THE GEOMETRY OF LINEAR REGULAR TYPES

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ABSTRACT. This paper is concerned with extending the results of [2] in the context of the solution set D of a regular $Lstp$ defined over $\emptyset$ in a simple theory T. In [7], a notion of p-weight is developed for regular types in stable theories. Here we show that the corresponding notion holds in simple theories and give a geometric analysis of associated structures $G(D)$ and $G(D)^{large}$, the former of which appears in [2]. We show that D is linear iff $G(D)$ and $G(D)^{large}$ (localized, resp) are both modular with respect to the p-closure operator cl_p. Finally, we show that modularity of $G(D)^{large}$ provides a local analogue of 1-basedness for the theory T.

1. PRELIMINARIES

For convenience we will assume that the ambient theory T is supersimple. In [4], Kim shows that forking inside simple theories satisfies the Independence Theorem over a model $\mathcal{M}$.

In order to apply the independence theorem over parameters, the notion of Lascar strong type is introduced. As is shown in [1], if T is supersimple, then T has elimination of hyperimaginaries and the notion of Lascar Strong Type simplifies to the following;

$Lstp(\bar{a}/A) = Lstp(\bar{b}/A)$ iff $tp(\bar{a}/acl(A)) = tp(\bar{b}/acl(A))$ iff $stp(\bar{a}/A) = stp(\bar{b}/A)$

where $acl(A)$ denotes the algebraic closure of A in $\mathcal{M}^{eq}$.

We can then apply the Independence Theorem for Lascar strong types;

If $\bar{c} \underset{A}{\downarrow} \bar{d}$, $Lstp(\bar{a}/A) = Lstp(\bar{b}/A)$ and $\bar{a} \underset{A}{\downarrow} \bar{c}$, $\bar{b} \underset{A}{\downarrow} \bar{d}$

then the nonforking extensions of $tp(\bar{a}/A) = tp(\bar{b}/A)$ to $A\bar{c}$ and $A\bar{d}$ respectively can be amalgamated.

Given a complete type $p(\bar{x}, b)$, where b denotes a possibly infinite set of parameters, we define the parallelism class of $p(\bar{x}, b)$ to be;

$$\mathfrak{B} = \{p(\bar{x}, c) : E(c, b)\}$$

where E is the transitive closure of the relation

$$R(c, b) \equiv p(\bar{x}, c) \cup p(\bar{x}, b) \text{ have a common non forking extension}$$

As is shown in [6], if b is an amalgamation base, then E is a type definable equivalence relation on $tp(b)$. If T is supersimple, then the parameter set of $stp(a/B)$ is an amalgamation base and we define the canonical base $C = Cb(Lstp(a/B))$ to be the E class of $acl(B)$. Again, assuming T supersimple, E is the intersection of definable equivalence relations on $tp(acl(B))$ and we may take C to be a possibly infinite set of parameters in $\mathcal{M}^{eq}$. Note the assumption that T is supersimple is not critical in what follows provided we work with $\mathcal{M}^{heq}$ instead of $\mathcal{M}^{eq}$. In general, we do not assume that our parameter sets are algebraically closed. As the notion of non-forking is invariant under algebraic closure in $\mathcal{M}^{eq}$, we often implicitly replace a parameter set B by its algebraic closure in $\mathcal{M}^{eq}$, hoping this will not cause confusion.

We will require the following facts about canonical bases as given in [6], [5] and [8];

Fact 1.1. *1. The Independence Theorem holds for the restriction of a Lstp over A to the base $C \subset acl(A)$.*

Let $A \subset B$ be sets and $\bar{a}$ a tuple, then;

2. $\bar{a} \underset{A}{\downarrow} B$ iff $Cb(Lstp(\bar{a}/B)) \subset acl(A)$. As a consequence, if $C = Cb(Lstp(\bar{a}/A))$, then $\bar{a} \underset{C}{\downarrow} A$ and $\bar{a} \underset{A}{\downarrow} C$

3. If $D = Cb(Lstp(\bar{a}/B))$ and $\bar{a} \underset{A}{\downarrow} B$, then, using the fact that C and D are amalgamation bases, $dcl(C) = dcl(D)$.

4. If $\{\bar{a}_i : i < \omega\}$ is a Morley sequence in $Lstp(\bar{a}/A)$, then $C = Cb(Lstp(\bar{a}/A)) \subset dcl(\bar{a}_i : i < \omega)$

Definition 1.2. *A pregeometry is a set S with a closure operation $cl : P(S) \to P(S)$ satisfying the following axioms found in [7];*

1. If $A \subseteq S$, then $A \subseteq cl(A), cl(A) = cl(cl(A))$.

2. If $A \subseteq B \subseteq S$, then $cl(A) \subseteq cl(B)$.

3. If $A \subseteq S$, $a, b \in S$, then $a \in cl(Ab) \setminus cl(A)$ implies $b \in cl(Aa)$.

4. If $a \in S$ and $a \in cl(A)$, then there is some finite $A_0 \subset A$ with $a \in cl(A_0)$.

We say that (S, cl) is modular if for A, B finite dimensional closed subsets of S, $dim(A \cup B) = dim(A) + dim(B) - dim(A \cap B)$.

Remarks 1.3. *A necessary and sufficient condition for modularity of a pregeometry (S, cl) is the following;*

Whenever $a, b \in S$, $B \subset S$ is closed and finite dimensional, $dim(ab) = 2$ and $dim(ab/B) \leq 1$, then there is $c \in cl(ab) \cap cl(B)$ with $c \notin cl(\emptyset)$ (*)

2. Regular Types and p-weight

Let p be a non-algebraic complete Lascar strong type over $\emptyset$. Recalling the definition of orthogonality in simple theories, see for example [8], we say that p is *regular* if it is orthogonal to all its forking extensions.

Lemma 2.1. *If p is regular, the realisations D of p form a pregeometry with the the closure operation cl given by $cl(A) = \{x \in p : x \not\!\downarrow_\emptyset A \}$.*

Proof. We check the axioms, 2 is trivial and 4 follows from the finite character of forking. 3 follows immediately from forking symmetry and all the work is in showing that 1 holds, namely we have to see that if $A \subset p$, $a, b_1 \ldots b_n$ is a tuple in p such that $b_i \not\!\downarrow_\emptyset A$ for each i and $a \not\!\downarrow_\emptyset b_1, \ldots, b_n$ then in fact $a \not\!\downarrow_\emptyset A$. Suppose not, so a

realises a non forking extension of ρ to A. Each b_i realises a forking extension of ρ to A so by definition of regularity, we must have that $a \underset{A}{\downarrow} b_1$. Now we just repeat the argument with Ab_1 replacing A, clearly $b_i \underset{\emptyset}{\not\downarrow} Ab_1$ for $i \geq 2$ and again using regularity $a \underset{Ab_1}{\downarrow} b_2$, so we get $a \underset{A}{\downarrow} b_1 b_2$. After n steps, using transitivity, we have that $a \underset{A}{\downarrow} b_1 \dots b_n$ and so, as $a \underset{\emptyset}{\downarrow} A$ we get $a \underset{\emptyset}{\downarrow} b_1 \dots b_n$. This contradicts the original hypothesis.

$\square$

Given a set of parameters A, we let $D_A = \{x \in D : x \underset{\emptyset}{\downarrow} A\}$ and define a closure operation cl_A by $cl_A(B) = \{x \in D_A : x \underset{A}{\not\downarrow} B\}$ for $B \subset D_A$. Imitating the above proof, one easily checks that (D_A, cl_A) forms a pregeometry which we refer to as the localisation p_A of p to A.

Given any pregeometry (S, cl), we use the standard notation (S', cl'), as in [2], for the associated geometry. For closed $B \subset S$, we have a notion of dimension $dim(B)$. For closed sets $B \subset C \subset S$, we define $dim(C/B) = dim(C) - dim(B)$ and for arbitrary sets $B, C \subset S$, we define $dim(C/B) = dim(cl(C \cup B)/cl(B))$. Recall that this notion is additive and the same holds for the localised analogue dim_A.

Let p_1 and p_2 be types over possibly different sets. We say that p_1 is *hereditarily orthogonal* to p_2 if every extension of p_1 is orthogonal to p_2. Now fix a regular complete Lstp p defined over $\emptyset$ and define a Lstp q over a domain B to be p-simple if, for all $a \in q$, there exists F with $B \subset F$, $F \underset{B}{\downarrow} a$ and $I \subset p_F$ such that $tp(a/FI)$ is hereditarily orthogonal to p. Given F as above, we say that F *witnesses the p-simplicity of q*. Note that by an automorphism argument and the fact that p was assumed to be defined over $\emptyset$, in order to check p-simplicity of a given type q, it is sufficient to verify it for any element realising q

We define a notion of p-weight for Lascar strong types q as follows;

$w_p(q) = min\{\kappa :$ there is $F \supset B$, $I \subset p_F$ as above with $dim_F(I) = \kappa\}$.

$w_p(a/B) = w_p(Lstp(a/B))$

We say that F *witnesses the p-weight of* q if F witnesses the p-simplicity of q and there exists I as above with $dim_F(I) = w_p(q)$.

Lemma 2.2. *If F witnesses the p-weight of $Lstp(a/B)$ and $G \supset F$ with $\underset{F}{\overset{G}{\downarrow}} a$ then G also witnesses the p-weight of $Lstp(a/B)$.*

Proof. We clearly have that $\underset{B}{\overset{G}{\downarrow}} a$. Let $I \subset p_F$ be independent with $Card(I) = w_p(Lstp(a/B))$ and $Lstp(a/FI)$ hereditarily orthogonal to p. Then still $Lstp(a/GI)$ is hereditarily orthogonal to p. Moreover, $I \subset p_G$ as if $c \in I$ with $\underset{F}{\overset{c}{\not\downarrow}} G$, then, by the finite character of forking, we can find a tuple $\bar{g}$ with $\underset{F}{\overset{\bar{g}}{\downarrow}} a$ and $\underset{F}{\overset{c}{\not\downarrow}} \bar{g}$, $\underset{F}{\overset{c}{\not\downarrow}} a$. As $Lstp(c/F)$ is regular, this contradicts the fact that $Lstp(c/F)$ has ordinary weight 1; a proof of this last fact can be found in for example [8]. Now, by definition of p-weight, we must have that $dim_G(I) = w_p(Lstp(a/B))$. $\square$

Lemma 2.3. *Suppose that $q=Lstp(a/B)$ is p-simple and F witnesses the p-weight of q. Let $Y = \{b : b \in p_F, \underset{F}{\overset{b}{\not\downarrow}} a \}$. Then $Lstp(a/FY)$ is hereditarily orthogonal to p and $w_p(q) = dim_F(Y)$.*

Proof. By the choice of F and regularity of p, we can find independent $I \subset p_F$ such that $Lstp(a/FI)$ is hereditarily orthogonal to p and $w_p(q) = Card(I)$. We can assume that $\underset{F}{\overset{b}{\not\downarrow}} a$ for all $b \in I$. Otherwise, choose $I_0 \subset I$ maximal with $\underset{F}{\overset{a}{\downarrow}} I_0$. Now replace F by FI_0 and I by $I \setminus I_0$, then the pair $(FI_0, I - I_0)$ work in the definition of $w_p(q)$, which is a contradiction unless $I_0 = \emptyset$. Hence $I \subset Y$. Moreover, I is a basis for Y over F. If not, we can find $y \in Y$ with

$y \underset{F}{\downarrow} I$ and $y \underset{F}{\not\downarrow} a$. If $y \underset{FI}{\downarrow} a$, then $y \underset{F}{\downarrow} aI$, contradict-

ing the definition of Y. Therefore, $y \underset{FI}{\not\downarrow} a$, contradicting the fact

that $Lstp(a/FI)$ is hereditarily orthogonal to p. $\square$

Theorem 2.4. *We have the following properties of p-weight, 0 only holding with the assumption that T is supersimple;*

  0. w_p is finite: If $Lstp(a/B)$ is p-simple, then $w_p(a/B)$ is finite.

  1. Non-Forking: If $B \subset C$, $a \underset{B}{\downarrow} C$ then $Lstp(a/B)$ is p-simple iff $Lstp(a/C)$ is p-simple and $w_p(a/B) = w_p(a/C)$.

  2. Extension: If $Lstp(a/B)$ is p-simple and $B \subset C$ then $Lstp(a/C)$ is p-simple and $w_p(a/C) \leq w_p(a/B)$

  3. Additivity: If $Lstp(a/B)$ and $Lstp(b/B)$ are p-simple, then so is $Lstp(ab/B)$ and $w_p(ab/B) = w_p(a/Bb) + w_p(b/B)$.

  4. Algebraicity: If $Lstp(a/B)$ is p simple and $b \in acl(aB)$ then $Lstp(b/B)$ is p-simple and $w_p(b/B) \leq w_p(a/B)$

  5. Finite Character: If $Lstp(a/B)$ is p-simple and $B \subset C$ then there exists a finite $\bar{c} \subset C$ such that $w_p(a/C) = w_p(a/B\bar{c})$

  6. Permutation: $Lstp(ab/B)$ is p-simple iff $Lstp(ba/B)$ is p-simple and $w_p(ab/B) = w_p(ba/B)$

Proof. These are adaptations of the corresponding properties in the stable case;

0. Let $F \supset B$ witness p-weight for $Lstp(a/B)$ and Y the set given by Lemma 2.3. Then, it's sufficient to prove that $dim_F(Y)$ is finite. Suppose not, then we can find an infinite sequence $\{c_i : 0 \leq i < \omega\}$ independent over F such that $c_i \underset{F}{\not\downarrow} a$. By transitivity of non-forking,

we have that $c_i \underset{Fc_0 \ldots c_{i-1}}{\not\downarrow} a$ for each i. This gives an infinite forking chain and contradicts the fact that T is supersimple.

1. Suppose $Lstp(a/B)$ is p-simple and let F witness the p-simplicity. Choose $F' \equiv_{aB} F$ with $F' \underset{aB}{\downarrow} C$. Then $F' \underset{B}{\downarrow} aC$ and $F' \underset{C}{\downarrow} a$. By automorphism, F' witnesses the p-simplicity of $Lstp(a/B)$, so we can find $I' \subset p_{F'}$ with $tp(a/F'I')$ hereditarily orthogonal to p. Then $tp(a/F'I'C)$ is hereditarily orthogonal to p which shows that $Lstp(a/C)$ is p-simple. Now suppose that F witnesses the p-simplicity of $Lstp(a/B)$ and $I \subset p_F$ with $dim_F(I) = \kappa$, $Lstp(a/FI)$ hereditarily orthogonal to p. By the same argument, we may assume that $F \underset{C}{\downarrow} a$. Then $tp(a/FCI)$ is hereditarily orthogonal to p and $dim_{FC}(I) \leq \kappa$, which shows that $w_p(a/C) \leq w_p(a/B)$. Conversely, suppose that F witnesses p-simplicity of $Lstp(a/C)$ and $I \subset p_F$ with $dim_F(I) = \kappa$, $Lstp(a/FI)$ hereditarily orthogonal to p. Then $a \underset{B}{\downarrow} F$ as $Lstp(a/C)$ is a non-forking extension of $Lstp(a/B)$. This shows that $Lstp(a/B)$ is p-simple and $w_p(a/B) \leq w_p(a/C)$.

2. Let B_1 witness p-weight for $Lstp(a/B)$. By the usual arguments, we may assume that $B_1 \underset{B}{\downarrow} aC$, hence $a \underset{C}{\downarrow} B_1$. Now choose $I_1 \subset p_{B_1}$ with $w_p(Lstp(a/B)) = dim_{B_1}(I_1)$ and $Lstp(a/B_1I_1)$ hereditarily orthogonal to p. Then still $Lstp(a/B_1CI_1)$ $(*)$ is hereditarily orthogonal to p. Now choose $I_1' \subset I_1$ with $I_1' \subset p_{B_1C}$ and let J_1 be a basis for I_1' over B_1C. Then we claim that $Lstp(a/B_1CJ_1)$ is hereditarily orthogonal to p. As $dim_{B_1C}(J_1) = dim_{B_1C}(I_1') \leq dim_{B_1}(I_1) = w_p(Lstp(a/B))$, this is sufficient to prove 2. In order to show the claim, let $F \supset B_1CJ_1$ and suppose that $d \in p_F$. Without loss of generality, we may assume that still $I_1' \subset p_F$ and J_1 is still a basis for I_1' over F, otherwise take the corresponding subsets. By regularity of p, $b \underset{F}{\downarrow} I_1 \setminus I_1'$, hence $b \in p_{F,I_1 \setminus I_1'}$. If $b \underset{F,I_1 \setminus I_1'}{\not\downarrow} I_1'$, then $b \in cl_{F,I_1 \setminus I_1'}(I_1')$ and $I_1' \subset cl_{F,I_1 \setminus I_1'}(J_1)$ implies $b \in cl_{F,I_1 \setminus I_1'}(J_1)$ by transitivity of $cl_{F,I_1 \setminus I_1'}$. Therefore, $b \underset{F,I_1 \setminus I_1'}{\not\downarrow} J_1$ contradicting the fact that $J_1 \subset F$ and we conclude that $b \in p_{FI_1}$. By $(*)$, we have that $b \underset{FI_1}{\downarrow} a$ and the previous argument shows $b \underset{F}{\downarrow} I_1$, hence $b \underset{F}{\downarrow} a$ as required.

3. For the first part of 3, suppose that $Lstp(a/B)$ and $Lstp(b/B)$ are p-simple and let $B_1, B_2 \supset B$ witness their p-simplicity. Choose $b' \equiv_{Ba} b$, with $b' \underset{Ba}{\smile} B_1$ and find $C \equiv_B B_2$ with C witnessing p-simplicity for $Lstp(b'/B)$. Choose $B_2' \equiv_{Bb'} C$ with $B_2' \underset{Bb'}{\smile} aB_1$, then, as $B_2' \underset{B}{\smile} b'$, we have that $B_2' \underset{B}{\smile} ab'B_1$. Therefore, $B_2' \underset{B_1a}{\smile} b'$ and $B_2' \underset{B_1}{\smile} a$. This gives $ab' \equiv_B ab$ and $ab' \underset{B}{\smile} B_1B_2'$ with B_1 witnessing p-simplicity for $Lstp(a/B)$ and B_2' witnessing p-simplicity for $Lstp(b'/B)$. Then we can find I_1, I_2 such that $Lstp(a/B_1I_1)$ and $Lstp(b'/B_2'I_2)$ are both hereditarily orthogonal to p. We then claim that $Lstp(ab'/B_1B_2'I_1I_2)$ is hereditarily orthogonal to p. If not, then we can find $F \supset B_1B_2'I_1I_2$ and $c \in p_F$ such that $ab' \underset{F}{\not\smile} c$. Then $c \underset{F}{\not\smile} b'$ or $c \underset{Fb'}{\not\smile} a$. In either case, we have a contradiction. Hence, $Lstp(ab'/B) = Lstp(ab/B)$ is p-simple.

In order to show that w_p is additive, let B_1 witness p-weight for $Lstp(a/B)$ with corresponding I_1. By the first part, $Lsp(ab/B)$ is p-simple so we can find B_2 witnessing the p-weight of $Lstp(ab/B)$. Similar to the above, we may assume that $B_2 \underset{B}{\smile} abI_1B_1$ and $ab \underset{B}{\smile} B_1B_2$. By Lemma 2.2, we may assume that B_1B_2 witness p-weight both for $Lstp(a/B)$ and $Lstp(ab/B)$. Now choose I_2 maximal with I_1I_2 independent over B_1B_2 and such that $c \underset{B_1B_2}{\not\smile} ab$ for $c \in I_2$. Using Lemma 2.3, we have that $Lstp(ab/B_1B_2I_1I_2)$ is hereditarily orthogonal to p and $w_p(ab/B) = dim_{B_1B_2}(I_1I_2) = dim_{B_1B_2}(I_1) + dim_{B_1B_2}(I_2) = Card(I_2)+w_p(a/B)$, the last equality following form the fact that B_1B_2 witnesses p-weight for $Lstp(a/B)$. It is therefore sufficient to prove that $w_p(b/aB) = Card(I_2)$. Choose parameters $B_3 \supset aB$ witnessing p-weight for $Lstp(b/aB)$. We may assume that $b \underset{aB}{\smile} B_1B_2B_3I_1$ and $B_3 \underset{aB}{\smile} bB_1B_2I_1I_2$. By Lemma 2.2, $B_1B_2B_3I_1$ witnesses the p-weight of $Lstp(b/aB)$ (1). We have that I_2 is independent over $aB_1B_2I_1$ (∗), as $I_2 \underset{B_1B_2I_1}{\smile} a$ by the fact that $Lstp(a/B_1I_1)$ is hereditarily

orthogonal to p and I_2 is independent over $I_1B_1B_2$ by definition. Hence, by choice of B_3, I_2 is independent over $B_1B_2B_3I_1$ (2). Now, if $c \in I_2$, then $c \underset{B_1B_2B_3I_1}{\not\downarrow} b$ (3). Otherwise, by $(*)$, $c \underset{B_1B_2I_1}{\downarrow} ab$. Now $c \underset{B_1B_2}{\downarrow} I_1$ which gives $c \underset{B_1B_2}{\downarrow} ab$. This contradicts the choice of I_2. Combining $(1),(2),(3)$ and Lemma 1.3, it's sufficient to show that if $d \in p_{B_1B_2B_3I_1I_2}$ $(**)$, then $d \underset{B_1B_2B_3I_1}{\downarrow} b$. We already have that $Lstp(ab/B_1B_2I_{1!}I_2)$ is hereditarily orthogonal to p, hence so is $Lstp(ab/B_1B_2B_3I_1I_2)$ and $Lstp(b/B_1B_2B_3I_1I_2)$. Now, if $(**)$ holds, we have that $d \underset{B_1B_2B_3I_1I_2}{\downarrow} b$. Then $d \underset{B_1B_2B_3I_1}{\downarrow} I_2$ implies $d \underset{B_1B_2B_3I_1}{\downarrow} b$ as required.

$\square$

Remarks 2.5. *Note that a trivial consequence of 3. is that for a tuple of elements $\bar{a} \subset p$ and parameters F, $w_p(\bar{a}/F) = dim_F(\bar{a}')$ where $\bar{a}' \subset \bar{a}$ is the subtuple of elements in p_F.*

4. Suppose $C \supset B$ witnesses p-weight for a, then, as $b \in acl(aB)$, $C \underset{B}{\downarrow} b$. Choose $I \subset p_C$ with $Lstp(a/CI)$ $(*)$ hereditarily orthogonal to p and $w_p(a/B) = Card(I)$. Then still $Lstp(b/CI)$ is hereditarily orthogonal to p. For suppose not, then there exists $D \supset CI$ and $d \in p_D$ with $d \underset{D}{\not\downarrow} b$, hence $d \underset{D}{\not\downarrow} a$, contradicting $(*)$. This clearly shows 4.

As a corollary of 4, we have the following;

Lemma 2.6. *Suppose that $Lstp(a/B)$ is p-simple, $B \subset C$ and $E = Cb(a/C)$, then $Lstp(E/B)$ is p-simple with finite p-weight.*

Proof. By 2 of Fact 1.1 and the assumption that T is supersimple, we can find a finite set $E_0 \subset E$ with $E \subset acl(E_0)$. By 4 of Fact 1.1, we can find a finite Morley sequence with $E_0 \subset dcl(a_i : 1 \le i \le n)$, hence $E \subset acl(a_i : 1 \le i \le n)$. Now, using 3,4 of Theorem 2.4, we have that $Lstp(E/B)$ is p-simple and $w_p(E/B) \le nw_p(a/B)$ as required.

$\square$

We also require the following, the proof is the same as for the stable case, given in [7];

Lemma 2.7. *Suppose $Lstp(a/B)$ is p-simple and $B \subset C$ with $w_p(a/B) = w_p(a/C) = n$. Then, if $E = Cb(a/C)$, $E \subset cl_p(B)$.*

Remarks 2.8. *We can consider C as a finite tuple in $\mathcal{M}^{eq}$ up to interalgebraicity.*

5. Choose $F \supset B$ witnessing p-weight for $Lstp(a/B)$ with $F \underset{B}{\downarrow} aC$.
Then we can find $I \subset p_F$ with $Card(I) = w_p(a/B)$ and $w_p(a/FI) = 0$.
Using 3, we also calculate $w_p(I/aF) = 0$. Now, by 1, $w_p(a/C) = w_p(a/FC)$. Using 2,3, $w_p(a/FC) = w_p(aI/FC) = w_p(I/FC)$. Hence, $w_p(a/C) = w_p(I/FC)$. Now, by the remarks after 3, we can find $J \subset I' \subset I$ with I' the subtuple belonging to p_{FC}, J a basis for I' over FC and $w_p(I/FC) = w_p(J/FC)$. Now, if $i \in I \setminus I'$, then $i \underset{\emptyset}{\not\downarrow} FC$
and if $i \in I' \setminus J$, then $i \underset{FC}{\not\downarrow} J$, hence, if $i \in I \setminus J$, then $i \underset{\emptyset}{\not\downarrow} FCJ$
($*$). By the finite character of forking, we can find a finite tuple $\bar{c} \subset C$ such that ($*$) holds replacing FCJ by $F\bar{c}J$. Then still I' is the subtuple of I belonging to $p_{F\bar{c}}$ and J is a basis for I' over $F\bar{c}$. Hence $w_p(I/FC) = w_p(I/F\bar{c})$. Using 2,3 again, $w_p(a/F\bar{c}) = w_p(I/F\bar{c})$ and by 1 again, $w_p(a/F\bar{c}) = w_p(a/B\bar{c})$ as required.

6. The proof of 6 is trivial by the definitions of p-simplicity, p-weight and the fact that for $B \subset C$, $ab \underset{B}{\downarrow} C$ iff $ba \underset{B}{\downarrow} C$.

In order to make the definitions of Section 3, we require one more notion.

Definition 2.9. *$Lstp(a/B)$ is p-pure if for every $B \subset C$, $w_p(a/B) = w_p(a/C)$ iff $a \underset{B}{\downarrow} C$.*

The fundamental results on p-pure types are the following;

Lemma 2.10. *Suppose that $Lstp(a/X)$ is p-simple and $w_p(e/X) \geq 1$ for all $e \in acl(aX) \setminus acl(X)$ with $Lstp(e/X)$ p-simple. Then $Lstp(a/X)$ is p-pure.*

Proof. Choose $F \supset X$ witnessing p-weight for $Lstp(a/X)$ and $I \subset p_F$ with $Lstp(a/FI)$ hereditarily orthogonal to p and $dim_F(I) = w_p(a/F)$.

We first show that a and I are equidominant over F, see [8] for the relevant definition. Suppose that $c \not\!\!\downarrow_F I$ $(*)$. As $w_p(a/FI) = w_p(I/Fa) = 0$, we have $w_p(a/Fc) = w_p(I/Fc)$ by 2,3 of Theorem 2.4. By $(*)$ and regularity of p, $w_p(I/Fc) < w_p(I/F)$. Hence, $w_p(a/Fc) < w_p(a/F)$, which, by 1 of Theorem 2.4, implies $c \not\!\!\downarrow_F a$. This shows that a dominates I over F. We want to see that I dominates a over F as well. Suppose not, then we find c with $c \downarrow_F I$ and $c \not\!\!\downarrow_F a$.

Let $E = Cb(Lstp(aI/cF))$, then, by $aI \not\!\!\downarrow_F c$ and 2. of Fact 1.1, $E \subset acl(cF)$ and we can find $e \in E \setminus acl(F)$ with $e \not\!\!\downarrow_F a$ $(*)$. By 4 of Fact 1.1, we can find a Morley sequence $\{a_1 I_1, \ldots, a_n I_n\}$ realising $Lstp(aI/cF)$ such that $e \in acl(a_1 I_1, \ldots, a_n I_n)$. Then, by 3,4 of Theorem 2.4, $Lstp(e/FI_1! \ldots I_n)$ is p-simple and $w_p(e/FI_1 \ldots I_n) = 0$. As $I_1 \ldots I_n \downarrow_F c$ and $e \in acl(cF)$, we have, by 1 of Theorem 2.4, that $Lstp(e/F)$ is p-simple and $w_p(e/F) = 0$. Finally, let $C' = Cb(Lstp(eF/aX))$, then, by $(*)$ and 2. of Fact 1.1, we can find $c' \in C'$ with $c' \in acl(aX) \setminus acl(X)$. As $Lstp(a/X)$ is p-simple, by 4 of Theorem 2.4 we have that $Lstp(c'/X)$ is p-simple. Again, by 4 of Fact 1.1, we can find a Morley sequence $\{e_1 F_1, \ldots, e_n F_n\}$ realising $Lstp(eF/aX)$ such that $c' \in acl(e_1 F_1, \ldots, e_n F_n)$. Then again by 3,4 of Theorem 2.4, $w_p(c'/XF_1 \ldots F_n) = 0$ and, as $c' \downarrow_X F_1 \ldots F_n$, by 1 of Theorem 2.4, $Lstp(c'/X)$ is p-simple and $w_p(c'/X) = 0$ as well. This contradicts the assumption of the Lemma, so a and I are equidominant over F. Finally, suppose that $X \subset Y$ and $w_p(a/X) = w_p(a/Y)$. Choose F as above with $F \downarrow_X aY$ and $I \subset p!_F$ such that a and I are equidominant over F. Then, by 1 of Theorem 1.4, $w_p(I/FY) = w_p(a/FY) = w_p(a/Y) = w_p(a/X) = w_p(I/F)$. Hence, $I \subset p_{FY}$ and $I \downarrow_F Y$ by regularity of p. Then $a \downarrow_F Y$ as I dominates a over F and $a \downarrow_X Y$. This shows that $Lstp(a/X)$ is p-pure as required.

$\square$

Lemma 2.11. *Suppose $Lstp(a/B)$ is p-simple and $w_p(a/B) = n$.*

Let $B_a^{reg} = \{b \in acl(aB) : w_p(b/B) = 0\}$, so $B \subset B_a^{reg}$, then $Lstp(a/B_a^{reg})$ is p-pure and for any c with $Lstp(c/B)$ p-simple, $w_p(c/B) = w_p(c/B_a^{reg})$.

Proof. We first claim that if $Lstp(e/B_a^{reg})$ is p-simple and $e \in acl(aB_a^{reg}) \backslash acl(B_a^{reg})$ then $w_p(e/B_a^{reg}) \geq 1$ (*). As $e \in acl(aB)$, by 4 of Theorem 1.4, $Lstp(e/B)$ is p-simple. Suppose (*) fails, then, using 5, we can find a finite tuple $\bar{b} \in B_a^{reg}$ with $w_p(e/B\bar{b}) = 0$. Now, using 2,3,6 and the definition of B_a^{reg}, we can easily calculate $w_p(e/B) = w_p(\bar{b}e/B) = w_p(e\bar{b}/B) = 0$. As $e \in acl(aB)$, we conclude that $e \in B_a^{reg}$, contradicting the assumption. Now by (*) and Lemma 2.10, we conclude that $Lstp(a/B_a^{reg})$ is p-pure. For the second part, again we can find a finite tuple $\bar{b} \in B_a^{reg}$ with $w_p(c/B_a^{reg}) = w_p(c/B\bar{b})$. Again, a simple weight calculation gives that $w_p(c/B\bar{b}) = w_p(c\bar{b}/B) = w_p(c/B)$ as required. □

Lemma 2.12. *Suppose that $Lstp(a/B)$ is p-simple,* $a \underset{B}{\downarrow} C$ *and $Lstp(a/C)$ is p-pure, then $Lstp(a/B)$ is p-pure.*

Proof. Choose $F \supset C$ witnessing the p-weight of $Lstp(a/C)$, so F also witnesses the p-weight of $Lstp(a/B)$. If $I \subset p_F$ is chosen with $Lstp(a/FI)$ hereditarily orthogonal to p and $dim_F(I) = w_p(a/F)$, then, as we have seen, a and I are weight equivalent over F. As is easily seen, $Lstp(I/F)$ is p-pure, hence, as $Lstp(a/F)$ is p-pure, an easy check gives that a and I are domination equivalent over F. Now repeat the proof at the end of Lemma 2.10. □

3. Linearity and 1-Basedness

Definition 3.1. *We say that D is linear if the following holds;*

If ab is a pair in D with $Lstp(ab/B)$ p-pure having p-weight 1, then $w_p(C/\emptyset) \leq 1$ where $C = Cb(Lstp(ab/B))$.

We also introduce the following 2 objects.

Definition 3.2. $G(D) = \{c : Lstp(c/\emptyset)$ *is p-simple of p-weight 1$\}$

and

$G(D)^{large} = \{c : Lstp(c/\emptyset) \text{ is } p\text{-simple of finite } p\text{-weight}\}$

with corresponding localised structures;

$G(D)_A = \{c \in G(D) : w_p(c/A) = 1\}$

and

$G(D)_A^{large} = \{c \in G(D)^{large} : w_p(c/A) \text{ is } finite\}$

Note that by 2 of Theorem 2.4, $G(D)_A^{large} = G(D)^{large}$. We define a closure operator cl_p on $G(D)^{large}$ by $cl_p(B) = \{c \in G(D)^{large} : w_p(c/B) = 0$. We also have a corresponding operator cl_p by restriction to $G(D)$ and localised operators $cl_{p,A}$ on $G(D)_A$ and $G(D)_A^{large}$.

Lemma 3.3. *The following properties hold for cl_p.*

1. *cl_p is transitive for $G(D)$(local.resp.) and $G(D)^{large}$(local.resp.)*

2. *cl_p is finite for $G(D)$(local.resp) and $G(D)^{large}$(local.resp.).*

3. *cl_p satisfies exchange on $G(D)$(local.resp).*

4. *$G(D)$ and $G(D)_A$ form pregeometries under cl_p and $cl_{p,A}$*

Proof. 1. For suppose that $\bar{a} \in cl_p(\bar{b})$ and $\bar{b} \in cl_p(\bar{c})$ then $w_p(\bar{a}\bar{b}\bar{c}/\emptyset) = w_p(\bar{c}/\emptyset)$ by 3. of Theorem 2.4 and $w_p(\bar{b}\bar{a}\bar{c}/\emptyset) = w_p(\bar{a}\bar{c}/\emptyset) = w_p(\bar{a}/\bar{c}) + w_p(\bar{c}/\emptyset)$, so $w_p(\bar{a}/\bar{c}) = 0$ and $\bar{a} \in cl_p(\bar{c})$. The localised proof is similar.

2. For suppose that $B \subset G(D)^{large}$ or $B \subset G(D)$ and $\bar{a} \in cl_p(B)$ then ,by property 4 of w_p, there is a finite $\bar{b} \subset acl(B)$ such that $\bar{a} \in cl_p(\bar{b})$. By transitivity of cl_p and the fact that algebraic types have p-weight 0, we can assume that $\bar{b} \in B$.

3. For suppose that $aBc \subset G(D)$ and $a \in cl_p(Bc)\backslash cl_p(B)$. Replacing B by B_c^{reg} and using Lemma 2.11 we may assume that $w_p(a/B_c^{reg}) = 1$ and $Lstp(c/B_c^{reg})$ is p-pure. Then, as $w_p(a/B_c^{reg}c) = 0$, by the extension property we must have that $a \underset{B_c^{reg}}{\not\smile} c$. Therefore, as $c \in G(D)$ and $Lstp(c/B_c^{reg})$ is p-pure, $w_p(c/B_c^{reg}a) = 0$, that is $c \in cl_p(B^{reg}a)$.

Then, by transitivity of cl_p on $G(D)^{large}$, $c \in cl_p(Ba)$.

4. The axioms 1,3 and 4 for a pregeometry in Definition 1.2 are easily verified from the previous parts of the lemma. 2 is a straightforward consequence of the extension property for w_p.

$\square$

Definition 3.4. *We say that $G(D)$ is linear if (ab) is a pair from $G(D)$ and $B \subset G(D)$ such that $Lstp(ab/B)$ is p-pure with $w_p(ab/B) = 1$, then $w_p(C) \leq 1$ where $C = Cb(Lstp(ab/B))$.*

We now prove the following;

Lemma 3.5. *If D is linear then $G(D)$ is linear.*

Proof: The proof will be similar to [2]. Let (ab) be a pair from $G(D)$ with $w_p(ab) = 2$, the case for $w_p(ab) = 1$ is easier, and suppose that $B \subset G(D)$ with $w_p(ab/B) = 1$ and $Lstp(ab/B)$ p-pure. Let F witness the p-weight of $Lstp(ab/B)$ with $\begin{smallmatrix} F & \downarrow & B \\ & ab & \end{smallmatrix}$. Then we can find $cd \subset p_F$ such that $w_p(ab/Fcd) = 0$, $w_p(cd/F) = 2$ and $w_p(ab/F) = 2$. By additivity of p-weight we must have $w_p(cd/abF) = 0$ as well.

Claim 1: $w_p(cd/FB) = 1$.

As $w_p(ab/B) = 1$, $F \downarrow_B ab$ and w_p is invariant under non forking extension we have $w_p(ab/FB) = 1$. Then

$$w_p(abcd/FB) = w_p(ab/cdFB) + w_p(cd/FB) = w_p(cd/FB) =$$

$$w_p(cd/abFB) + w_p(ab/FB) = 0 + 1 = 1$$

giving the claim.

Now replace FB by $FB^{reg}_{cd} = \{b \in acl(cdFB) : w_p(b/FB) = 0\}$

By Lemma 2.11, $Lstp(cd/FB^{reg}_{cd})$ is p-pure with p-weight 1 and, by linearity of D, $w_p(C) \leq 1$ where $C = Cb(Lstp(cd/FB^{reg}_{cd}))$. Then

Claim 2: $w_p(cd/cl_p(cd) \cap acl(FB^{reg}_{cd})) = 1$

We have that $cd \downarrow_C FB$, hence $w_p(cd/C) = 1$. Then, by additivity of w_p and linearity of D, we calculate $w_p(C/cd) = w_p(C) - 1 = 0$, therefore $C \subset cl_p(cd)$ As $C \subset cl_p(cd) \cap acl(FB^{reg})$ and $cl_p(cd) \cap acl(FB^{reg}) \subset acl(FB^{reg}_{cd})$ the claim is shown.

Claim 3: $w_p(ab/W) = 1$, where $W = cl_{p,F}(ab) \cap acl(FB^{reg}_{cd}) \cup F$

We clearly still have that $w_p(cd/cl_{p,F}(cd) \cap acl(FB^{reg}) \cup F) = 1$. Using additivity, $w_p(ab/cl_{p,F}(cd)) \cap acl(FB^{reg}) \cup F) = 1$. By transitivity of p-closure, we must have that $cl_{p,F}(cd) = cl_{p,F}(ab)$, hence $w_p(ab/W) = 1$ as required.

Now let $C' = Cb(Lstp(ab/B))$. Then

Claim 4: $w_p(ab/WC') = 1$

If not, then as $C' \in acl(B)$, $ab \in cl_p(FB^{reg}_{cd})$. Again by transitivity of p-closure and the definition of FB^{reg}_{cd} we must have $ab \in cl_p(FB)$. Then, as $ab \downarrow_B F$, $ab \in cl_p(B)$, contradicting the fact $w_p(ab/B) = 1$ and giving the claim.

Now $ab \downarrow_{C'} B$ so still $w_p(ab/C') = 1$ and moreover, by Lemma 2.11, $Lstp(ab/C')$ is still p-pure. Then, by definition of p-purity, we must have that $ab \downarrow_{C'} W$ and so $C' \in acl(W)$. Then $C' \subset cl_{p,F}(ab)$ and as $C' \downarrow_{ab} F$, we must have $C' \subset cl_p(ab)$. Now, by calculating $w_p(abC')$, we have that $w_p(C') = 1$ as required.

Lemma 3.6. *If $G(D)$ is linear then $G(D)$ is modular.*

Proof: As $G(D)$ forms a pregeometry, it is sufficient to check the criterion $(*)$ in Remarks 1.3. So choose x_1x_2 in $G(D)$ with $w_p(x_1x_2) = 2$ and Y closed finite dimensional such that $w_p(x_1x_2/Y) = 1$. By finiteness, we can find $\bar{y} \subset Y$ such that $w_p(x_1x_2/\bar{y}) = 1$ and $cl_p(\bar{y}) = Y$. Replace $\bar{y}$ by $\bar{y}^{reg}_{x_1x_2}$, so we can assume that $Lstp(x_1x_2/\bar{y}^{reg}_{x_1x_2})$ is p-pure, though $\bar{y}^{reg}_{x_1x_2}$ may no longer be contained in $G(D)$. Now, using 3. of Fact 1.1 and Lemma 2.12, we can replace the parameters $\bar{y}^{reg}_{x_1x_2}$ by a Morley sequence $W \subset G(D)$ such that $Lstp(x_1x_2/W)$ is still p-pure and the canonical base is preserved. By linearity of $G(D)$, we have that $C = Cb(Lstp(x_1x_2/W)) = Cb(Lstp(x_1x_2/\bar{y}^{reg}_{x_1x_2}) \subset cl_p(x_1x_2) \cap cl_p(\bar{y})$. As $w_p(C) = 1$, we can find a tuple $\bar{c} \in G(D)$ such that $\bar{c}$ and $! C$ are interalgebraic. Then in fact $\bar{c}$ witnesses the criterion $(*)$ as required.

The 2 lemmas combine to give the following theorem.

Theorem 3.7. *If D is linear then $G(D)$ is modular.*

Even though $G(D)^{large}$ is not a pregeometry it still makes sense to talk of the dimension of a closed set.

Definition 3.8. *Given $X, Y \subset G(D)^{large}$ closed*

$dim(X/Y) = max\{w_p(\bar{a}/Y) : \bar{a} \in X\}$ *if this set is bounded.*

and $dim(X/Y) = \infty$ otherwise.

Definition 3.9. *$G(D)^{large}$ is modular if the following holds ;*

For finite dimensional closed $X, Y \subset G(D)^{large}$ $dim(X/Y) = dim(X/X \cap Y)$.

We now prove the following;

Theorem 3.10. *If D is linear then $G(D)^{large}$ is modular*

Here the problem is made more difficuly by the fact that $G(D)^{large}$ is not a pregeometry.

Proof. We first reduce the problem to a finite one, as in in general $cl_p(X)$ will be a very large set! Suppose $G(D)^{large}$ is not modular, then there exists closed sets X and Y such that $dim(X/Y) < dim(X/X \cap Y)$. Taking $\bar{x} \in X$ so that $w_p(\bar{x}/Y)$ is maximal, by definition we have that $w_p(\bar{x}/X \cap Y) < w_p(\bar{x}/Y)$. By finiteness, I can find $\bar{c} \subset X \cap Y$ and $\bar{y} \subset Y$ such that $w_p(\bar{x}/\bar{c}) < w_p(\bar{x}/\bar{y})$ and, moreover, as weight is preserved on both sides, we can take $\bar{c}$ and $\bar{y}$ such $cl_p(\bar{c}) = X \cap Y$ and $cl_p(\bar{y}) = Y$. Therefore, it is sufficient to prove that

$w_p(\bar{x}/\bar{y}) = w_p(\bar{x}/\bar{c})$ where $cl_p(\bar{c}) = cl_p(\bar{x}) \cap cl_p(\bar{y})$ (*)

We show (*) by induction on $w_p(\bar{x}/\bar{y})$ for $\bar{x}$ and $\bar{y}$ finite tuples from $G(D)^{large}$.

Base Case. $w_p(\bar{x}/\bar{y}) = 1$.

Suppose $w_p(\bar{x}) = n$, then I can find $F \downarrow \bar{x}\bar{y}$ and $z_1 \ldots z_n \in p_F$ such that $\bar{x}$ and $z_1 \ldots z_n$ are weight equivalent over F (*). As before, one checks that $w_p(z_1 \ldots z_n/F\bar{y}) = 1$. Without loss of generality, we can assume that $w_p(z_i/F\bar{y}) = 1$ for each i. Now, adding parameters $e_1 \ldots e_n \subset cl_p(\emptyset)$, we may assume that $Lstp(z_i/e_1 \ldots e_n)$ is p-pure for all i and all the conditions are preserved with $F\bar{y}e_1 \ldots e_n$ replacing $F\bar{y}$. We must have that $w_p(z_1 z_i/F\bar{y}e_1 \ldots e_n) = 1$ for all i, hence by linearity of D, we can find $c_i \in G(D)$ for $i \geq 2$ with $cl_p(c_i) = cl_p(z_1 z_i) \cap cl_{p,F}(\bar{y}e_1 \ldots e_n)$. Clearly, $e_i \subset cl_p(c_i)$, so without loss of generality $e_i \subset c_i$. Now $w_p(c_i/z_1 z_i) = 0$ and $w_p(c_i/z_1) = 1$, otherwise $z_1 \underset{e_1}{\not\downarrow} c_i$ and $z_1 \subset cl_{p,F}(\bar{y})$. Hence $z_i \underset{z_1 e_i}{\not\downarrow} c_i$. As $w_p(z_i/e_i z_1) = 1$ and $Lstp(z_i/e_i z_1)$ is p-pure, we have that $w_p(z_i/z_1 c_i) = 0$ so $z_i \subset cl_p(z_1 c_i)$. We want to show that $w_p(c_2 \ldots c_n) = n - 1$ from which, taking $\bar{c} = c_2 \ldots c_n$, we clearly have that $w_p(z_1 \ldots z_n/\bar{c}) = w_p(z_1 \ldots z_n/F\bar{y})$ and $cl_p(\bar{c}) = cl_p(z_1 \ldots z_n) \cap cl_{p,F}(\bar{y})$. Suppose not, say $c_n \subset cl_p(c_2 \ldots c_{n-1})$, then as $z_n \subset cl_p(z_1 c_n)$ and $c_2 \ldots c_{n-1} \subset cl_p(z_1 \ldots z_{n-1})$, we have that $z_n \subset cl_p(z_1 \ldots z_{n-1})$ contradicting the fact that $z_1 \ldots z_n$ are independent realisations of p_F. Now, $cl_p(\bar{z}) \cap cl_{p,F}(\bar{y}) \subset cl_{p,F}(\bar{z}) \cap cl_{p,F}(\bar{y}) \cup F$. Therefore, using 3. of Theorem 2.4 to check that $w_p(z_1 \ldots z_n/F\bar{y}) = w_p(z_1 \ldots z_n/cl_{p,F}(\bar{y}) \cup F)$, we have that $w_p(\bar{z}/F\bar{y}) = w_p(\bar{z}/cl_{p,F}(\bar{z}) \cap cl_{p,F}(\bar{y}) \cup F)$. Let $W = cl_{p,F}(\bar{x}) \cap cl_{p,F}(\bar{y}) \cup F$, then, using (∗), $w_p(\bar{x}/W) = w_p(\bar{x}/\bar{y}) = 1$. Finally, we can assume that $Lstp(\bar{x}/\bar{y})$ is p-pure and one checks that $w_p(\bar{x}/WC) = 1$, where $C \subset G(D)^{large}$ is $Cb(Lstp(\bar{x}/\bar{y}))$. As in Lemma 1.18, this forces $C \subset cl_p(W)$ and then $C \subset cl_{p,F}(\bar{x})$, and then $C \subset cl_p(\bar{x})$. This gives the result, as $w_p(\bar{x}/\bar{y}) = w_p(\bar{x}/C) = w_p(\bar{x}/cl_p(\bar{x}) \cap cl_p(\bar{y}))$ and clearly $cl_p(C) = cl_p(\bar{x}) \cap cl_p(\bar{y})$, otherwise we could find $z \in cl_p(\bar{x}) \setminus cl_p(C)$ such that $w_p(\bar{x}/C) = w_p(\bar{x}/Cz)$, which contradicts 3 of Theorem 2.4

Induction Step.

We now inductively assume the result for $\bar{x}$ and $\bar{y}$ with $w_p(\bar{x}/\bar{y}) = m$ and suppose that $w_p(\bar{x}/\bar{y}) = m + 1$. Now again we can find $F \downarrow \bar{x}\bar{y}$ and $z_1 \ldots z_n \in p_F$ such that $z_1 \ldots z_n$ is weight equivalent to $\bar{x}$ over F. Then still $w_p(\bar{x}/F\bar{y}) = m + 1$ and we may assume $z_i \notin cl_p(F\bar{y})$ for some i, otherwise $\bar{x} \in cl_p(F\bar{y})$ which is not the case. Using the fact that $w_p(z_1/F\bar{y}) = 1$ say, then by a weight calculation we have that $w_p(\bar{x}/z_1 F\bar{y}) = m$. We now temporarily add F to the language, and take p-closure to include F. Then, working in $G(D)_F^{large}$, we have that $w_p(\bar{x}/\bar{y}) = m + 1$ and $w_p(\bar{x}/z_1 \bar{y}) = m$. Applying the induction

hypothesis to $G(D)^{large}_F$, we can find c in $G(D)^{large}_F$ such that $cl_p(c) = cl_p(\bar{x}) \cap cl_p(z_1\bar{y})$. Then $w_p(cz_1/\bar{y}) = 1$ as $c \in cl_p(z_1\bar{y})$ and $z_1 \notin cl_p(\bar{y})$. Therefore, we can find $d \in G(D)^{large}_F$ such that $cl_p(d) = cl_p(cz_1) \cap cl_p(\bar{y})$ and moreover $w_p(d) = w_p(cz_1) - 1 = w_p(c) - 1 = w_p(\bar{x}) - m - 1$. As $cl_p(cz_1) \cap cl_p(\bar{y}) = cl_p(\bar{x}) \cap cl_p(\bar{y})$, this tells us exactly that $w_p(\bar{x}/F\bar{y}) = w_p(\bar{x}/Fd)$ where $cl_{p,F}(d) = cl_{p,F}(\bar{x}) \cap cl_{p,F}(\bar{y})$. Now letting $C' = Cb(Lstp(\bar{x}/\bar{y}))$ and assuming as usual that $Lstp(\bar{x}/\bar{y})$ is p-pure, we have that $w_p(\bar{x}/FdC') = m$ otherwise as $C' \in acl(\bar{y})$ then $w_p(\bar{x}/F\bar{y}) < m$ which is not the case. Hence, by p-purity, we have that $C' \in cl_p(F\bar{x})$ and then as $F \downarrow_{\bar{x}} C'$, $C' \in cl_p(\bar{x})$. This proves the result. $\qquad\square$

So we have,

Theorem 3.11. *If D is linear then $G(D)$ and $G(D)^{large}$ are both modular.*

The following result is an easy adaptation of the proof in the stable case, given in [7], p269, $(iii) \rightarrow (iv)$. Here, observe that the corresponding notation to $D(p, A_0)$ is $G(D)^{large}$ as we have taken $A_0 = \emptyset$ and that Lemma 2.1 there corresponds to Lemma 2.7 in this paper.

Theorem 3.12. *If $G(D)^{large}$ is modular, then D is linear.*

Also, the following result is a straightforward adaptation of Theorem 3.10;

Theorem 3.13. *If $G(D)$ is modular, then $G(D)^{large}$ is modular.*

Combining these results gives that

Theorem 3.14. *D is linear iff $G(D)$ and $G(D)^{large}$ are modular.*

Remarks 3.15. *The localised analogues of Theorem 3.14 replace $G(D)$ and $G(D)^{large}$ by $G(D)_A$ and $G(D)^{large}_A$. For $G(D)^{large}_A$ there is nothing to prove and that $G(D)_A$ modular implies $G(D)$ linear is straightforward.*

Modularity of $G(D)^{large}$ can be seen as a local analogue of 1-basedness for the theory T. More precisely, we say that a simple theory T with elimination of hyperimaginaries is 1-based if, for any sets A and B in a big model $\mathcal{M}$, we have that $A \underset{acl(A) \cap acl(B)}{\downarrow} B$ where acl is taken in the sense of $\mathcal{M}^{eq}$. This is equivalent to the following condition (*) on canonical bases.

For any tuple $\bar{a}$ and parameters $B \subset \mathcal{M}^{eq}$, then $Cb(\bar{a}/B) \subset acl(\bar{a})$ (*)

The proof is fairly straightforward; Suppose that T is 1-based, then $\bar{a} \underset{acl(\bar{a}) \cap B}{\overset{\downarrow}{}} B$. By 2. of Fact 1.1, we have that (*) holds. Conversely, suppose that (*) holds, then given $\bar{a}, B$, by 2. of Fact 1.1 again, we must have that $\bar{a} \underset{acl(a) \cap acl(B)}{\overset{\downarrow}{}} B$. Now, T must be 1-based by the finite character of forking.

If T is a simple 1-based theory and D denotes the solution set of any regular type, then, if X, Y are p-closed subsets of $G(D)^{large}$ (therefore algebraically closed), by 1-basedness we must have that $X \underset{X \cap Y}{\overset{\downarrow}{}} Y$. Using 1. of Theorem 2.4, we get that $w_p(X/Y) = w_p(X/X \cap Y)$, so $G(D)^{large}$ is modular and, in particular, by Theorem 3.14, D is linear. The converse, in general, is false, there are examples of *stable* theories all of whose regular types are linear but which are not 1-based, see [7]. However, we can show the following, which should be compared with ($*$) above;

Theorem 3.16. *If D is linear, then given $a \in G(D)^{large}$ and parameters B such that $Lstp(a/B)$ is p-pure, we have $Cb(Lstp(a/B)) \subset cl_p(a)$, where cl_p denotes the p-closure operator on $G(D)^{large}$.*

Proof. The proof is a rather immediate consequence of the main Theorem 3.14. Let $C = Cb(Lstp(a/B))$, then $C \subset G(D)^{large}$ by Lemma 2.6. Let $E = cl_p(a) \cap cl_p(C)$. Then by modularity of $G(D)^{large}$, $w_p(a/C) = w_p(a/cl_p(C)) = w_p(a/E) = w_p(a/EC)$. Now, by p-purity of $Lstp(a/C)$, we must have that $a \underset{C}{\overset{\downarrow}{}} E$, therefore by 3. of Fact 1.1, $C = Cb(Lstp(a/EC))$. By Lemma 2.7, $C \subset cl_p(E)$ and hence $C \subset cl_p(a)$ as required. $\qquad\square$

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