We have to exclude the case that g and G are constant. Suppose for contradiction

[8] We require the following lemma;

Lemma 4.1. *Let $U \subset \mathcal{R}^n$ be open, and $f \in C^\infty((U), \mathcal{R})$, then, if $p \in U$;*

$$f(p) = lim_{\overline{h}\to\overline{0}} \frac{1}{h_1 \ldots h_n} \int_{[p_1-,p_1+h_1]\times\ldots\times[p_n,p_n+h_n]} f(\overline{x})d\overline{x}$$

This is easily proved by induction on n. The case $n = 1$ follows from the FTC. Suppose true for $n = k$. Let;

$$g_{h_{k+1}}(x_1, \ldots, x_k) = \int_{[p_{k+1},p_{k+1}+h_{k+1}]} f(x_1, \ldots, x_k, s)ds$$

Then $lim_{h_{k+1}\to 0} g_{h_{k+1}}(x_1, \ldots, x_k) = f(x_1, \ldots, x_k, p_{k+1})$, using the fact that, for $(x_1, \ldots, x_k) \in pr_k(U)$, $f_{(x_1,\ldots,x_k)}(s) \in C^\infty(pr_{x_{k+1}}((U \cap pr^{-1}((x_1, \ldots, x_k)))), \mathcal{R})$. Then;

$$lim_{\overline{h}\to\overline{0}} \frac{1}{h_1 \ldots h_{k+1}} \int_{[p_1,p_1+h_1]\times\ldots\times[p_{k+1},p_{k+1}+h_{k+1}]} f(\overline{x})d\overline{x}$$

$$= lim_{\overline{h}\to\overline{0}} \frac{1}{h_1 \ldots h_k} [\int_{[p_1,p_1+h_1]\times\ldots\times[p_k,p_k+h_k]} (lim_{h_{k+1}\to 0} g_{h_{k+1}}(x_1, \ldots, x_k))d(x_1 \ldots x_k)]$$

$$= lim_{\overline{h}\to\overline{0}} \frac{1}{h_1 \ldots h_k} [\int_{[p_1,p_1+h_1]\times\ldots\times[p_k,p_k+h_k]} (f(x_1, \ldots, x_k, p_{k+1}))d(x_1 \ldots x_k)]$$

$$= f(p_1, \ldots, p_k, p_{k+1})$$

---

by the induction hypothesis applied to $f_{p_{k+1}}(x_1,\ldots,x_k)=f(x_1,\ldots,x_k,p_{k+1})$, *with* $f_{p_{k+1}}\in C^\infty(pr_{1,k}(pr^{-1}(p_{k+1})\cap U),\mathcal{R})$.

Definition 4.2. *Given a tuple* $\overline{h}=(h_1,\ldots,h_n)$, *and* $\overline{p}=(p_1,\ldots,p_n)\in\mathcal{R}^n$, *we let;*

$I_{\overline{p},\overline{h}}=[p_1,p_1+h_1]\times\ldots[p_n,p_n+h_n]$

Given a rotation $r\in O(n,R)$, *and the translation* $tr_{-\overline{p}}$, *with* $tr_{-\overline{p}}(\overline{p})=\overline{0}$, *we let* $r_{\overline{p}}=(tr_{\overline{p}}\circ r\circ tr_{-\overline{p}})$, $I_{r,\overline{p},\overline{h}}=r_{\overline{p}}(I_{\overline{p},\overline{h}})$

Lemma 4.3. *Let* $f\in C^\infty(\mathcal{R}^n,\mathcal{R})$, *then, if* $p\in\mathcal{R}^n$, *and* $r\in O(n,R)$;

$f(p)=lim_{\overline{h}\rightarrow\overline{0}}\frac{1}{h_1.\ldots.h_n}\int_{I_{r,\overline{p},\overline{h}}}f(\overline{x})d\overline{x}$

Proof. It is a simple exercise to show that, $|det(Jac(r_{\overline{p}}))|=|det(Jac(r_{\overline{0}}))|=|det(Jac(r))|=1$, $(*)$,then;

$\int_{I_{r,\overline{p},\overline{h}}}f(\overline{x})d\overline{x}$

$=\int_{I_{\overline{p},\overline{h}}}(r_{\overline{p}}\circ f)(\overline{x})d\overline{x}$

So, using the previous lemma;

$lim_{\overline{h}\rightarrow\overline{0}}\frac{1}{h_1.\ldots.h_n}\int_{I_{r,\overline{p},\overline{h}}}f(\overline{x})d\overline{x}$

$=lim_{\overline{h}\rightarrow\overline{0}}\frac{1}{h_1.\ldots.h_n}\int_{I_{\overline{p},\overline{h}}}(r_{\overline{p}}\circ f)(\overline{x})d\overline{x}=(r_{\overline{p}}\circ f)(p)=f(p)$

□

Lemma 4.4. *Let* $f\in C^\infty(\mathcal{R}^n,\mathcal{R})$, *with the property that, for all* $\{\overline{p},\overline{h}\}\subset\mathcal{R}^n$, $r\in O(n,\mathcal{R}^n)$;

$\int_{I_{\overline{p},\overline{h}}}f(\overline{x})d(\overline{x})=\int_{I_{r(\overline{p}),\overline{h}}}f(\overline{x})d(\overline{x})$, $(*)$

Then f *is* $O(n)$*-invariant.*

Proof. We have that, for $p\in\mathcal{R}^n$, $r\in O(n)$, then, using Lemmas 4.1, the assumption $(*)$, the observation $(*)$ in 4.3, and, the result of 4.3;

$f(p)=lim_{\overline{h}\rightarrow\overline{0}}\frac{1}{h_1.\ldots.h_n}\int_{I_{\overline{p},\overline{h}}}f(\overline{x})d(\overline{x})$

$=lim_{\overline{h}\rightarrow\overline{0}}\frac{1}{h_1.\ldots.h_n}\int_{I_{r(\overline{p}),\overline{h}}}f(\overline{x})d(\overline{x})$

$=lim_{\overline{h}\rightarrow\overline{0}}\frac{1}{h_1.\ldots.h_n}\int_{I_{r^{-1},\overline{p},\overline{h}}}(f\circ r))(\overline{x})d(\overline{x})$

$=((f\circ r)(p))$

This shows that f is $O(n)$-invariant, as required.

that $G = L$ with $L \in \mathcal{R}_{>0}$. Then as g is $O(3)$ invariant, and $g(v_1, v_2, v_3)$ is a probability distribution, g must be supported on a sphere, centred at $\overline{0}$, with radius $R = \frac{3}{4\pi L}^{\frac{1}{3}}$, contradicting the fact that G is differentiable.

We have that;

$\frac{\partial g}{v_i} = \frac{dG}{d|\overline{v}|}\frac{\partial|\overline{v}|}{\partial v_i} = \frac{dG}{d|\overline{v}|}\frac{v_i}{|\overline{v}|}$ $(1 \leq i \leq 3$

Dividing by v_i, $1 \leq i \leq 3$, $v_i \neq 0$, we obtain;

$\frac{1}{v_i}\frac{\partial g}{v_i} = \frac{1}{|\overline{v}|}\frac{dG}{d|\overline{v}|}$ $(1 \leq i \leq 3)$ $(**)$

Using $(*)$ again, we have that;

$\frac{\partial g}{v_i} = \prod_{j \neq i} f(v_j)\frac{df}{dv_i}$ $(1 \leq i \leq 3)$, $(***)$

and, using $(**), (***)$, we obtain;

$\frac{1}{v_1}f(v_2)f(v_3)\frac{df}{dv_1} = \frac{1}{v_2}f(v_1)f(v_3)\frac{df}{dv_2} = \frac{1}{v_3}f(v_1)f(v_2)\frac{df}{dv_3}$

Dividing by $\prod_{1 \leq j \leq 3} f(v_j)$, $f(v_i) \neq 0$, $1 \leq i \leq 3$, we obtain;

$\frac{1}{v_1 f(v_1)}\frac{df}{dv_1} = \frac{1}{v_2 f(v_2)}\frac{df}{dv_2} = \frac{1}{v_3 f(v_3)}\frac{df}{dv_3}$

Hence, as $f \neq 0$, letting $U = \{x : f(x) \neq 0\}$,
we obtain that;

$\frac{df}{dv_i} = Cv_i f(v_i)$, $(1 \leq i \leq 3)$ on $U \setminus \{v_i = 0\}$

This gives;

$f(v_i) = Ae^{\frac{Cv_i^2}{2}}$ on $\mathcal{R}$

Using the assumption (iii), we have $\int_{\mathcal{R}} f(v_i) = 1$. Hence, we have that, $C < 0$. Letting $b = -C$

□

$f(v_i) = (\frac{b}{2\pi})^{\frac{1}{2}} e^{\frac{-bv_i^2}{2}}$, as $1 = A(\frac{2\pi}{b})^{\frac{1}{2}}$ (†), ([9]), $(1 \leq i \leq 3)$

$g(\overline{v}) = (\frac{b}{2\pi})^{\frac{3}{2}} e^{\frac{-b|\overline{v}|^2}{2}}$

We now derive the probability distribution h, for the absolute velocity $\overline{v}$.

We have, for $\{\alpha_1, \alpha_2\} \subset \mathcal{R}$, that, if $v_1 \geq \alpha_1$, $v_2 \geq \alpha_2$;

$|\overline{v}| \geq \alpha$ iff $v_3 \geq (\alpha^2 - (v_1^2 + v_2^2)^2)^{\frac{1}{2}}$ or $v_3 \leq -(\alpha^2 - (v_1^2 + v_2^2)^2)^{\frac{1}{2}}$, for $\alpha \geq (v_1^2 + v_2^2)^{\frac{1}{2}}$

Hence, as the probability distribution f is symmetric, given $\{v_1, v_2\}$,;

$P(|\overline{v} \geq \alpha) = 2P(v_3 \geq (\alpha^2 - (v_1^2 + v_2^2)^2)^{\frac{1}{2}})$, if $\alpha \geq ((v_1^2 + v_2^2)^2)^{\frac{1}{2}}$

$=0$, if $\alpha < ((v_1^2 + v_2^2)^2)^{\frac{1}{2}}$

It follows that the conditional cumulative distribution function $H(|\overline{v}||v_1, v_2)$, for molecular speed, is given by;

$H(|\overline{v}||v_1, v_2)$

$= 2F((|\overline{v}|^2 - (v_1^2 + v_2^2)^2)^{\frac{1}{2}})$, if $|\overline{v}| \geq ((v_1^2 + v_2^2)^2)^{\frac{1}{2}}$

$= 0$, if $|\overline{v}| < ((v_1^2 + v_2^2)^2)^{\frac{1}{2}}$

---

[9]**??** Letting;
$D = \int_{\mathcal{R}} e^{\frac{-bx^2}{2}} dx$

we have that $D^2 = \int_{\mathcal{R}^2} e^{\frac{-b(x^2+y^2)}{2}} dxdy$

$= \int_{\mathcal{R}^2 \setminus \{0\}} e^{\frac{-br^2}{2}} r dr d\theta$

$= \frac{2\pi}{-b} [e^{\frac{-br^2}{2}}]_0^{\infty}$

$= \frac{2\pi}{b}$

$D = \frac{\sqrt{2\pi}}{\sqrt{b}}$

where F is the cumulative distribution function associated to f. As f is smooth, we can differentiate the above expression, to get the conditional distribution function $h(|\overline{v}||v_1,v_2)$;

$h(|\overline{v}||v_1,v_2)$

$= 2f((|\overline{v}|^2-(v_1^2+v_2^2)^2)^{\frac{1}{2}})$, if $|\overline{v}| \geq ((v_1^2+v_2^2)^2)^{\frac{1}{2}}$

$= 0$, if $|\overline{v}| < ((v_1^2+v_2^2)^2)^{\frac{1}{2}}$

By Bayes Theorem, (†), and footnote **??** we have that;

$h(|\overline{v}|) = \int_{\mathcal{R}^2} h(|\overline{v}||v_1,v_2)f(v_1)f(v_2)dv_1dv_2$

$= \int_{((v_1^2+v_2^2)^2)^{\frac{1}{2}} \leq |\overline{v}|} 2f((|\overline{v}|^2-(v_1^2+v_2^2)^2)^{\frac{1}{2}})f(v_1)f(v_2)dv_1dv_2$

$= 2\pi|\overline{v}|^2(\frac{b}{2\pi})^{\frac{3}{2}}e^{\frac{-b|\overline{v}|^2}{2}}$, ([10])

Letting N denote the number of molecules in the system, we have that the energy $E = \frac{Nm}{2} < |\overline{v}|^2 >$, where $< |\overline{v}|^2 >= E(V_1^2+V_2^2+V_3^2) = E(V_1^2)+E(V_2^2)+E(V_3^2)$, and $\{V_1,V_2,V_3\}$ are the random variables governing the distribution of the velocity components, ([11]). Hence, using the expression (†) and footnote 10;

$E = \frac{Nm}{2} < |\overline{v}|^2 >= \frac{3Nm}{2}Var(V_1) = \frac{3Nm}{2}\int_{\mathcal{R}} v_1^2 e^{\frac{-bv^2}{2}} = \frac{3Nm}{2}\frac{b}{2\pi}^{\frac{1}{2}}\frac{\sqrt{2\pi}}{b^{\frac{3}{2}}} = \frac{3Nm}{2b}$

Substituting $b = \frac{3Nm}{2E}$ into (†), we obtain;

$f(v_i) = (\frac{3Nm}{4\pi E})^{\frac{1}{2}}e^{\frac{-3Nmv_i^2}{4E}}$

$g(\overline{v}) = (\frac{3Nm}{4\pi E})^{\frac{3}{2}}e^{\frac{-3Nm|\overline{v}|^2}{4E}}$

---

[10] To see this is a probability distribution, observe that, integrating by parts;

$\int_{\mathcal{R}} x^2 e^{\frac{-bx^2}{2}}dx = \frac{1}{b}\int_{\mathcal{R}} e^{\frac{-bx^2}{2}}dx = \frac{1}{b}\frac{\sqrt{2\pi}}{\sqrt{b}} = \frac{\sqrt{2\pi}}{b^{\frac{3}{2}}}$

so $\int_{\mathcal{R}} h(|\overline{v}|) = 1$

[11] Observe that, as $\{V_1,V_2,V_3\}$ are independent, $< |\overline{v}|^2 = E((V_1+V_2+V_3)^2) = Var(V_1+V_2+V_3)$

5. Appendix: Infinitesimal Differentials

Definition 5.1. *We let;*

$C^{\infty}(\mathcal{R}^n)=\{f:\mathcal{R}^n\rightarrow\mathcal{R}:\frac{\partial^I f}{\partial x^I}\in C(\mathcal{R}^n),\ for\ all\ I\in(\mathcal{Z}_{\geq 0})^n\}$

where, for a multi-index $I=(i_1,\ldots,i_n)$, *we let* $\frac{\partial^I f}{\partial x^I}=\frac{\partial^{i_1}\ldots\partial^{i_n}}{\partial x_1^{i_1}\ldots\partial x_n^{i_1}}$.

We define;

$C^{\infty}(\mathcal{R}^n,\mathcal{R}^m)=\{F:\mathcal{R}^n\rightarrow\mathcal{R}^m:F_j\in C^{\infty}(\mathcal{R}^n),1\leq j\leq n\}$.

For $1\leq l\leq n$, *we let;*

$\frac{\partial F}{\partial x_l}=(\frac{\partial F_1}{\partial x^l},\ldots,\frac{\partial F_j}{\partial x^l},\ldots,\frac{\partial F_m}{\partial x^l})^t$

We fix a non-zero infinitesimal $\epsilon\in(\mu(0)\cap{}^*\mathcal{R})$, $\{\epsilon_1$, *and let* $\eta=\frac{1}{\epsilon}$, *For* $n\in\mathcal{Z}_{\geq 1}{}^n$, *and* $1\leq l\leq n$, *we let* $\overline{\epsilon_l}=(\epsilon)^n\overline{e_l}$, *where* $\{\overline{e_1},\ldots,\overline{e_l},\ldots,\overline{e_n}\}$ *is the standard basis for* $\mathcal{R}^n$.

For $F\in C^{\infty}(\mathcal{R}^n,\mathcal{R}^m)$, *we let* ${}^*F:({}^*\mathcal{R})^n\rightarrow({}^*\mathcal{R})^m$ *denote the transfer of* F. *Then, for* $F\in C^{\infty}(\mathcal{R}^n,\mathcal{R}^m)$, *we let* $d({}^*F):({}^*\mathcal{R})^n\rightarrow(({}^*\mathcal{R})^{nm}\cap\mu(\overline{0}))$, $dF:(\mathcal{R})^n\rightarrow(({}^*\mathcal{R})^m\cap\mu(\overline{0}))$ *be defined by;*

$d({}^*F)(x_1',\ldots,x_n')=\sum_{l=1}^n{}^*(\frac{\partial F}{\partial x_l})|_{(x_1',\ldots,x_n')}\overline{\epsilon_l}$

$dF=d({}^*F)|_{\mathcal{R}^n}$

We let $J=([0,1])$, ${}^*J={}^*([0,1])$, ν *be the counting measure on* ${}^*J$, *defined by* $\nu((x',x'+\frac{1}{\eta}))=\frac{1}{\eta}$ *and define;*

$\int_{({}^*J)^n}d({}^*F)={}^*\sum_{0\leq s_1\leq\eta_1-1,\ldots,0\leq s_l\leq\eta_l-1,\ldots,0\leq s_n\leq\eta_n-1}d({}^*F)(\frac{s_1}{\eta_1},\ldots,\frac{s_l}{\eta_l},\ldots,\frac{s_n}{\eta_n})$

$\int_{(J)^n}dF={}^{\circ}(\int_{({}^*J)^n}d^*F)$

For $F\in C^{\infty}(\mathcal{R}^n,\mathcal{R}^m)$, *we define* $grad(F)\in C^{\infty}(\mathcal{R}^n,\mathcal{R}^{nm})$, *by;*

$grad(F)_{i,j}=\frac{\partial F_i}{\partial x_j}$

for $1\leq i\leq m$, $1\leq j\leq n$.

For $F \in C^\infty(\mathcal{R}^n, \mathcal{R}^m)$, we let $\delta({}^*F) : ({}^*\mathcal{R})^n \rightarrow (({}^*\mathcal{R})^m \cap \mu(\overline{0}))$, $\delta F : (\mathcal{R})^n \rightarrow (({}^*\mathcal{R})^m \cap \mu(\overline{0}))$ be defined by;

$\delta({}^*F)(x_1', \ldots, x_n') = \sum_{l=1}^n {}^*(\frac{\partial F}{\partial x_l})|_{(x_1', \ldots, x_n')}\epsilon$, ([12])

$\delta F = \delta({}^*F)|_{\mathcal{R}^n}$

Observe that $\delta({}^*F) = \epsilon^{1-n}(d({}^*F) \centerdot \overline{p})$ and $\delta(F) = \epsilon^{n-1}(dF \centerdot \overline{p})$, where $p = (1, \ldots, 1)^t$.

For $F \in C^\infty(\mathcal{R}^n, \mathcal{R})$, we let $grad({}^*F) = \epsilon^{-1}d({}^*F)$, and $grad(F) = grad({}^*F)|_{\mathcal{R}^n}$

Given $H \in C^\infty(\mathcal{R}^n, \mathcal{R}^n)$, we define the infinitesimal vector field of order ϵ,associated to H, to be ${}^*H_\epsilon = \epsilon{}^*H$, and, of order ϵ^2, to be ${}^*H_{\epsilon^2} = \epsilon^{2*}H$, and, of order ϵ^n, to be ${}^*H_{\epsilon^n} = \epsilon^{n*}H$

For a smooth contour γ in $\mathcal{R}^2$, enclosing a simple open surface S, with intrinsic parametrisation $\overline{r} : [0, a] \rightarrow \mathcal{R}^2$, where, $a = length(\gamma)$, and transfer ${}^*\gamma$ in ${}^*\mathcal{R}^2$, $\overline{{}^*r} : {}^*[0, a] \rightarrow {}^*\mathcal{R}^2$, we define, for a vector field $H \in C^\infty(\mathcal{R}^n, \mathcal{R}^n)$;

$\int_{{}^*\gamma} {}^*H \centerdot {}^*(d\overline{r}) = {}^*\sum_{0 \leq t' \leq [\eta a]-1} {}^*H_\epsilon|_(r_1(\frac{t'}{\eta}), r_2(\frac{t'}{\eta})) \centerdot ({}^*(r_1')(\frac{t'}{\eta}), {}^*(r_2')(\frac{t'}{\eta}))$, ([13])

[12]For later use, we fix linearly independent infinitesimals, (over $\mathcal{R}$), $\{\epsilon_1, \ldots, \epsilon_n\} \subset {}^*\mathcal{R}$, and let $\nu_{\overline{\epsilon}}$ be the counting measure on ${}^*\mathcal{R}^n$, defined by $\nu_{\overline{\epsilon}}((\frac{i_1}{\eta_1}, \frac{i_1+1}{\eta_1}) \times \ldots \times (\frac{i_n}{\eta_n}, \frac{i_n+1}{\eta_n})) = \frac{1}{(\eta_1 \centerdot \ldots \centerdot \eta_n})= \epsilon_1 \centerdot \ldots \centerdot \epsilon_n$. For $F \in C^\infty(\mathcal{R}^n, \mathcal{R}^m)$, we let $F_{\overline{\epsilon}} : (\mathcal{R}_{\overline{\epsilon}})^n \rightarrow (({}^*\mathcal{R})^m)$ be the $\nu_{\overline{\epsilon}}$ measurable counterpart of F. Then we can also define $\delta({}^*F)(x_1', \ldots, x_n') = \sum_{l=1}^n (\frac{\partial F}{\partial x_l})_{\overline{\epsilon}}|_{(x_1', \ldots, x_n')}\epsilon_1$. Similarly, for a vector field $W \in C^\infty(\mathcal{R}^n, \mathcal{R}^n)$, we define;

$\delta({}^*W)(x_1' \ldots, x_n') = \sum_{l=1}^n (W_l)_{\overline{\epsilon}}|_{(x_1', \ldots, x_n')}\epsilon_1$

[13]For a smooth contour γ, and smooth vector field H, with associated infinitesimal increment $H_{\epsilon_1, \epsilon_2}$ of order (ϵ_1, ϵ_2), we can also define, (see also footnote 2);

$\int_{{}^*\gamma} \delta({}^*H) = \int_{{}^*\gamma} H_{\epsilon_1, \epsilon_2} = {}^*\sum_{[\eta_1 \overline{r}_1(x)], [\eta_2 \overline{r}_2(x)] : x \in {}^*[0,1]} H_{\epsilon_1, \epsilon_2}$

$\int_\gamma (H) = {}^\circ(\int_{{}^*\gamma} \delta({}^*H))$

$\int_{^*\gamma} grad(^*F) \centerdot ^*(d\bar{r}) = ^*\sum_{0\leq t'\leq[\eta a]-1} grad(^*F) \centerdot (^*(r_1')(\frac{t'}{\eta}), ^*(r_2')(\frac{t'}{\eta}))$

We define;

$curl(^*H_\epsilon) = \epsilon(\bigtriangledown \times ^*H_\epsilon)$

Given a simple open surface $S \subset \mathcal{R}^2$, and $V \in C^\infty(S, \mathcal{R}^3)$, we define;

$\int_{^*S}(^*V).d(^*S) = ^*\sum_{(s',t')\in^*\mathcal{Z}^2,(\frac{s'}{\eta},\frac{t'}{\eta})\in(^*S)} {^*V_{\epsilon^2}}.(0,0,1)$

Lemma 5.2. *Integration by Increments*
For $F \in C^\infty(\mathcal{R}, \mathcal{R})$, we have;

$\int_J dF = \int_J (\frac{dF}{dx})dx$

For $F \in C^\infty(\mathcal{R}^2, \mathcal{R})$, we have;

$\int_{J^2} dF = \int_{J^2} grad(F)dx_1dx_2$

For $F \in C^\infty(\mathcal{R}^n, \mathcal{R})$, we have;

$\int_{(J)^n} dF = \int_{(J)^n} grad(F)(dx_1 \ldots dx_l \ldots dx_n)$

Proof. We have, by Definition 5.1, the fact that $^*(\frac{dF}{dx})$ is S-integrable with respect to ν, (as $\nu(J)$ is finite, $^*(\frac{dF}{dx})$ is measurable $d\nu$, and $^*(\frac{dF}{dx})$ is continuous), results of [?] or [?], and $sp^*(dx) = L(\nu)$, $sp^*(\frac{dF}{dx}) = {^\circ}(^*(\frac{dF}{dx}))$, (as $\frac{dF}{dx}$ is continuous), where $sp : {^*J} \rightarrow J$ is the specialisation map, that;

$\int_J dF = {^\circ}(\int_{(^*J)} d^*F) = \frac{1}{\eta}[^* \sum_{0\leq s\leq\eta-1} {^*(\frac{dF}{dx})}(\frac{s}{\eta})]$

$= {^\circ}(\int_{(^*J)} {^*(\frac{dF}{dx})} d\nu)$

$= \int_{(^*J)} {^\circ}(^*(\frac{dF}{dx}))dL(\nu)$

$= \int_J (\frac{dF}{dx})dx$

$\int_{J^2} dF = {^\circ}(\int_{(^*J)^2} d^*F) = {^\circ}(^* \sum_{0\leq s_1\leq\eta-1, 0\leq s_2\leq\eta-1} d(^*F)(\frac{s_1}{\eta}, \frac{s_2}{\eta}))$

We have that;

$d({}^{*}F)(\frac{s_1}{\eta},\frac{s_2}{\eta}) = {}^{*}(\frac{\partial F}{\partial x_1})(\frac{s_1}{\eta},\frac{s_2}{\eta})\overline{\epsilon} + {}^{*}(\frac{\partial F}{\partial x_2})(\frac{s_1}{\eta},\frac{s_2}{\eta})\overline{\epsilon}$

$= \epsilon^2[{}^{*}(\frac{\partial F}{\partial x_1})(\frac{s_1}{\eta},\frac{s_2}{\eta})\overline{e_1} + {}^{*}(\frac{\partial F}{\partial x_2})(\frac{s_1}{\eta},\frac{s_2}{\eta})\overline{e_2}]$

Hence;

$\int_{J^2} dF = {}^{\circ}(\epsilon^2[{}^{*}\sum_{0\leq s_1\leq \eta-1, 0\leq s_2\leq \eta-1}{}^{*}(\frac{\partial F}{\partial x_1})(\frac{s_1}{\eta},\frac{s_2}{\eta})\overline{e_1} + {}^{*}(\frac{\partial F}{\partial x_2})(\frac{s_1}{\eta},\frac{s_2}{\eta})\overline{e_2}])$

For $0 \leq s_1 \leq \eta - 1$, let;

$H(\frac{s_1}{\eta}) = (H_1(\frac{s_1}{\eta}), H_2(\frac{s_1}{\eta}))^t = \epsilon({}^{*}\sum_{0\leq s_2\leq \eta-1}{}^{*}(\frac{\partial F}{\partial x_1})(\frac{s_1}{\eta},\frac{s_2}{\eta})\overline{e_1} + {}^{*}(\frac{\partial F}{\partial x_2})(\frac{s_1}{\eta},\frac{s_2}{\eta})\overline{e}_2)$

If $x_1' \sim x_1''$, and $0 \leq s_2 \leq \eta - 1$, we have that ${}^{*}(\frac{\partial F}{\partial x_1})(x_1',\frac{s_2}{\eta}) \sim {}^{*}(\frac{\partial F}{\partial x_1})(x_1'',\frac{s_2}{\eta})$, and, similarly, ${}^{*}(\frac{\partial F}{\partial x_2})(x_1',\frac{s_2}{\eta}) \sim {}^{*}(\frac{\partial F}{\partial x_2})(x_1'',\frac{s_2}{\eta})$, as $\{\frac{\partial F}{\partial x_1},\frac{\partial F}{\partial x_2}\} \subset C(J^2)$. In particular, for any $\delta \in \mathcal{R}_{>0}$, $i \in \{1,2\}$, we have that;

$|H_i(x_1') - H(x_1'')|$

$\leq ({}^{*}\sum_{0\leq s_2\leq \eta-1}|{}^{*}(\frac{\partial F}{\partial x_i})(x_1',\frac{s_2}{\eta}) - {}^{*}(\frac{\partial F}{\partial x_i})(x_1'',\frac{s_2}{\eta})|\epsilon$

$\leq ({}^{*}\sum_{0\leq s_2\leq \eta-1}\delta\epsilon)$

$\leq \eta\epsilon(\delta) = \delta$

Hence, as $\delta > 0$ was arbitrary, $H_i(x_1') \sim H_i(x_1'')$, for $1 \leq i \leq 2$. It follows that;

$({}^{\circ}H_i(\frac{s_1}{\eta})) = ({}^{\circ}H_i({}^{\circ}(\frac{s_1}{\eta}))$. For $x_1 \in \mathcal{R}$, we have that;

$({}^{\circ}H(x_1)) = {}^{\circ}(\epsilon^{*}\sum_{0\leq s_2\leq \eta-1}{}^{*}(\frac{\partial F}{\partial x_1})(\frac{s_1}{\eta},\frac{s_2}{\eta})\overline{e_1} + {}^{*}(\frac{\partial F}{\partial x_2})(\frac{s_1}{\eta},\frac{s_2}{\eta})\overline{e_2})$

$= (\int_J(\frac{\partial F}{\partial x_1}(x_1,x_2)dx_2, \frac{\partial F}{\partial x_2}(x_1,x_2))dx_2)^t$

$= (h_1(x_1), h_2(x_1))^t = h(x_1)$

so, for $x_1' \in {}^{*}J$, we have that ${}^{\circ}H(x_1') = h({}^{\circ}x_1')$, $(\dagger)$.

Then;

$\int_{J^2} dF = {}^{\circ}({}^{*}\sum_{0\leq s_1\leq \eta-1}(\epsilon H(\frac{s_1}{\eta}))$

$= \int_J h(x_1)dx_1$

$= \int_J(\int_J(\frac{\partial F}{\partial x_1}(x_1,x_2),\frac{\partial F}{\partial x_2}(x_1,x_2))dx_2)dx_1$

$= \int_{J^2} grad(F)dx_1dx_2$

□

Remarks 5.3. *Observe that if $F \in C^\infty(\mathcal{R}^2,\mathcal{R})$, then, for $\{\overline{a},\overline{b}\} \subset \mathcal{R}^2$;*

$F(\overline{b}) - F(\overline{a}) = \int_\gamma grad(F).d\overline{r}$

for any smooth path γ, parameterised by $\overline{r}$. This follows from the fact that, for any smooth closed contour $\mathfrak{C}$, enclosing a simple open surface S, we have, by Stokes Theorem, that;

$\oint_{\mathfrak{C}} grad(F).d\overline{r} = \int_S curl(grad(F)).dS = 0$

as $curl(grad(F)) = \frac{\partial}{\partial x_2}(\frac{\partial F}{\partial x_1}) - \frac{\partial}{\partial x_1}(\frac{\partial F}{\partial x_2}) = 0$

and, without loss of generality, assuming $a_1 < b_1$, and $a_2 < b_2$, where $\overline{a} = (a_1,a_2)$, $\overline{b} = (b_1,b_2)$, then;

$F(b_1,b_2) - F(b_1,a_2) = \int_{l_{\overline{c},\overline{b}}} grad(F).d\overline{r}_2 = \int_0^{b_2-a_2} \frac{\partial F}{\partial x_2}(b_1,a_2+t)dt$

$F(b_1,a_2) - F(a_1,a_2) = \int_{l_{\overline{a},\overline{c}}} grad(F).d\overline{r}_1 = \int_0^{b_1-a_1} \frac{\partial F}{\partial x_1}(a_1+t,a_2)dt$

where $\{\overline{r}_1,\overline{r}_2\}$ are intrinsic parametrisations of $\{l_{\overline{a},\overline{c}}, l_{\overline{c},\overline{b}}\}$

and $\overline{c} = (b_1,a_2)$.

We now reformulate a nonstandard version of this result.

Lemma 5.4. *Let $F \in C^\infty(\mathcal{R}^2,\mathcal{R})$, then for $\{\overline{a},\overline{b}\} \subset \mathcal{R}^2$;*

$F(\overline{b}) - F(\overline{a}) = {}^\circ(\int_{{}^*\gamma} grad({}^*F) \centerdot {}^*(d\overline{r}))$

for any smooth simple path γ, parameterised by $\overline{r}$

Proof. Again, wlog, assume that $a_1 < b_1$ and $a_2 < b_2$, $\overline{c} = (b_1,a_2)$. Then, if $d_2 = b_2 - a_2 = length(l_{\overline{c},\overline{b}})$, $d_1 = b_1 - a_1 = length(l_{\overline{a},\overline{c}})$ with intrinsic parameterisation, $r_1(t) = (b_1,a_2+t))$, $r_1'(t) = (0,1)$,

$r_2(t)=(a_1+t,a_2)$, $r_1'(t)$, $r_2'(t)=(1,0)$

$I_1=(\int_{{}^*l_{\bar{c},\bar{b}}}grad({}^*F)\centerdot{}^*(d\bar{r}))$

$={}^*\sum_{0\leq t'\leq[\eta(b_2-a_2)]-1}grad({}^*F)|_{(r_1(\frac{t}{\eta}),r_2(\frac{t}{\eta}))}\centerdot({}^*(r_1')(\frac{t'}{\eta}),{}^*(r_2')(\frac{t'}{\eta}))$

$={}^*\sum_{0\leq t'\leq[\eta(b_2-a_2)]-1}\epsilon(({}^*\frac{\partial F}{\partial x_1})(b_1,a_2+\frac{t'}{\eta}),({}^*\frac{\partial F}{\partial x_2})(b_1,a_2+\frac{t'}{\eta}))\centerdot(0,1)$

$={}^*\sum_{0\leq t'\leq[\eta(b_2-a_2)]-1}\epsilon({}^*\frac{\partial F}{\partial x_2})(b_1,a_2+\frac{t'}{\eta})$

${}^\circ(I_1)={}^\circ(\int_0^{b_2-a_2}\frac{\partial F}{\partial x_2}(b_1,a_2+t)dt)$

$=F(\bar{b})-F(\bar{c})$

Similarly, if;

$I_2=(\int_{{}^*l_{\bar{c},\bar{b}}}grad({}^*F)\centerdot{}^*(d\bar{r}))$

${}^\circ(I_2)=F(\bar{c})-F(\bar{a})$

We now claim that if $\gamma\subset\mathcal{R}^2$ is any smooth closed contour, then;

$\int_{{}^*\gamma}{}^*H\centerdot{}^*(d\bar{r})\simeq\int_{{}^*S}curl({}^*H_\epsilon).d({}^*S)$

([14]) □

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[14]We assume that $\bar{r}$ is analytic, (†). Let $A=\{t\in\mathcal{R}:r_2'(t)=0\}$, then, if A is infinite, it follows, by analyticity, that $r_2'|_{[0,1]}\equiv 0$, contradicting the simplicity assumption. Similarly, $B=\{t\in\mathcal{R}:r_1'(t)=0\}$ is finite, and there exists finitely many points $\{(a_1,b_1),\ldots(a_r,b_r)\}$, $\{(a_1',b_1'),\ldots(a_s',b_s')\}$, defining the vertical and horizontal tangents of γ, respectively. Let $\{l_{a_1'},\ldots,l_{a_{r'}'}\}$, $\{l_{b_1'},\ldots,l_{b_{s'}'}\}$, with $r'\leq r$, $s'\leq s$, $a_1'<\ldots<a_{r'}'$, $b_1'<\ldots<b_{s'}'$, denote the vertical and horizontal tangents lines respectively. By compactness of γ, each $(\gamma\cap l_{a_i'})$ consists of finitely many points $V_i=\{(a_i',c_j'):1\leq j\leq v(i)\}$, $1\leq i\leq r'$, and, similarly, we obtain finite sets $H_j=\{(d_i',b_j'):1\leq i\leq h(j)\}$, $1\leq j\leq s'$, consisting of the intersections $(\gamma\cap l_{b_j'})$, for $1\leq j\leq s'$. Let S_0 be a connected component of $S\cap W_{a_{i_0}',a_{i_0+1}'}$, for some $1\leq i_0\leq r'-1$, where $W_{a_{i_0}',a_{i_0+1}'}=\{(x_1,x_2)\in\mathcal{R}^2:a_{i_0}'<x_1<a_{i_0+1}'\}$. Then, $\delta(S_0)\cap(l_{a_{i_0}'}\cup l_{a_{i_0+1}'})$, consists either of (i), 2 points $\{p_1,p_2\}$, $p_1\in l_{a_{i_0}'}$, $p_2\in l_{a_{i_0+1}'}$ (ii), 2 closed intervals $\{I_1,I_2\}$, $I_1\subset l_{a_{i_0}'}$, $I_2\subset l_{a_{i_0+1}'}$, (iii), a point and an interval $\{p_1,I_2\}$, (iv), an interval and a point $\{I_1,p_2\}$, (††). In order to see this, suppose that $\delta(S_0)\cap(l_{a_{i_0}'})$ is either finite, with at least 3 points, or non empty after removing a closed interval I_1, $(*)$, then there exist at least 3 points $\{p_{1,1},p_{1,2},p_{1,3}\}\subset\delta(S)\cap(l_{a_{i_0}'})\subset\delta(S_0)\cap(l_{a_{i_0}'})$, with $p_{1,1}<p_{1,2}<p_{1,3}$. Suppose,

Lemma 5.5. *For any smooth vector field W;*

$\int_\gamma \delta(W) = \int_\gamma W.d\overline{r}$.

Lemma 5.6. *Let $W \in C^\infty(\mathcal{R}^2, \mathcal{R}^2)$, then;*

$\int_\gamma \delta(W) = 0$, $(*)$

for any smooth closed path γ, iff $\frac{\partial W_1}{\partial x_2} = \frac{\partial W_2}{\partial x_1}$, $(**)$

Proof. Suppose $(**)$ holds, then we have that $curl(W_\epsilon)$=0, hence, by the infinitesimal version of Stokes Theorem;

$\int_{^*\gamma} \delta(^*W_\epsilon) \simeq 0$, and, therefore, $\int_\gamma \delta(W) = 0$.

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wlog, that $\overline{r}(t_i) = p_{1,i}$, with $1 \leq i \leq 3$, and $\overline{r}(t_i + k_i) = p_{2,i}$, with $\{p_{2,1}, p_{2,2}, p_{2,3}\} \subset \delta(S_0) \cap (l_{a'_{i_0}+1})$, (as if such k_i did not exist, we would obtain l_i, with $r'_1(t_i + l_i) = 0$, and $a'_{i_0} < r_1(t_i + l_i) < a'_{i_0+1}$), and $p_{2,1} \leq p_{2,2} \leq p_{2,3}$. By simplicity of the contour γ, we can assume that $p_{2,1} < p_{2,2} \leq p_{2,3}$. Choosing $p'_{1,3} \in (S_0 \cap B(p_{1,3}, \epsilon))$, $p'_{2,1} \in (S_0 \cap B(p_{2,1}, \epsilon_1))$, for sufficiently small ϵ, it is clear that any path $\gamma' \subset S_0$ crosses $\overline{r}|_{[t_2,t_2+k_2)}$, contradicting connectedness of S_0. Suppose that $\delta(S_0) \cap (l_{a'_{i_0}})$ consists of exactly one closed interval I_1 and $\delta(S_0) \cap (l_{a'_{i_0+1}})$ consists of exactly 2 points $\{p_{2,1}, p_{2,2}\}$, $(**)$. Consider $Z = \{s \in [0, a'_{i_1} - a'_{i_0}] : S_0 \cap l_{a'_{i_0}+t}$ *is not path connected*$\}$, and let $s_0 = \mu(s)(s \in Z)$, $0 < s_0$. Then, we can find $(a'_{i_0} + s_0)$ lying on the contour $(\gamma \cap \delta(S_0))$, and we can find t_0, with $a'_{i_0} < \overline{r}(t_0) < a'_{i_0+1}$, such that $\overline{r}(t_0) \in \delta(S_0)$. We have that $r'_1(t_0) \neq 0$, hence there exists an open interval $(s_0 - \epsilon, s_0 + \epsilon) \subset Z$, contradicting the choice of t_0. Suppose that both $\delta(S_0) \cap (l_{a'_{i_0}})$ and $\delta(S_0) \cap (l_{a'_{i_0+1}})$ consists of exactly 2 points $\{p_{2,1}, p_{2,2}\}$, $\{p_{1,1}, p_{1,2}\}$ $(***)$. By the same argument as in $(**)$, we can assume that $\delta(S_0) \cap (l_s)$ has exactly 2 connected components for $s \in [a'_{i_0}, a'_{i_0+1}]$, then, the assumption $(\dagger\dagger)$ holds for each $s \in (a'_{i_0}, a'_{i_0+1})$, which we excluded. Hence, we are left with the 4 cases outlined above. In each of these cases, using the argument to exclude $(\dagger\dagger)$, we can assume that $\delta(S_0) \cap (l_s)$ has exactly 1 connected components for each $s \in [a'_{i_0}, a'_{i_0+1}]$, then, it follows easily, using the arguments above, the assumption that $W_{a'_{i_0}, a'_{i_0+1}}$ contains no vertical or horizontal tangents, and simplicity of the contour, that; in case (i), $\delta(S_0))$ consists of the smooth closed curve consisting of 2 curve seqments $\overline{r}|_{[t_1,t_2]}$, $\overline{r}|_{[t_2,t_3]}$, with $r'_1(t_i) = 0$, for $1 \leq i \leq 3$, and, wlog, $\overline{r}(t_1) = \overline{r}(t_3) = p_2$, and $S_0 = Int(\delta(S_0))$ (this, of course, implies that $Im(\overline{r}|_{[t_1,t_3]}) = \gamma$; in case (ii), $\delta(S_0)$ consists of the piecewise smooth closed curve contour consisting of 4 curve segments $l_{q_1,q_3} \cup l_{q_2,q_4} \cup \overline{r}_{[t_1,t_2]} \cup \overline{r}_{[t_3,t_4]}$, with, wlog, $\overline{r}(t_i) = q_i$, for $1 \leq i \leq 4$, and $S_0 = Int(\delta(S_0))$; in case (iii), $\delta(S_0)$ consists of the piecewise smooth closed curve contour consisting of 3 curve segments $l_{q_2,q_4} \cup \overline{r}_{[t_1,t_2]} \cup \overline{r}_{[t_2,t_3]}$, with, wlog, $\overline{r}(t_1) = q_2$, $\overline{r}(t_2) = p_1$ and $\overline{r}(t_3) = q_4$, and $S_0 = Int(\delta(S_0))$. Case (iv) is similar.)

Conversely, suppose that $(*)$ holds, then we have, for any surface S, bounded by a smooth contour γ, that;

$\int_{^*S} curl(W_\epsilon) \simeq 0$. We clearly have that;

$^\circ(\int_{^*S} curl(W_\epsilon))d(^*S) = \int_S curl(W)dS = 0$

This obviously implies that $\frac{\partial W_1}{\partial x_2} = \frac{\partial W_2}{\partial x_1}$.

□

Lemma 5.7. *Chain Rule Let* $F_1 \in C^\infty(\mathcal{R}^n, \mathcal{R}^m)$, $F_2 \in C^\infty(\mathcal{R}^m, \mathcal{R}^s)$, *then;*

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