SOME NOTES ON QUANTIFIER ELIMINATION AND MODEL COMPLETENESS

These notes are intended to clarify some remarks I made during the model theory classes.

Definition 0.1. *A theory T, which needn't be complete, in a language with at least one constant symbol, has quantifier elimination if;*

For all $\phi(\bar{x}) \in \mathcal{L}$, there exists a quantifier free $\psi(\bar{x}) \in \mathcal{L}$, such that;

$T \vdash \forall x(\phi(\bar{x}) \leftrightarrow \psi(\bar{x}))$

Lemma 0.2. *Let $\phi(\bar{x}) \in \mathcal{L}$, then the following are equivalent;*

(i). There exists a quantifier free formula $\psi(\bar{x})$ such that $T \vdash \forall \bar{x}(\phi(\bar{x}) \leftrightarrow \psi(\bar{x}))$

(ii). For $\mathcal{M}, \mathcal{N} \models T$ $\mathcal{M}_0 \subseteq \mathcal{M}$ and $\mathcal{M}_0 \subseteq \mathcal{N}$, $\forall \bar{c} \in \mathcal{M}_0$;

$\mathcal{M} \models \phi(\bar{c})$ *iff* $\mathcal{N} \models \phi(\bar{c})$

Proof. Suppose (i) holds, then
$$\begin{aligned}\mathcal{M} \models \phi(\bar{c}) &\Leftrightarrow \mathcal{M} \models \psi(\bar{c})\\ &\Leftrightarrow \mathcal{M}_0 \models \psi(\bar{c}),\ \mathcal{M}_0 \subseteq \mathcal{M}\\ &\Leftrightarrow \mathcal{N} \models \psi(\bar{c}),\ \mathcal{M}_0 \subseteq \mathcal{N}\\ &\Leftrightarrow \mathcal{N} \models \phi(\bar{c})\end{aligned}$$

so (ii) holds. Suppose (ii) holds, let $\Sigma(\bar{x})$ be the set of all quantifier free formulae $\psi(\bar{x})$ such that;

$T \vdash \forall \bar{x}(\phi(\bar{x}) \rightarrow \psi(\bar{x}))$

It is sufficient to prove that;

$T \cup \Sigma(\bar{x}) \vdash \phi(\bar{x})$

Suppose not, then there exists $\mathcal{M} \models T$, $\bar{a} \in \mathcal{M}$, $\mathcal{M} \models \Sigma(\bar{a})$ and $\mathcal{M} \models \neg\phi(\bar{a})$. Let $\Gamma(\bar{x}) = qftp_M(\bar{a})$, then $T \cup \Gamma(\bar{x}) \cup \phi(\bar{x})$ is consistent. Otherwise, $T \cup \{\phi(\bar{x})\} \vdash \neg\gamma(\bar{x})$, for some $\gamma \in \Gamma$, whereby $\neg\gamma \in \Sigma$. Hence, there exists a model $\mathcal{N} \models T$, and $\bar{b}$ from $\mathcal{N}$, with $qftp_N(\bar{b}) = qftp_M(\bar{a})$, such that $\mathcal{N} \models \phi(\bar{b})$. Let $< \bar{a} >$ and $< \bar{b} >$ be the structures generated by the tuples $\bar{a}$ and $\bar{b}$. As $qftp_M(\bar{a}) = qftp_N(\bar{b})$, we have that there exists an isomorphism $< \bar{a} > \cong < \bar{b} >$. It follows that there exists $\mathcal{M}_0 \subseteq M$, $\mathcal{M}_0 \subseteq N$, containing $\bar{a}$, with $\mathcal{M} \models \neg\phi(\bar{a})$ and $\mathcal{N} \models \phi(\bar{a})$, this contradicts (ii), hence, (i) holds.

□

Remarks 0.3. *For the lemma, T need not be complete. Also observe that quantifier elimination requires that T can eliminate quantifiers from sentences of the language, so we need at least one constant symbol. As explained by Franziska, the lemma shows that the incomplete theory $DTFAG$ does not have quantifier elimination. However, the completion obtained by adding $\exists v(v \neq 0)$ does, you can use the theorem below.*

We observe that;

Lemma 0.4. *T has quantifier elimination if all formulae of the form $\exists y\phi(y, \bar{x})$ where, $\phi(y, \bar{x})$ is quantifier free, are equivalent to quantifier free formulae.*

Proof. By routine induction. □

Then,

Theorem 0.5. *Suppose that T is complete. T has quantifier elimination, iff for all $\mathcal{M}, \mathcal{N} \models T$, where $\mathcal{N}$ is $|T|^+$ saturated, finitely generated $\mathcal{M}_0 \subseteq \mathcal{M}, \mathcal{N}$, and $b \in \mathcal{M} \setminus \mathcal{M}_0$, there exists $b' \in \mathcal{N}$, and an isomorphism;*

$$f :< \mathcal{M}_0, b > \rightarrow < \mathcal{M}_0, b' >$$

with $f|\mathcal{M}_0 = id$ and $f(b) = b'$. $(*)$

Proof. Suppose T has quantifier elimination, then there exists $b' \in \mathcal{N}$, with $tp_N(b'/\mathcal{M}_0) = tp_M(b/\mathcal{M}_0)$. To see this, let $\phi(x, \bar{a}) \in tp_M(b/\mathcal{M}_0)$, $\bar{a} \in \mathcal{M}_0$, then;

$$T \vdash \exists x\phi(x, \bar{y}) \leftrightarrow \psi(\bar{y}), \text{ with } \psi(\bar{y}) \text{ quantifier free.}$$

$$\mathcal{M} \models \exists x\phi(x, \bar{a}) \Leftrightarrow \mathcal{M} \models \psi(\bar{a})$$

$$\Leftrightarrow \mathcal{M}_0 \models \psi(\bar{a}), \text{ as } \mathcal{M}_0 \subseteq \mathcal{M}$$

$$\Leftrightarrow \mathcal{N} \models \psi(\bar{a}), \text{ as } \mathcal{M}_0 \subseteq \mathcal{N}$$

$$\Leftrightarrow \mathcal{N} \models \exists x \phi(x, \bar{a}).$$

By compactness and $|T|^+$ saturation of $\mathcal{N}$, there exists b' with the required properties. It is straightforward to see that $< \mathcal{M}_0, b > \cong < \mathcal{M}_0, b' >$; we have that, for $\bar{a} \in \mathcal{M}_0$, $\psi(x, \bar{y})$ quantifier free;

$$< \mathcal{M}_0, b > \models \psi(\bar{a}, b) \Leftrightarrow \mathcal{M} \models \psi(\bar{a}, b)$$

$$\Leftrightarrow \mathcal{N} \models \psi(\bar{a}, b')$$

$$\Leftrightarrow < \mathcal{M}_0, b' > \models \psi(\bar{a}, b')$$

Conversely, suppose that $(*)$ holds. In order to prove quantifier elimination, we use Lemma 0.4. to show (ii) of Lemma 0.2. Suppose that $\mathcal{M}_0 \subseteq \mathcal{M}, \mathcal{N}$ with $\mathcal{M}, \mathcal{N} \models T$. We may assume that $\mathcal{N} \prec \mathcal{N}'$, for $\mathcal{N}'$, $|T|^+$-saturated. Let $\bar{c} \in \mathcal{M}_0$. By Lemma 0.4, it is enough to check (ii) of Lemma 0.2, for $\exists y \phi(y, \bar{x})$, with $\phi(y, \bar{x})$ quantifier free. We have that;

$$\mathcal{M} \models \exists y \phi(y, \bar{c}) \Rightarrow \mathcal{M} \models \phi(b, \bar{c})$$

$$\Rightarrow \mathcal{N}' \models \phi(b', \bar{c})$$

$$\Rightarrow \mathcal{N}' \models \exists y \phi(y, \bar{c})$$

$$\Rightarrow \mathcal{N} \models \exists y \phi(y, \bar{c}) \quad \text{as } \mathcal{N} \prec \mathcal{N}'$$

The converse is similar. □

Definition 0.6. *A theory T, which need not be complete, is model complete if $\mathcal{M}, \mathcal{N} \models T$ and $\mathcal{M} \subseteq N$ implies that $\mathcal{M} \prec \mathcal{N}$,*

Lemma 0.7. *The following are equivalent;*

(i). T is model complete.

(ii). Every $\phi(\bar{x}) \in \mathcal{L}$ is equivalent to an existential formula, mod T.

(iii). Every $\phi(\bar{x}) \in \mathcal{L}$ is equivalent to a universal formula, mod T.

Proof. $(ii) \Rightarrow (iii)$. We have that;

$T \vdash \phi(\bar{x}) \leftrightarrow \exists \bar{y} \psi(\bar{y}, \bar{x})$

$\therefore T \vdash \phi(\bar{x}) \leftrightarrow \neg \forall \bar{y} \neg \psi(\bar{y}, \bar{x})$

$\therefore T \vdash \phi(\bar{x}) \leftrightarrow \neg \exists \bar{z} \sigma(\bar{z}, \bar{x})$

$\therefore T \vdash \phi(\bar{x}) \leftrightarrow \forall \bar{z} \neg \sigma(\bar{z}, \bar{x})$

$(iii) \Rightarrow (ii)$ is similar.

$(ii), (iii) \Rightarrow (i)$

$\mathcal{M}, \mathcal{N} \models T$ and $\mathcal{M} \subseteq N$, $\phi(\bar{x}) \in \mathcal{L}$, $\bar{a} \in \mathcal{M}$;

$T \vdash \phi(\bar{x}) \leftrightarrow \exists \bar{y} \psi(\bar{y}, \bar{x}) \leftrightarrow \forall \bar{z} \sigma(\bar{z}, \bar{x})$

$\mathcal{M} \models \phi(\bar{a}) \Rightarrow \mathcal{M} \models \exists \bar{y} \psi(\bar{y}, \bar{a})$

$\Rightarrow \mathcal{M} \models \psi(\bar{b}, \bar{a})$

$\Rightarrow \mathcal{N} \models \psi(\bar{b}, \bar{a})$, as $\bar{b} \in \mathcal{M}$ and $\mathcal{M} \subseteq \mathcal{N}$

$\Rightarrow \mathcal{N} \models \exists \bar{y}(\bar{y}, \bar{a})$

$\Rightarrow \mathcal{N} \models \phi(\bar{a})$

$\mathcal{N} \models \phi(\bar{a}) \Rightarrow \mathcal{N} \models \forall \bar{z} \sigma(\bar{z}, \bar{a})$

$\Rightarrow \mathcal{N} \models \sigma(\bar{b}, \bar{a})$ for all $\bar{b} \in \mathcal{M}$

$\Rightarrow \mathcal{M} \models \sigma(\bar{b}, \bar{a})$, for all $\bar{b} \in \mathcal{M}$

$\Rightarrow \mathcal{M} \models \forall \bar{z} \sigma(\bar{z}, \bar{a})$

$\Rightarrow \mathcal{M} \models \phi(\bar{a})$ again

$(i) \Rightarrow (iii)$. Suppose that $\phi(\bar{x}) \in \mathcal{L}$ and let;

$\Gamma(\bar{x}) = \{\psi(\bar{x}) \in \mathcal{L} : \psi(\bar{x}) \textit{ universal in } \mathcal{L}, T \vdash \forall \bar{x}(\phi(\bar{x}) \rightarrow \psi(\bar{x}))\}$

We claim that $T+\Gamma\vdash\phi(\bar{x})$, $(*)$. If so, then;

$T\vdash(\bigwedge\psi_i(\bar{x}))\leftrightarrow\phi(\bar{x})$

$\therefore T\vdash\forall\bar{z}(\psi_1(\bar{x},\bar{z}))\wedge\forall\bar{y}(\psi_2(\bar{x},\bar{y}))\wedge\ldots\leftrightarrow\phi(\bar{x})$

$\therefore T\vdash\forall\bar{z}\forall\bar{y}\ldots(\psi_1(\bar{x},\bar{z})\wedge\psi_2(\bar{x},\bar{y})\wedge\ldots)\leftrightarrow\phi(\bar{x})$

and we are done. If not, then there exists a model $\mathcal{M}$ of $T+\Gamma(\bar{d})+\neg\phi(\bar{d})$. It is a straightforward exercise to check that for a theory T, the models of $T_\forall$, the universal consequences of T, are exactly those which have an extension that models T. Now;

$\mathcal{M}_{\bar{d}}\models(T+\phi(\bar{d}))_\forall\equiv\Gamma(\bar{d})$

Hence, $\mathcal{M}_{\bar{d}}\subseteq\mathcal{N}_{\bar{d}}\models T+\phi(\bar{d})$

By (i), $\mathcal{M}_{\bar{d}}\prec\mathcal{N}_{\bar{d}}$, so $\mathcal{M}\models\phi(\bar{d})$, a contradiction.

□

Definition 0.8. *If $\mathcal{M}\subseteq\mathcal{N}$, we say that $\mathcal{M}$ is existentially closed in $\mathcal{N}$, if for any existential formula $\phi(\bar{x})\in\mathcal{L}$, and $\bar{a}\in\mathcal{M}$, if $\mathcal{N}\models\phi(\bar{a})$, then $\mathcal{M}\models\phi(\bar{a})$*

Theorem 0.9. *Robinson's Test*

T is model complete iff for $\mathcal{M},\mathcal{N}\models\mathcal{T}$, $\mathcal{M}\subseteq\mathcal{N}$, $\mathcal{M}$ is existentially closed in $\mathcal{N}$

Proof. We first prove the following observation;

If $\mathcal{M}\subseteq\mathcal{N}$, then $\mathcal{M}\subseteq_{e.c}\mathcal{N}$ iff there exists $\mathcal{M}\prec\mathcal{M}'$ with $\mathcal{N}\subseteq\mathcal{M}'$ $(*)$

Suppose $(*)$ holds, and $\mathcal{N}\models\exists\bar{y}\phi(\bar{y},\bar{b})$, with $\phi(\bar{y},\bar{b})$ quantifier free

then $\quad\mathcal{M}'\models\exists\bar{y}\phi(\bar{y},\bar{b})$, as $\mathcal{N}\subseteq\mathcal{M}'$

and $\quad\mathcal{M}\models\exists\bar{y}\phi(\bar{y},\bar{b})$, as $\mathcal{M}\prec\mathcal{M}'$

so $\mathcal{M}\subseteq_{e.c}\mathcal{N}$. Conversely, suppose that $\mathcal{M}\subseteq_{e.c}\mathcal{N}$, then we show that;

$T'=Th(\mathcal{M},M)\cup q.f.diag(\mathcal{N},N)$ in $\mathcal{L}(\mathcal{N})$

is consistent. Suppose not, then;

$Th(\mathcal{M}, M) \vdash \neg\phi(\bar{a}, \bar{b})$ for $\phi(\bar{x}, \bar{y}) \in \mathcal{L}$, $\bar{a} \in \mathcal{M}$, $\bar{b} \in (\mathcal{N} \setminus \mathcal{M})$

Therefore, $Th(\mathcal{M}, M) \vdash \forall\bar{y}\neg\phi(\bar{a}, \bar{y})$

$$Th(\mathcal{M}, M) \vdash \neg\exists\bar{y}\phi(\bar{a}, \bar{y})$$

so $\mathcal{M} \not\models \exists\bar{y}\phi(\bar{a}, \bar{y})$, a contradiction, as now $\mathcal{M}$ is not existentially closed in $\mathcal{N}$. Robinson's result now follows easily, one direction follows trivially from the definition of model completeness. For the other direction, suppose that;

$\mathcal{M}_0 \subseteq \mathcal{N}_0$ with $\mathcal{M}_0, \mathcal{N}_0 \models T$. By hypothesis, we have that;

$\mathcal{M}_0 \subseteq_{e.c} \mathcal{N}_0 \subseteq_{e.c} \mathcal{M}_1 \subseteq_{e.c} \mathcal{N}_1$ $(**)$

where, by the previous observation, $\mathcal{M}_0 \prec \mathcal{M}_1 \models T$, and $\mathcal{N}_0 \prec \mathcal{N}_1 \models T$.

Continuing the chain $(**)$, we have that $\mathcal{M}_k \prec \mathcal{M}_{k+1}$ and $\mathcal{N}_k \prec \mathcal{N}_{k+1}$. Then $\mathcal{M}_0 \prec \bigcup \mathcal{M}_k = \bigcup \mathcal{N}_k \succ \mathcal{N}_0$. Hence, $\mathcal{M}_0 \prec \mathcal{N}_0$ as required. □

Remarks 0.10. *Be careful not to confuse Robinson's test for model completeness with the Tarski-Vaught test for elementary substructure. This says that if $\mathcal{M} \subseteq \mathcal{N}$, then $\mathcal{M} \prec \mathcal{N}$ iff for every formula $\exists y\phi(\bar{x}, y)$ and $\bar{a} \in \mathcal{M}$, if $\mathcal{N} \models \exists y\phi(\bar{a}, y)$, then there is $b \in \mathcal{M}$ with $\mathcal{M} \models \phi(\bar{a}, b)$. It is proved easily using induction on the complexity of formulae.*